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QUASICLASSICAL APPROXIMATION IN IONIZATION PROBLEMS

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The paper is devoted to the development of a complex trajectory technique (description of the tunnel effect in the quasiclassical case with the help of trajectories corresponding to imaginary "time"). The following problems are considered: ionization of a bound level in collisions with a charged particle; ionization of atoms by a constant electric field in the presence of a magnetic field; evaluation of the adiabatic corrections to the probability for the tunneling through a potential barrier of arbitrary shape.

1. INTRODUCTION

THE method of complex trajectories (called MCT in the following) was first formulated for the solution of the problem of the ionization of an atom in the field of a strong light wave.[1,2,3] However, the MCT has a more general significance and is in principle applicable to the calculation of the probability of tunneling w through an arbitrary barrier V(**r**, t) which changes with time (but satisfying, of course, the usual conditions for the use of quasiclassical methods, i.e., being sufficiently broad and smooth). The essence of the MCT consists in finding a sub-barrier trajectory which formally satisfies the classical equations of motion but corresponds to imaginary "time." The MCT is the natural generalization of quasiclassical approximation to the nonstationary case, which is particularly clear from the Feynman representation of the wave function as an integral over paths.[4,5] In the quasiclassical case the action S is large ($S/\hbar \gg 1$) and the main contribution to this integral comes from a narrow beam of trajectories close to the classical one. In problems connected with tunneling transitions there are no real trajectories of a classical particle going from the initial state (bound level in the atom) to infinity. This is also the reason why one has to go to imaginary "times" (strictly speaking this procedure must be regarded as an analytic continuation of the Feynman integral for $\psi(\mathbf{r}, t)$ to the saddle point).

In the original formulation of the MCT it was assumed that the external field acting on the system is periodic in time.[1,3] This is not necessary, however. In Sec. 2 we use the MCT in the problem of the ionization of a bound state (deuteron, atom, ion, etc.) in the collision with a charged particle. Since this problem is stationary it may be solved by the usual quasiclassical methods.[6,7] However, the consideration of sub-barrier trajectories is useful since it makes the problem more perspicuous. Formulas (17) and (18) are obtained for the probability of ionization of a bound state in the case of arbitrary masses. These formulas reduce in special cases to expressions for the probabilities of various processes: deuteron disintegration in the Coulomb field of the nucleus, ionization of negative ions in the collision with electrons, etc.

In Sec. 3 we consider the ionization of an atom in the presence of a magnetic field, and in Sec. 4 we discuss the adiabatic corrections to the probability for tunneling. The calculation is exact up to a factor in front of the exponential, which permits us to restrict ourselves to the consideration of the sub-barrier trajectory which minimizes Im S (in the following this trajectory will be called extremal).

2. IONIZATION OF A BOUND LEVEL IN COULOMB SCATTERING

Let the particle M be a bound state of two particles m_1, m_2 with binding energy ϵ:

$$M = m_1 + m_2 - \varepsilon, \quad \varepsilon = \varkappa^2/2\mu_{12} \ll m_1, m_2 \tag{1}$$

($\mu_{12} = m_1 m_2/(m_1 + m_2)$, $\hbar = 1$). The collision of M with charged particles can lead to the process $M \to m_1 + m_2$. An example for such a process may be (1) the disintegration of a light nucleus with small binding energy (for example, the deuteron or H^3) in the scattering by the Coulomb field of a heavy nucleus, (2) the ionization of an atom or an ion in the collision with electrons. We shall call all these processes ionization of a bound level in Coulomb scattering, although in the first example (for deuterons) this terminology is usually not employed. We find the ionization probability w with the help of the MCT, restricting ourselves for simplicity to the case where the Coulomb interaction between m_1 and m_2 vanishes at large distances (here m_1 is the mass of the charged particle and m_2 is the mass of the neutral particle). Because of this limitation our results refer to the ionization of negative ions (processes of the type $He^- + e \to He + 2e$) and to the disintegration of the deuteron.

It is convenient to begin with the case where the motion of the compound particle M occurs in a given external field. Such is the case for the scattering of deuterons on nuclei with large Z.[1)] Below, we shall therefore, for definiteness, speak of a "deuteron" although the masses m_1 and m_2 are not assumed equal to one another. The main contribution to the disintegration probability comes from the extremal trajectory corresponding to a head-on collision. Owing to the

1)The probability for the disintegration of the deuteron in the Coulomb field of a heavy nucleus was calculated in [6-8]. The connection with the results of these papers is discussed below.

Coulomb repulsion from the nucleus, the deuteron is stopped at the return point $x_0 = Ze^2/E$ (Z is the charge of the nucleus) and then begins the sub-barrier motion, which is of main interest for the disintegration process. Going over to an imaginary "time" $\tau = -it$ in the sub-barrier motion and taking account of the fact that in this case the potential $V(x) = Z/x$ changes sign,[3] we get the known expressions for the trajectory:[9]

$$x = x_0 \cos^2(\xi/2), \quad \tau = {}^1\!/_2 x_0 v_0^{-1}(\xi + \sin\xi),$$
$$\dot{x} = dx/d\tau = -v_0 \operatorname{tg}({}^1\!/_2\xi), \quad \operatorname{Im}\tilde{S}(\xi) = \eta(\xi - \sin\xi). \tag{2}$$

Here $E = Mv_0^2/2$ is the kinetic energy of the incoming deuterons $\eta = Ze^2/\hbar v_0$ is the Coulomb parameter for the deuteron, ξ is a variable which determines the sub-barrier trajectory, where $0 \le \xi < \pi$ ($\xi = \pi$ corresponds to the case where the deuteron falls into the nucleus), and $\tilde{S}$ is the reduced action.[3,9] At the point of return of the deuteron the energy-momentum laws are satisfied;[7] moreover, in the zero-range approximation for the nuclear forces one must take account of the continuity of the coordinates at the instant of disintegration ($x_d = x_p = x_n$). Setting the energy of the neutron $E_n = 0$ [this corresponds to the minimum of Im $\tilde{S}$, cf. (9)], we have the following equations for the determination of the point of disintegration:

$$E \operatorname{tg}^2\frac{\xi_0}{2} + \varepsilon = E_1 \operatorname{tg}^2\frac{\xi_1}{2},$$
$$p \operatorname{tg}\frac{\xi_0}{2} = p_1 \operatorname{tg}\frac{\xi_1}{2},$$
$$\frac{1}{E}\cos^2\frac{\xi_0}{2} = \frac{1}{E_1}\cos^2\frac{\xi_1}{2}, \quad E_1 = E - \varepsilon. \tag{3}$$

The sub-barrier motion of the proton is also described by (2), where we must replace x_0 and v_0 by $x_1 = Ze^2/E_1$ and $v_1 = \sqrt{2E_1/m_1}$. Here E_1 and p_1 are the energy and momentum of the proton at infinity, and ξ_0 and ξ_1 are the values of the parameter ξ for the deuteron and the proton at the disintegration point.

Equations (3) have the unique solution ($\xi_1 > \xi_0$)

$$\xi_0 = 2\operatorname{arc\,tg}\sqrt{m_1\varepsilon/m_2E}, \quad \xi_1 = 2\operatorname{arc\,tg}\sqrt{M\varepsilon/m_2E_1}. \tag{4}$$

The probability for a tunneling transition of the deuteron from a bound state to a state in the continuum is equal to (up to factor in front of the exponential)

$$w \sim \exp\{-2\operatorname{Im}(\tilde{S}_p + \tilde{S}_d)\}, \tag{5}$$

where

$$\tilde{S}_d = -i\eta(\xi_0 - \sin\xi_0),$$
$$\tilde{S}_p = i\eta_1(\xi_1 - \sin\xi_1), \quad \eta_1 = Ze^2/\hbar v_1, \tag{6}$$

(Im $\tilde{S}_d$ is negative since the deuteron moves into the depth of the barrier). Then

$$w \sim \exp\left\{-\frac{2Ze^2}{\hbar v}\left(\frac{\xi_1}{\sin\xi_1} - \frac{\xi_0}{\sin\xi_0}\right)\right\}, \tag{7}$$

where, by definition,

$$v = (m_1\varepsilon + m_2E)\,[2m_1m_2(m_1 + m_2)\,\varepsilon]^{-1/2}. \tag{8}$$

Besides the total disintegration probability, one is also interested in the energy spectrum of the outgoing particles. To this end one must consider the sub-barrier trajectories with $E_n \neq 0$ (E_n is the energy of the neutron after the disintegration) and expand $\tilde{S}(E_n)$ for small E_n. Omitting details of the calculation, we find

$$dw(E_n) = \text{const}\cdot e^{-\alpha(E)E_n}dE_n, \tag{9}$$
$$\alpha(E) = \eta_1(2\xi_1 - \sin 2\xi_1)\,/\,2E_1. \tag{10}$$

With increasing E_n the probability drops rapidly, i.e., the extremal trajectory corresponds to $E_n = 0$.[2)] The energy of the outgoing protons E_p is found from the relation $E_p + E_n = E - \epsilon$. In the case of equal masses ($m_1 = m_2$) the expressions (7) to (10) agree with the result of Lifshitz [cf. formulas (2.12), (2.13), and (3.4) in[7]].

However, it is seen from (4), (7), and (8) that the transition to the case of arbitrary masses does not reduce to the replacement of some mass by a reduced mass. The reason for this is that the masses m_1 and m_2 enter in (3), and the position of the disintegration point for the deuteron depends essentially on them. We note that the exact formula for w (with the correct coefficient), obtained by Landau and Lifshitz,[8] also refers only to the case of equal masses.

Let us now show in which way the MCT differs from the method of[7]. According to Lifshitz, the sub-barrier "trajectory" corresponds to complex values of the coordinate x; in particular, the disintegration point of the deuteron lies in the complex plane. In our method x remains real, but the time t becomes imaginary. For stationary problems both approaches are equivalent. However, it has been shown[3] that the transition to an imaginary time also allows one to solve nonstationary problems (for example, the problem of the many-photon ionization of an atom or ion by a strong light wave, where the potential $V(x, t) = -Fx\cos\omega t$ depends explicitly on time).

Let us formulate the condition for the applicability of our formulas. First, it is necessary that Im $\tilde{S} \gg 1$, where $\tilde{S} = \tilde{S}_p + \tilde{S}_d$. Since Im $\tilde{S}$ decreases with increasing E, this is a restriction on the energy E from above. From (7) we obtain for $E \gg \epsilon$

$$w \sim \exp\left\{-C\left(\frac{\varepsilon}{E}\right)^2\right\}, \quad C = \frac{4}{3}\frac{Z}{137}\sqrt{2\left(1 + \frac{m_1}{m_2}\right)\frac{m_1}{\varepsilon}}. \tag{7a}$$

Hence, the quasiclassical method is applicable for

$$\varepsilon \leqslant E < C^{1/2}\varepsilon, \tag{11}$$

and it is necessary that $C \gg 1$. This is guaranteed by a large Z (heavy nuclei) and a small binding energy ϵ. For example, for the deuteron $\epsilon = 2.2$ MeV and

$$C = \frac{8}{3}\frac{Z}{137}\left(\frac{m}{\varepsilon}\right)^{1/2} = 0.4Z$$

(m is the mass of the nucleon). Second, in writing down the conservation laws (3) we have neglected[3)] the radius of the deuteron $r_0 = \hbar/\sqrt{2\mu\epsilon}$, which requires L_d, $L_p \gg r_0$. Here L_d and L_p are the widths of the barrier for the deuteron and the proton:

$$L_d = x_0 - x_d = x_0\sin^2(\xi_0/2), \quad L_p = x_1\sin^2(\xi_1/2). \tag{12}$$

[2)] As shown in [7], Im S(E_n) has another minimum for $E < 2\epsilon$, which is located in the region $E_n < 0$. We do not consider the corresponding sub-barrier trajectory since it corresponds to another reaction: the capture of a neutron by the nucleus. A sub-barrier study of the (d,p) reaction on heavy nuclei was carried out by Ter-Martirosyan [10] (cf. also [11,12]).

[3)] This approximation was also used in [7-10].

For $E \gg \epsilon$ we obtain from (2) and (4)

$$L_d = \frac{Ze^2\varepsilon}{E^2}\frac{m_1}{m_2}, \quad L_p = \frac{Ze^2\varepsilon}{E^2}\left(1+\frac{m_1}{m_2}\right), \tag{12a}$$

which implies that the conditions L_d, $L_p \gg r_0$ are equivalent to (11) and do not impose new restrictions on the deuteron energy E. Finally, it is necessary that the disintegration point x_d lie outside the nucleus:

$$x_d = x_0 \cos^2(\xi_0/2) > R. \tag{13}$$

This leads to the inequalities

$$\frac{Ze^2}{\varepsilon}\frac{m_2}{M} > R \text{ for } E = \varepsilon, \quad \frac{Ze^2}{E} > R \text{ for } E \gg \varepsilon, \tag{13a}$$

from where for $m_2 \sim m_1$

$$E < Cr_0\varepsilon / R. \tag{14}$$

Since $R \approx$ (1 to 2)r_0, this condition is weaker than (11). Thus our consideration is valid in the threshold region of E, from $E = \epsilon$ to the values given by (11). For larger values of E the exponent in (7) becomes comparable with unity and the quasiclassical method is no longer applicable.

Near the threshold ($E \to \epsilon$) we have

$$w \sim \exp\left\{-\frac{2Ze^2}{\hbar}\sqrt{\frac{m_1}{2(E-\varepsilon)}}\right\} = \exp\left(-\frac{2Ze^2}{\hbar v_1}\right), \tag{15}$$

where v_1 is the velocity of the proton at infinity. This behavior of w is explained by the fact that $\operatorname{Im}\tilde{S}_d \to$ const but $\operatorname{Im}\tilde{S}_p \to \infty$ (the width of the barrier for protons is increased without limit).

Up to now we have neglected the recoil of the nucleus, regarding it as infinitely heavy. Let us now consider the general case where all three masses m_1, m_2, and M' (M' is the mass of the nucleus) are finite. Going over to the system of the center of inertia of the deuteron-nucleus system and writing the conservation laws for the disintegration, we obtain equations analogous to (3). Their solution has the form

$$\operatorname{tg}\frac{\xi_0}{2} = \left[\left(1+\frac{M}{M'}\right)\frac{m_1\varepsilon}{m_2E}\right]^{1/2}, \quad \operatorname{tg}\frac{\xi_1}{2} = \left[\left(1+\frac{m_1}{M'}\right)\frac{M\varepsilon}{m_2(E-\varepsilon)}\right]^{1/2}. \tag{16}$$

The probability for disintegration is

$$\frac{dw}{dE_n} = \exp\left\{-\frac{2Ze^2}{\hbar v}\left(\frac{\xi_1}{\sin\xi_1} - \frac{\xi_0}{\sin\xi_0}\right) - \frac{2Ze^2}{\hbar v'}(2\xi_1 - \sin 2\xi_1)\frac{E_n}{4(E-\varepsilon)}\right\}, \tag{17}$$

where

$$v = \frac{(1+M/M')m_1\varepsilon + m_2E}{(2m_1m_2M\varepsilon)^{1/2}},$$

$$v' = \frac{m_1+M'}{M+M'}\left[2\frac{m_1+M'}{m_1M'}(E-\varepsilon)\right]^{1/2}, \tag{18}$$

and E and E_n are the kinetic energy of the incoming particles and the neutron in the c.m.s. The value $E_n = 0$ is the most probable, as before. For $M' \to \infty$ formula (17) goes over into the earlier expression.

Formula (17) includes also another limiting case, when the particle M' moves and M is at rest. This occurs, for example, in the ionization of atoms and ions by electrons. In this case $m_1 = M' = m$ (the mass of the electron), $m_2 \approx M$ (the mass of the atom). Since we do not take account of the Coulomb interaction between the particles m_1 and m_2, our formulas refer strictly to the ionization of negative ions of the type He^-, I^-, etc. Denoting the energy of the electron in the l.s. by E and taking into account that $Z = 1$, $m \ll M$, we find

$$\frac{dw(E_n)}{dE_n} = \exp\left\{-\frac{2e^2\varkappa}{E+\varepsilon}\left[\left(\frac{\xi_1}{\sin\xi_1} - \frac{\xi_0}{\sin\xi_0}\right) + \frac{M}{4m}\left(\frac{\xi_1}{\sin\xi_1} - \frac{\sin 2\xi_1}{2\sin\xi_1}\right)\frac{E_n}{E-\varepsilon}\right]\right\} \tag{19}$$

where

$$\xi_0 = 2\operatorname{arc\,tg}\sqrt{\varepsilon/E}, \quad \xi_1 = 2\operatorname{arc\,tg}\sqrt{2\varepsilon/(E-\varepsilon)}, \tag{19a}$$

and E_n is the kinetic energy of the neutral atom (He, I, etc.) remaining after the ionization.

The exponential in (19) can be written in a form analogous to (5):

$$w \sim \exp\{-2\operatorname{Im}(\tilde{S}_1 + \tilde{S}_0)\}, \tag{20}$$

where $\tilde{S}_1 = i\kappa R(\xi_1/\sin\xi_1 - 1)$ is the action for the two electrons after the disintegration of the ion; $\tilde{S}_0 = -i\kappa R(\xi_0/\sin\xi_0 - 1)$ is the action for the incident electron as it moves into the depth of the barrier; $R = e^2/(E+\epsilon)$. Since the masses of the incident particle and the bound electron are comparable (in the given case $M' = m_1$), $R < R_0 = e^2/E$ (R_0 is the distance of closest approach between the electron and the negative ion). For the same reason the quantity $\operatorname{Im}\tilde{S}_0$ cannot be neglected compared to $\operatorname{Im}\tilde{S}_1$. In order to obtain the probability for tunneling w one must consider the simultaneous motion of the two electrons inside the barrier, as was done above.[4)]

For $E \gg \epsilon$ the exponential in (19) takes the form

$$w \sim \exp\left\{-C\frac{\varepsilon^2}{E^2}\right\}, \quad C = \frac{4}{3}\frac{e^2\varkappa}{\varepsilon}. \tag{21}$$

Introducing $I_0 = e^2/2a_0 = 13.6$ eV (I_0 is the ionization potential of the hydrogen atom) we rewrite (21) in the form

$$C = {}^8/_3\sqrt{I_0/\varepsilon}. \tag{21a}$$

The condition $C \gg 1$ is satisfied owing to the smallness of the binding energy of the negative ion as compared to I_0. Thus $\epsilon = 0.075$ eV and $C = 36$ for the He^- ion; and $\epsilon = 3.07$ eV and $C = 5.7$ for the I^- ion. The question may arise why the expression (21a) for C does not contain the small factor $\alpha = 1/137$ [as in (7a) for $Z = 1$]. The point is that $C \sim Z\alpha(m/\epsilon)^{1/2}$ and for the atom $m\alpha^2 \sim I_0$. Thus this small parameter enters already in the definition of I_0.

In the framework of the MCT we can also consider not-head-on collisions of particles where the sub-barrier trajectory is no longer one-dimensional.[14] The study of such trajectories allows one to find the angular distribution of the particles formed after the disintegration. We restrict ourselves here to the simplest case of far collisions: $\rho \gg \kappa^{-1}$, where ρ is the im-

4)We note that in the consideration of the ionization of negative ions by electron collisions [13] an approximation was used which reduces to the account of the sub-barrier motion of only the bound electron (for fixed position of the incident electron). This approximation is not applicable in the given problem because of the equality of the masses of the electrons. Therefore the expression for the ionization probability w obtained in [13] [cf. formula (20) (of [13]) is incorrect.

pact parameter and κ^{-1} is the radius of the bound state ($\epsilon = \kappa^2/2$). Then the problem simply reduces to the ionization of a bound level by a uniform electric field and can be solved by the method developed in [3]. Neglecting the curvature of the trajectory of the incident particle,[5)] we have an equation for the sub-barrier motion of the electron: $\ddot{\mathbf{r}} = \mathbf{R}/R^3$ where $\mathbf{r}$ is the radius vector of the bound electron and $\mathbf{R}(t) = \{vt, \rho, 0\}$ is the radius vector of the incident electron. Introducing the characteristic frequency $\omega = v/\rho$ and going over to the imaginary time $\tau = -i\omega t$, we have

$$\frac{d^2x}{d\tau^2} = -i\frac{F}{\omega^2}\frac{\tau}{(1-\tau^2)^{3/2}}, \quad \frac{d^2y}{d\tau^2} = -\frac{F}{\omega^2(1-\tau^2)^{3/2}}, \qquad (22)$$

where $F = \rho^{-2}$ (the origin of the coordinate system is put at the center of the atom, the x axis is parallel to the velocity of the incident particles). Using the boundary conditions for the extremal trajectory,[3]

$$\mathbf{r}(-\tau_0) = 0, \quad \mathbf{r}^2(-\tau_0) = \varkappa^2, \quad x(0) = \dot{y}(0) = 0, \qquad (22a)$$

we find an equation for the initial moment τ_0:

$$4\lambda/\sin 2\lambda - (\lambda/\sin\lambda)^2 - 1 = \gamma^2, \qquad (23)$$

where

$$\tau_0 = \sin\lambda, \quad \gamma = \omega\varkappa/F = \rho/\rho_0, \quad \rho_0 = (E\varepsilon)^{-1/2}, \qquad (23a)$$

The parameter $\gamma = \omega/\omega_t$ has the same physical meaning as for the ionization of an atom by a light wave. Calculating the action $\tilde{S}$ along the sub-barrier trajectory we find for the ionization probability

$$w \sim \exp\left\{-\frac{2F_0}{3F}g(\gamma)\right\}, \quad F_0 = \varkappa^3, \qquad (24)$$

where

$$g(\gamma) = \frac{3}{2\gamma}\frac{\sin\lambda - \lambda\cos\lambda}{1 - {}^1/_2\cos\lambda\,(\sin\lambda/\lambda + \lambda/\sin\lambda)} = \begin{cases} 1 - {}^4/_{15}\gamma^2, & \gamma \ll 1 \\ {}^3/_2\gamma^{-1}, & \gamma \gg 1 \end{cases}. \qquad (24a)$$

For $\gamma \ll 1$ formula (24) agrees with the ionization probability in a constant field F: $w \sim \exp(-2F_0/3F)$. For $\gamma \gg 1$ we have $w \sim \exp(-2\omega_0/\omega) = \exp(-\kappa^2\rho/v)$. This result can also be obtained with the help of nonstationary perturbation theory.

The approximation of a uniform field is applicable for $\rho \gg r_0$, where

$$r_0 = \frac{\varkappa^2}{2F}\left(\frac{2}{\gamma}\sin\frac{\lambda}{2}\right)^2$$

is the width of the barrier. Using (23) we transform this condition to

$$\rho \gg E^{-1}, \quad \text{if} \quad \gamma \gg 1 \quad (\rho \gg \rho_0)$$

$$\varepsilon^{-1/2} \ll \rho \ll \varepsilon^{-1}, \quad \text{if} \quad \gamma \ll 1 \quad (\rho \ll \rho_0). \qquad (25)$$

Since $E > \epsilon$, the uniform field approximation is manifestly correct in the region $\gamma \gg 1$. The adiabatic regime $\gamma \ll 1$ presupposes $\epsilon \ll E \ll 1$. When this inequality is fulfilled, formula (24) holds in the whole region $\rho \gg \kappa^{-3/2}$.

5)This is correct for $\rho \gg E^{-1}$. For definiteness we consider below the collision of a negative ion with an electron, where the atomic system of units is used: e = m = ħ = 1. The description of the motion of the incident electron with the help of the classical orbit is correct for $\rho \gg E^{-1/2} = \kappa^{-1}(\epsilon/E)^{1/2}$.

3. EFFECT OF A MAGNETIC FIELD ON THE IONIZATION

Let us consider the ionization of an atom under the action of constant fields—an electric field **F** and a magnetic field **H**. In order to apply the MCT we must solve the equation for the subbarrier motion,

$$\ddot{\mathbf{r}} = \mathbf{F} + \frac{1}{c}[\mathbf{vH}] - \frac{Z}{r^3}\mathbf{r}. \qquad (26)^*$$

The sub-barrier trajectory is easily found for Z = 0 (thus we are in fact considering below the ionization of a level bound by short-ranged forces). Taking account of the boundary conditions

$$\mathbf{r}(t_0) = 0, \quad \dot{\mathbf{r}}^2(t_0) = -\varkappa^2, \quad \operatorname{Im}\mathbf{r}(0) = 0, \qquad (27)$$

as usual for a δ potential,[3] we find the trajectory

$$x = \frac{i}{\omega}[a(\operatorname{sh}\tau_0 + \operatorname{sh}\tau) - b(\tau_0 + \tau)], \quad y = \frac{a}{\omega}(\operatorname{ch}\tau_0 - \operatorname{ch}\tau),$$

$$z = \frac{F\cos\alpha}{2\omega^2}(\tau_0^2 - \tau^2) - i\frac{v_z}{\omega}(\tau_0 + \tau), \quad -\tau_0 \leqslant \tau \leqslant 0. \qquad (28)$$

Here a and v_z are constants of integration, $b = cFH^{-1}\sin\alpha$, $\omega = eH/mc$ is the Larmor frequency, $\tau = i\omega t$, and τ_0 is the initial moment of the sub-barrier motion; the y axis is taken along **F**, the x axis is perpendicular to the (**F**, **H**) plane, and α is the angle between **F** and **H**.

Use of the condition Im **r**(0) = 0, which singles out the extremal among the classical trajectories, yields

$$b/a = \operatorname{sh}\tau_0/\tau_0, \quad v_z = 0. \qquad (29)$$

Substituting (28) and (29) in the condition $\dot{\mathbf{r}}^2(t_0) = -\kappa^2$, we arrive at a transcendental equation for the determination of τ_0:

$$\tau_0^2[1 - \sin^2\alpha(\operatorname{cth}\tau_0 - 1/\tau_0)^2] = \gamma^2, \qquad (30)$$

where

$$\gamma = \omega/\omega_t = \varkappa H/cF, \quad \omega_t = F/\varkappa \qquad (31)$$

(in atomic units c = 137). Then

$$\tau_0 = \tau_0(\gamma, \alpha) = \begin{cases} \gamma(1 + {}^1/_{18}\gamma^2\sin^2\alpha + \ldots) & \text{for } \gamma \ll 1 \\ \gamma/|\cos\alpha| & \text{for } \gamma \gg 1 \quad (\alpha \neq \pi/2) \end{cases} \qquad (32)$$

The meaning of the parameter γ is clear from the equation

$$\gamma = 2r_0/r_L, \qquad (33)$$

where $r_0 = \kappa^2/2F$ is the width of the barrier in the uniform field F and $r_L = c\kappa/eH$ is the Larmor radius. The magnetic field which bends the trajectory of the electron hinders the tunneling. This effect is noticeable for $r_L \lesssim r_0$, i.e., for $\gamma \gtrsim 1$.

The action $\tilde{S}$ along the extremal trajectory is

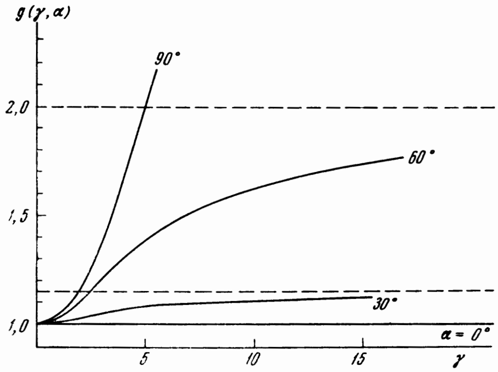
$$\tilde{S} = \int_{t_0}^{0}({}^1/_2\dot{\mathbf{r}}^2 + c^{-1}\mathbf{A}\dot{\mathbf{r}} - \varphi - {}^1/_2\varkappa^2)\,dt = iF_0 g(\gamma, \alpha)/3F, \qquad (34)$$

$$g(\gamma, \alpha) = \frac{1 + {}^1/_2\sin^2\alpha\,[1 - 3\operatorname{cth}\tau_0(\operatorname{cth}\tau_0 - 1/\tau_0)]}{[1 - \sin^2\alpha(\operatorname{cth}\tau_0 - 1/\tau_0)^2]^{3/2}} \qquad (35)$$

[for $\gamma \ll 1$, $g(\gamma, \alpha) = 1 + (1/30)\gamma^2\sin^2\alpha$]. The probability for ionization of the s level with binding energy $I = \kappa^2/2$ is equal to

$$w(\mathbf{F}, \mathbf{H}) = \omega_0|A|^2\frac{F}{F_0}\exp\left\{-\frac{2F_0}{3F}g(\gamma, \alpha)\right\}, \qquad (36)$$

*[**vH**] ≡ **v** × **H**.



The function g(γ,α) which determines the probability for ionization of an atom in the presence of electric and magnetic fields

where $\omega_0 = I/\hbar$, and A is a numerical coefficient in the asymptotic wave function of the bound state:

$$\psi(\mathbf{r}) \sim A\sqrt{\frac{\varkappa}{2\pi}}\frac{e^{-\varkappa r}}{r} \quad \text{for } \varkappa r \gg 1 \tag{37}$$

(for the δ potential A = 1). In (36) we have neglected the effect of the magnetic field on the factor in front of the exponential. The function $g(\gamma, \alpha)$ is shown in the figure. The inclusion of the magnetic field decreases the probability of ionization except in the case $\mathbf{H} \parallel \mathbf{F}$ [for $\alpha = 0$, $\tau_0 = \gamma$, and $g(\gamma, 0) = 1$].

For an estimate of the numerical value of the parameter γ we write (31) in the form

$$\gamma = (H/H_0)(F_0/F), \quad F_0 = \varkappa^3, \quad H_0 = c\varkappa^2. \tag{38}$$

For the hydrogen atom in the ground state $F_0 = 5.14 \times 10^9$ V/cm and $H_0 = 2.35 \times 10^9$ Oe. Thus values $\gamma \sim 1$ are attained, for example for $F \sim 10^5$ V/cm and $H \sim 10^5$ Oe. At the moment of leaving the sub-barrier region the electron has the momentum **p** directed along the vector $\mathbf{F} \times \mathbf{H}$, where

$$p = \varkappa \sin\alpha\,(1/\tau_0 - 1/\operatorname{sh}\tau_0)\,[1 - \sin^2\alpha\,(\operatorname{cth}\tau_0 - 1/\tau_0)^2]^{-1/2}. \tag{39}$$

The point of leaving the sub-barrier region is not the point where the particle is stopped. In this respect our problem differs from the motion in a potential field. The presence of a magnetic field leads to a "twisting" of the sub-barrier trajectory, which to some extent is analogous to the case of a light wave with electric polarization.[1] For a plane electromagnetic wave F = H, and $\gamma = \kappa/c = (2I/mc^2)^{1/2} \ll 1$; therefore the effect of the magnetic field of a wave is negligibly small. In the case of a constant field one must have $H \gg F$, in which case the effect of H becomes important.

4. ADIABATIC CORRECTIONS TO THE IONIZATION PROBABILITY

The transition to a variable field complicates considerably the calculation of the tunneling probability w. However, sometimes it is of interest to determine w for $\omega \ll \omega_t$ (ω is the oscillation frequency of the barrier, and ω_t is the frequency of tunneling through the barrier at rest). If the trajectory in the constant field is known, then the determination of the adiabatic corrections to the probability w reduces to quadratures.

Let

$$U(\mathbf{r}, t) = V + \delta V, \quad \delta V \sim \mu V, \quad \mu \ll 1. \tag{40}$$

Then the correction to the barrier penetrability has the form[3]

$$w \sim \exp\{-2\,\mathrm{Im}[\bar{W} + \delta\bar{W}_1 + \delta\bar{W}_2 + O(\mu^3)]\}, \tag{41}$$

where

$$\delta\bar{W}_1 = -\int_{t_0}^{0} \delta V(\mathbf{r}_0(t))\,dt,$$

$$\delta\bar{W}_2 = \frac{1}{2}\left[\delta V(t_0)\,\delta t_0 - \int_{t_0}^{0} \frac{\partial \delta V}{\partial \mathbf{r}_0}\mathbf{r}_1\,dt\right]. \tag{42}$$

Using (42) one can find the general form of the adiabatic correction to the barrier penetrability. Let

$$V(\mathbf{r}, t) = V_0(\mathbf{r}) + \omega t V_1(\mathbf{r}) + {}^1/_2(\omega t)^2 V_2(\mathbf{r}) + \ldots. \tag{43}$$

Setting

$$\mathbf{r}(t) = \mathbf{r}_0(t) + \omega\mathbf{r}_1(t) + {}^1/_2\omega^2\mathbf{r}_2(t) + \ldots, \tag{44}$$

where $\mathbf{r}_0(t)$ is the extremal trajectory in the constant field $V_0(\mathbf{r})$, and $\mathbf{r}(t)$ is the corresponding trajectory in the field $V(r, t)$, we find

$$\delta\bar{W} = -I_1\omega - {}^1/_2 I_2\omega^2 + O(\omega^3), \tag{45}$$

where

$$I_1 = \int_{t_0}^{0} V_1(\mathbf{r}_0(t))\,t\,dt, \tag{46}$$

$$I_2 = \int_{t_0}^{0}\left\{V_2(\mathbf{r}_0(t))\,t^2 + \dot{\mathbf{r}}_1^2(t) + \left(\mathbf{r}_1\frac{\partial}{\partial\mathbf{r}_0}\right)^2 V_0(\mathbf{r}_0)\right\}dt. \tag{47}$$

Thus the determination of the corrections $\sim\gamma^2$ inclusively, requires the knowledge of the trajectory $\mathbf{r}_0(t)$ and the first correction to it, $\mathbf{r}_1(t)$. The latter satisfies the equation

$$\ddot{\mathbf{r}}_1 = -t\nabla V_1, \quad \mathrm{Im}\,\mathbf{r}_1(0) = 0. \tag{48}$$

We illustrate the application of these formulas by the example of the ionization of an atom by elliptically polarized light (the Coulomb interaction is neglected). In this case $V(\mathbf{r}, t) = -F(x\cos\omega t + \epsilon y \sin\omega t)$, therefore $V_0 = -V_2 = -Fx$, and $V_1 = -\epsilon F y$. Then ($t_0 = i\kappa/F$):

$$\mathbf{r}_0(t) = \{{}^1/_2F(t^2 - t_0^2), 0, 0\},$$
$$\mathbf{r}_1(t) = \{0, {}^1/_6\varepsilon F t(t^2 - t_0^2), 0\}$$
$$I_1 = 0, \quad I_2 = {}^1/_{15}F^2 t_0^5(1 - {}^1/_3\varepsilon^2). \tag{49}$$

The ionization probability $w(F, \omega)$ for $\gamma \ll 1$ has the form ($\gamma = \omega/\omega_t$)

$$w(F,\omega) \sim \exp\{-2\,\mathrm{Im}(\bar{W} + \delta\bar{W})\}$$
$$= \exp\left\{-\frac{2F_0}{3F}\left[1 - \frac{1}{10}\left(1 - \frac{\varepsilon^2}{3}\right)\gamma^2 + \ldots\right]\right\}. \tag{50}$$

This expression agrees with the expansion (for $\gamma \ll 1$) of the exact formula for w in the case of elliptical polarization of light as obtained in[1]. However, the calculation of δW by (45) to (47) is much simpler than the corresponding calculation in the case of arbitrary values of γ.

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