THERMODYNAMICS OF A WEAKLY RELATIVISTIC PLASMA

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We find relativistic corrections to the thermodynamic functions of a completely ionized plasma caused by electromagnetic interactions. The plasma itself is described by Darwin's Lagrangian which enables us to take relativistic effects into account up to terms of order v^2/c^2. We show that the simplified Darwin Hamiltonian can not be applied and that the papers by Krisan and Havas which use that Hamiltonian are incorrect.

1. INTRODUCTION

THE aim of the present paper is a study of relativistic effects in a plasma. It is well known that for a rigorous description of a system of interacting charged particles we must consider the electromagnetic field to be an independent object and must introduce apart from the variables $\mathbf{r}_i$, $\mathbf{v}_i$ which refer to the particles also the generalized coordinates for the field. If, however, the particle velocities are small compared to the velocity of light and if there is no external field, in which we include electromagnetic waves incident from the outside, we can describe the system up to terms of order $(v/c)^2$ using the Darwin Lagrangian:[1]

$$L^D = -\sum_i m_i c^2 \sqrt{1-\beta_i^2} - \sum_{i<j} \frac{e_i e_j}{r_{ij}} \left(1 - \frac{1}{2} \widehat{\beta_i \beta_j} : \tilde{\delta}_{ij}\right). \quad (1.1)$$

Here $\beta = v/c$, $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$. The dots between the tensors indicate contraction over the coordinate indices, the caret over the characters indicates a tensor, and the tensor $\tilde{\delta}_{ij}$ is equal to $(\alpha, \beta = x, y, z)$

$$(\tilde{\delta}_{ij})_{\alpha\beta} = \delta_{\alpha\beta} + (n_{ij})_\alpha (n_{ij})_\beta, \quad \mathbf{n}_{ij} = \mathbf{r}_{ij} / |\mathbf{r}_{ij}|. \quad (1.2)$$

For the first term in (1.1) we retain the exact relativistic value in order to be able to compare the results with those for a perfect relativistic gas later on.

The Lagrangian (1.1) contains only the variables $\mathbf{r}_i$, $\mathbf{v}_i$ referring to the particles while the field is taken into account through the second term which describes the interaction between the particles up to relativistic corrections of order $(v/c)^2$. There is here therefore no "independent electromagnetic field." This procedure is justified also by the fact that for a sufficiently rarefied plasma the (average) mean free path of a photon is usually appreciably larger than the dimensions of the system and the radiation leaves the plasma freely. We shall in the following consider just such "transparent" systems (typical example--plasma in thermonuclear devices).

From (1.1) follows an expression for the generalized momentum

$$\mathbf{P}_i = \frac{\partial L}{\partial \mathbf{v}_i} = \frac{m_i \mathbf{v}_i}{\sqrt{1-\beta_i^2}} + \sum_{j\neq i} \frac{e_i e_j}{r_{ij}} \frac{\mathbf{v}_j}{2c^2} \cdot \tilde{\delta}_{ij} \quad (1.3)$$

and the energy conservation law

$$E = \sum_i \frac{m_i c^2}{\sqrt{1-\beta_i^2}} + \sum_{i<j} \frac{e_i e_j}{r_{ij}} \left(1 + \frac{1}{2} \widehat{\beta_i \beta_j} : \tilde{\delta}_{ij}\right) = \text{const.} \quad (1.4)$$

It is convenient to introduce instead of the velocities $\mathbf{v}_i$ the simple relativistic particle momenta $\mathbf{p}_i = m_i \mathbf{v}_i / \sqrt{(1-\beta_i^2)}$. We can then write Eqs. (1.3) and (1.4) in the form

$$\mathbf{P}_i = \mathbf{p}_i + \sum_{j\neq i} \frac{e_i e_j}{r_{ij}} \frac{\mathbf{v}_i(\mathbf{p})}{2c^2} \cdot \tilde{\delta}_{ij}, \quad (1.5)$$

$$E(r, \mathbf{p}) = \sum_i \sqrt{m_i^2 c^4 + c^2 p_i^2} + \sum_{i<j} \frac{e_i e_j}{r_{ij}} \left(1 + \frac{1}{2} \beta_i(\mathbf{p}) \beta_j(\mathbf{p}) : \tilde{\delta}_{ij}\right). \quad (1.6)$$

Here $v(p) = (p/m)/\sqrt{(1 + (p/mc)^2)}$.

If we express the energy in terms of the generalized momenta we obtain the "total Darwin Hamiltonian" $H^D(r, P)$ which is, however, difficult to write down explicitly. This difficulty is connected with the impossibility of a simple inversion of Eq. (1.5). Indeed, even if in the second term in (1.5) we put approximately $v(p) \to p/m$ (which is legitimate by virtue of the approximation taken in (1.1)) solving through the method of successive approximations the relation

$$\mathbf{p}_i = \mathbf{P}_i - \sum_{j\neq i} \frac{e_i e_j}{r_{ij} 2 m_j c^2} \tilde{\delta}_{ij} \cdot \mathbf{p}_j, \quad (1.7)$$

we get an infinite series for $p = p(P_{gen})$:

$$\mathbf{p}_i = \mathbf{P}_i - \sum_{j\neq i} \frac{e_i e_j}{r_{ij} 2 m_j c^2} \tilde{\delta}_{ij} \cdot \mathbf{P}_j + \sum_{j\neq i} \sum_{k\neq j} \frac{e_i e_j e_j e_k}{r_{ij} 2 m_j c^2 r_{jk} 2 m_k c^2} \tilde{\delta}_{ij} \cdot \tilde{\delta}_{jk} \cdot \mathbf{P}_k - \ldots \quad (1.8)$$

One usually breaks this series off after the first two terms assuming that the others are small because of the higher powers of $1/c$. The Hamiltonian obtained through such a procedure from the energy (1.6)

$$H^D_{\text{simpl}}(r, \mathbf{P}) = \sum_i \sqrt{m_i^2 c^4 + c^2 P_i^2} + \sum_{i<j} \frac{e_i e_j}{r_{ij}} \left(1 - \frac{\widehat{\mathbf{P}_i \mathbf{P}_j} : \tilde{\delta}_{ij}}{2 m_i m_j c^2}\right) \quad (1.9)$$

we shall call the "simplified Darwin Hamiltonian".[1] Landau and Lifshitz, for instance, introduced such a simplified Hamiltonian[1)] in [2] and it was used in a number of papers to construct a thermodynamics and kinetics of a plasma.

This procedure, however, is valid only for systems containing a small number of interacting particles. For instance, a simplified Hamiltonian of the type (1.9) with quantum-mechanical generalizations can be used to consider the positronium atom.[7] It is, however, completely inapplicable to describe a plasma and results obtained using it are erroneous. To see this we con-

1)Usually the first term in (1.9) is also expanded up to terms of order $1/c^2$.

sider in more detail the infinite series (1.8). Neglecting relativistic effects for ions we shall assume that the summation in (1.8) is only over the electrons and we write the series in the form

$$\mathbf{p}_i = \mathbf{P}_i - \varepsilon \sum_{j\neq i} \frac{\tilde{\delta}_{ij}}{r_{ij}} \cdot \mathbf{P}_j + \varepsilon^2 \sum_{j\neq i}\sum_{k\neq j} \frac{\tilde{\delta}_{ij}\cdot\tilde{\delta}_{jk}}{r_{ij}r_{jk}} \cdot \mathbf{P}_k - \varepsilon^3 \sum_{j\neq i}\sum_{k\neq j}\sum_{l\neq k} \frac{\tilde{\delta}_{ij}\cdot\tilde{\delta}_{jk}\cdot\tilde{\delta}_{kl}}{r_{ij}r_{jk}r_{kl}} \cdot \mathbf{P}_l + \ldots, \quad (1.10)$$

where $\epsilon = e^2/2mc^2$. If we consider the electron component of the plasma as a continuous fluid we can consider the momenta $\mathbf{p}$ and $\mathbf{P}$ to be functions of the coordinates $\mathbf{p} = \mathbf{p}(\mathbf{r})$ and $\mathbf{P} = \mathbf{P}(\mathbf{r})$. Assuming in first approximation the particles to be independent and replacing the sum by an integral $(\sum_j \to n_e \int d\mathbf{r}_j)$ we get

$$\mathbf{p}(\mathbf{r}_1) = \mathbf{P}(\mathbf{r}_1) - (\varepsilon n_e)\int \frac{\tilde{\delta}_{12}}{r_{12}} \cdot \mathbf{P}_2\, d\mathbf{r}_2 + (\varepsilon n_e)^2 \int \frac{\tilde{\delta}_{12}\cdot\tilde{\delta}_{23}}{r_{12}r_{23}} \cdot \mathbf{P}_3\, d\mathbf{r}_2\, d\mathbf{r}_3 - \ldots \quad (1.11)$$

Changing to the Fourier representation

$$\{\mathbf{p}(\mathbf{r}), \mathbf{P}(\mathbf{r})\} = \int \{\mathbf{p}_k, \mathbf{P}_k\}\, e^{i\mathbf{k}\mathbf{r}}\, d\mathbf{k}, \quad \frac{\tilde{\delta}_{ij}}{r_{ij}} = \int \left(\hat{\delta} - \frac{\widehat{\mathbf{k}\mathbf{k}}}{k^2}\right) e^{i\mathbf{k}\mathbf{r}_{ij}} \frac{d\mathbf{k}}{\pi^2 k^2}, \quad (1.12)$$

we have

$$\mathbf{p}_k = \mathbf{P}_k - \frac{(\hat{\delta} - \widehat{\mathbf{k}\mathbf{k}}/k^2)}{k^2 d_c^2} \cdot \mathbf{P}_k \left\{1 - \frac{1}{k^2 d_c^2} + \frac{1}{(k^2 d_c^2)^2} - \frac{1}{(k^2 d_c^2)^3} + \ldots\right\}. \quad (1.13)$$

We have here introduced $1/d_c^2 = 4\pi n_e e^2/mc^2$; the parameter d_c has the dimensions of a length. The sum inside the braces is equal to $k^2 d_c^2/(k^2 d_c^2 + 1)$. The parameter $1/c^2$ occurs thus in (1.13) in combination with the electron density n_e and the real expansion parameter turns out to be $1/kd_c$. Performing the subsequent integration over k we see easily that it is impossible to limit ourselves in the expansion (1.13) to one or a few of the first terms (which all turn out to be diverging) and it is necessary to sum the whole series. It is therefore impossible to retain in Eq. (1.10) only the first two terms dropping the remainder and hence the simplified Darwin Hamiltonian (1.9) also turns out to be inapplicable to a plasma. In particular, the thermodynamic functions of a plasma determined, for instance, in the paper by Krisan and Havas [3] using the Hamiltonian (1.9) are incorrect (see below).

On the other hand, the initial Darwin Lagrangian (1.1) correctly takes into account relativistic corrections connected with magnetic interactions between the particles and with retardation effects.

In the following we calculate the thermodynamic functions of a weakly relativistic plasma on the basis of the Lagrangian (1.1).

2. RELATIVISTIC CORRECTIONS IN A PLASMA

Krisan and Havas [3] determined the thermodynamic potential of a plasma $\Omega(\Theta, V, \mu_e, \mu_i) = -pV$ starting from a grand ensemble:

$$dW_{N_e, N_i} = \frac{1}{N_e!\, N_i!} \exp\frac{1}{\Theta}(\Omega + \mu_e N_e + \mu_i N_i - H_{N_e+N_i})\, d\Gamma_{N_e, N_i}, \quad \sum_{N_e, N_i} \int dW_{N_e, N_i} = 1. \quad (2.1)$$

For $H_{N_e+N_i}$ they used here the simplified Darwin Hamiltonian (1.9) which as we shall show can not be applied to a plasma.

For our purpose it is convenient to evaluate the free energy

$$F(\Theta, V, N_e, N_i) = -\Theta \ln Z, \quad Z = \frac{1}{N_e!\, N_i!} \int \exp\left\{-\frac{1}{\Theta} H_{N_e+N_i}\right\} d\Gamma^{\text{gen}}_{N_e, N_i} \quad (2.2)$$

rather than the potential $\Omega(\Theta, V, \mu)$. We shall understand here by $H_{N_e+N_i}$ the "total Darwin Hamiltonian" corresponding to the original Lagrangian (1.1). In (2.2) the quantity $d\Gamma^{\text{gen}} = \Pi((2\pi\hbar)^{-3} d\mathbf{P} d\mathbf{r})$ is the number of states in the phase space of the coordinates and the generalized momenta. As it is difficult to write the "total Darwin Hamiltonian" out explicitly it is expedient to change in (2.2) from the generalized to the simple momenta $\mathbf{p} = m\mathbf{v}/\sqrt{(1 - \beta^2)}$ and then

$$d\Gamma^{\text{gen}} = J\left(\frac{P}{p}\right) d\Gamma^{\text{simp}}, \quad Z = \frac{1}{N_e!\, N_i!} \int \exp\left\{-\frac{1}{\Theta} E_{N_e+N_i}\right\} J\left(\frac{P}{p}\right) d\Gamma^{\text{simp}}. \quad (2.3)$$

Here $E_{N_e+N_i} = E(\mathbf{r}, \mathbf{p})$ is the energy of the system defined by Eq. (1.6), $J(P/p)$ is the Jacobian of the transition from generalized to simple momenta, and the quantity $d\Gamma^{\text{simp}} = \Pi((2\pi\hbar)^{-3} d\mathbf{p}\, d\mathbf{r})$ also refers to the simple momenta. Introducing the notation

$$\Lambda^\alpha = \int e^{-\varepsilon_\alpha^0/\Theta} d\mathbf{p}, \quad \varepsilon_\alpha^0 = \sqrt{m_\alpha^2 c^4 + c^2 p^2}, \quad \alpha = e, i, \quad (2.4)$$

we can write (2.3) in the form $Z = Z^{\text{id}}_{\text{rel}} Z^{\text{int}}$, where $Z^{\text{id}}_{\text{rel}}$ is the partition function of a relativistic gas neglecting interactions and Z^{int} the correction connected with the Coulomb and relativistic interactions:

$$Z^{\text{int}} = \int \prod_{i+e} \frac{d\mathbf{r}}{V} \prod_e \frac{d\mathbf{p}}{\Lambda_e} \prod_i \frac{d\mathbf{p}}{\Lambda_i} J\left(\frac{P}{p}\right) \times \exp\left[-\frac{1}{\Theta}\left(\sum_i \varepsilon_i^0 + \sum_e \varepsilon_e^0 + U^{\text{int}}_{\text{Coul}} + U^{\text{int}}_{\text{rel}}\right)\right]. \quad (2.5)$$

Here

$$U^{\text{int}}_{\text{Coul}} = \sum_{k<j} \frac{e_k e_j}{r_{kj}}, \quad U^{\text{int}}_{\text{rel}} = \sum_{k<j} \frac{e_k e_j}{2r_{kj}} \widehat{\beta_k \beta_j} : \tilde{\delta}_{kj} \quad (2.6)$$

(the summation is over the ions and the electrons). In the simplest case we can restrict ourselves to take the relativistic corrections into account for the electrons only. If we write for them, moreover, Eq. (1.5) for the generalized momentum approximately in the form

$$\mathbf{P}_e = \mathbf{p}_e + \sum_{e\neq e'} \frac{e^2}{2mc^2} \mathbf{p}_{e'} \cdot \frac{\tilde{\delta}_{ee'}}{r_{ee'}}, \quad (2.7)$$

the Jacobian $J(P/p)$ will refer only to the electrons and will then be independent of the momenta. Then (2.5) becomes

$$Z^{\text{int}} = \int \prod_{e+i} \frac{d\mathbf{r}}{V} e^{-U^{\text{int}}_{\text{Coul}}/\Theta} J_{ee}\left(\frac{P}{p}\right) \int \prod_e \frac{d\mathbf{p}}{\Lambda_e} \exp\left[-\frac{1}{\Theta}\left(\sum_e \varepsilon_e^0 + U^{\text{int}}_{ee\,\text{rel}}\right)\right]. \quad (2.8)$$

We consider the last integral. As first approximation we can take the non-relativistic expression for the eigen energy of the electrons:[2)] $\epsilon^0 = \sqrt{(m^2c^4 + c^2p^2)} = mc^2 + p^2/2m$. We then get

[2)]One can show that taking the next terms of this expansion into account would exceed the accuracy of the Darwin approximation for the interaction. However, in the Appendix we give an evaluation of Z^{int} also with the exact value of ϵ^0.

$$S_e \equiv \int \prod_e \frac{d\mathbf{p}}{\Lambda_e} \exp\left[-\frac{1}{\Theta}\left(\sum_e \varepsilon_e{}^0 + U_{ee\ \mathrm{rel}}^{\mathrm{int}}\right)\right]$$
$$\cong \int \prod_e \frac{d\mathbf{p}}{(2\pi m\Theta)^{3/2}} \exp\left[-\frac{1}{2\Theta m}\left(\sum_e p_e{}^2 + \frac{r_0}{2}\sum_{e\neq e'}\sum \widehat{\mathbf{p}_e\mathbf{p}_{e'}} : \frac{\widetilde{\delta}_{ee'}}{r_{ee'}}\right)\right], \quad (2.9)$$

where $r_0 = e^2/mc^2$ is the "classical electron radius." Equation (2.7) corresponds to a symmetric Jacobian matrix:

$$J\left(\frac{P}{p}\right) = |J_{\lambda\mu}| = \left|\frac{\partial P_{e\alpha}}{\partial p_{e'\beta}}\right| = \begin{vmatrix} 1 & 0 & 0 & J_{1x,2x} & J_{1x,2y} & J_{1x,2z} & J_{1x,3x} & \cdot \\ 0 & 1 & 0 & J_{1y,2x} & J_{1y,2y} & J_{1y,2z} & J_{1y,3x} & \cdot \\ 0 & 0 & 1 & J_{1z,2x} & J_{1z,2y} & J_{1z,2z} & J_{1z,3x} & \cdot \\ J_{2x,1x} & J_{2x,1y} & J_{2x,1z} & 1 & 0 & 0 & J_{2x,3x} & \cdot \\ J_{2y,1x} & J_{2y,1y} & J_{2y,1z} & 0 & 1 & 0 & J_{2y,3x} & \cdot \\ J_{2z,1x} & J_{2z,1y} & J_{2z,1z} & 0 & 0 & 1 & J_{2z,3x} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}. \quad (2.10)$$

The indices $\lambda\mu$ run here through the set (1x, 1y, 1z; 2x, 2y, 2z; 3x, ... ; N_ex, N_ey, N_ez) and the off-diagonal elements of this matrix are according to (2.7) equal to $J_{\mu\lambda} = J_{e,\alpha;e'\beta} = \frac{1}{2} r_0 (\widetilde{\delta}_{ee'})_{\alpha\beta}/r_{ee'}$.

One notes easily that we can use this matrix to write the index of the exponential in (2.9) in the form

$$\sum_e p_e{}^2 + \frac{r_0}{2}\sum_{e\neq e'}\sum \widehat{p_e p_{e'}} : \frac{\widetilde{\delta}_{ee'}}{r_{ee'}} = \sum_{\lambda=1}^{3N_e}\sum_{\mu=1}^{3N_e} p_\lambda J_{\lambda\mu} p_\mu. \quad (2.11)$$

The integral (2.9) then turns out to be equal to

$$S_e = \int \prod_e \frac{d\mathbf{p}}{(2\pi m\Theta)^{3/2}} \exp\left(-\frac{1}{2m\Theta}\sum_\lambda\sum_\mu p_\lambda J_{\lambda\mu} p_\mu\right)$$
$$= (\det\|J_{\lambda\mu}\|)^{-1/2} = \frac{1}{\sqrt{J_{ee}(P/p)}}, \quad (2.12)$$

and for the correction Z^{int} defined by Eq. (2.8) we have[3)]

$$Z^{\mathrm{int}} = \int \exp\left\{-\frac{U_{\mathrm{Coul}}^{\mathrm{int}}}{\Theta}\right\} \sqrt{J_{ee}\left(\frac{P}{p}\right)} \prod_{i+e} \frac{d\mathbf{r}}{V}. \quad (2.13)$$

As we have here implied the transition to the limits N_i, $N_e \to \infty$, $V \to \infty$ with $N_i/V = n_i$ and $N_e/V = n_e$ the Jacobian $J_{ee}(P/p)$ is a determinant of infinitely high order of the form (2.10). One can show that for such determinants the following relation holds:

$$\det(1 + \hat{\alpha}) = \exp[\mathrm{Sp}\ln(1 + \hat{\alpha})], \quad (2.14)$$

and using this we get

$$\lim_{N_e\to\infty} J_{ee}(P/p) = \exp\left\{-\tfrac{1}{2}\sum_\lambda\sum_\mu J_{\lambda\mu}J_{\mu\lambda} + \tfrac{1}{3}\sum_\lambda\sum_\mu\sum_\nu J_{\lambda\mu}J_{\mu\nu}J_{\nu\lambda} + \ldots\right\}. \quad (2.15)$$

In the sums which occur here, e.g., ($e_1 \neq e_2 \neq \ldots \neq e_m$)

$$\sigma_m = \sum_{\lambda_1}\sum_{\lambda_2}\ldots\sum_{\lambda_m} J_{\lambda_1\lambda_2}J_{\lambda_2\lambda_3}\ldots J_{\lambda_m\lambda_1} = \left(\frac{r_0}{2}\right)^m \sum_{e_{1,2,\ldots,m}=1}^{N_e\to\infty} \frac{\mathrm{Sp}\,\widetilde{\delta}_{e_1e_2}\cdot\widetilde{\delta}_{e_2e_3}\ldots\widetilde{\delta}_{e_me_1}}{r_{e_1e_2}r_{e_2e_3}\ldots r_{e_me_1}}, \quad (2.16)$$

we can approximately change the summation over the particles to an integration ($\sum_{e_1} \to n_e \int d\mathbf{r}_1$) and we then get

$$\sigma_m = \left(\frac{n_e e^2}{2mc^2}\right)^m \underbrace{\int\int\ldots\int}_{m} \frac{d\mathbf{r}_1 d\mathbf{r}_2\ldots d\mathbf{r}_m}{r_{12}r_{23}\ldots r_{m1}} \mathrm{Sp}(\widetilde{\delta}_{12}\cdot\widetilde{\delta}_{23}\ldots\widetilde{\delta}_{m1}). \quad (2.17)$$

The trace operation refers here to the coordinate indices x, y, z of the tensor $(\widetilde{\delta}_{ee'})_{\alpha\beta}$. Using moreover the Fourier representation

$$\frac{(\widetilde{\delta}_{ee'})_{\alpha\beta}}{r_{ee'}} = \frac{\delta_{\alpha\beta}}{r_{ee'}} + \frac{(r_{ee'})_\alpha (r_{ee'})_\beta}{r_{ee'}{}^3} = \int \frac{d\mathbf{k}}{\pi^2 k^2} e^{i\mathbf{k}\mathbf{r}_{ee'}}\left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right), \quad (2.18)$$

we get from (2.17)

$$\sigma_m = \frac{V}{4\pi^3}\int \frac{d\mathbf{k}}{(k^2 d_c{}^2)^m} \qquad \frac{1}{d_c{}^2} = \frac{4\pi n_e e^2}{mc^2}. \quad (2.19)$$

We shall call the length d_c the "relativistic Debye radius."

We get thus for the Jacobian

$$J_{ee'}\left(\frac{P}{p}\right) = \exp\left(-\frac{\sigma_2}{2} + \frac{\sigma_3}{3} - \frac{\sigma_4}{4} + \ldots\right)$$
$$\cong \exp\left[-\frac{V}{4\pi^3}\int d\mathbf{k}\sum_{m=2}^{\infty}\frac{1}{m}\left(\frac{-1}{k^2 d_c{}^2}\right)^m\right] = \exp\left(-\frac{V}{3\pi d_c{}^3}\right). \quad (2.20)$$

The convergence of the series which occurs here indicates the screening of the transverse interactions in a plasma over lengths of order d_c. Using (2.20) we can write the correction (2.13) in the form

$$Z^{\mathrm{int}} = Z_{\mathrm{rel}}^{\mathrm{int}} Z_{\mathrm{Coul}}^{\mathrm{int}} \text{ где } Z_{\mathrm{rel}}^{\mathrm{int}} \cong \sqrt{J_{ee'}(P/p)} = e^{-V/6\pi d_c{}^3}, \quad (2.21)$$

while we show in the Appendix how one can evaluate the usual Debye correction $Z_{\mathrm{Coul}}^{\mathrm{int}}$

$$Z_{\mathrm{Coul}}^{\mathrm{int}} = \int \exp\left\{-\frac{U_{\mathrm{Coul}}}{\Theta}\right\} \prod_{i+e} \frac{d\mathbf{r}}{V} = \exp\left(\frac{V}{12\pi d^3}\right), \quad (2.22)$$

We can thus write the free energy of the plasma in the form

$$F = -\Theta\ln Z = -\Theta\ln(Z_{\mathrm{rel}}^{\mathrm{id}} Z_{\mathrm{Coul}}^{\mathrm{int}} Z_{\mathrm{rel}}^{\mathrm{int}}) = F_{\mathrm{rel}}^{\mathrm{id}} + \Delta F_{\mathrm{Coul}}^{\mathrm{int}} + \Delta F_{\mathrm{rel}}^{\mathrm{int}}, \quad (2.23)$$

where $F_{\mathrm{rel}}^{\mathrm{id}}$ is the free energy of a relativistic ideal gas while the correction terms (Debye and relativistic) are equal to

$$\Delta F_{\mathrm{Coul}}^{\mathrm{int}} = -\frac{V\Theta}{12\pi d^3}, \quad \Delta F_{\mathrm{rel}}^{\mathrm{int}} = \frac{V\Theta}{6\pi d_c{}^3}. \quad (2.24)$$

Here $d_c = (4\pi n_e e^2/mc^2)^{-1/2} = c/\omega_{0,e}$ is the relativistic and $d = (\sum_{e,i} 4\pi n_\alpha e_\alpha^2/\Theta)^{-1/2}$ is the usual Debye radius. Using Eqs. (2.23) and (2.24) one easily determines the corrections to the other thermodynamic functions. In particular, we get for the pressure

$$p = -\frac{\partial F}{\partial V} = p_{\mathrm{rel}}^{\mathrm{id}} + \Delta p_{\mathrm{Coul}}^{\mathrm{int}} + \Delta p_{\mathrm{rel}}^{\mathrm{int}}, \quad (2.25)$$

where

$$\Delta p_{\mathrm{Coul}}^{\mathrm{int}} = -\frac{\Theta}{24\pi d^3}, \quad \Delta p_{\mathrm{rel}}^{\mathrm{int}} = +\frac{\Theta}{12\pi d_c{}^3}. \quad (2.26)$$

For the energy we have correspondingly

[3)]The square root of the determinant of a square matrix such as (2.11) is usually called a "Pfaffian."

$$E=-\Theta^2\frac{\partial}{\partial\Theta}\left(\frac{F}{\Theta}\right)=E^{\text{id}}_{\text{rel}}+\Delta E^{\text{int}}_{\text{Coul}}+\Delta E^{\text{int}}_{\text{rel}},\quad \Delta E^{\text{int}}_{\text{Coul}}=-\frac{V\Theta}{8\pi d^3}. \tag{2.27}$$

The correction $\Delta E^{\text{int}}_{\text{rel}}$ turns out to vanish and this corresponds to dropping the radiation field in the Darwin Lagrangian.

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APPENDIX

A MORE RIGOROUS CONSIDERATION OF THE RELATIVISTIC CORRECTIONS

In Sec. 2 we evaluated the relativistic corrections in the first non-vanishing approximation in v/c. In particular, we used in Eq. (2.9) the non-relativistic approximation $\epsilon^0 \cong mc^2 + p^2/2m$ for the eigen energy of the particles. We shall now consider the original Darwin Lagrangian (1.1) as a model Lagrangian for the plasma without assuming that v/c is small. This way of stating the problem was used, in particular, in the paper by Krisan and Havas[3] who, however, erroneously used the simplified Darwin Hamiltonian (1.9) which made it impossible to calculate correctly relativistic effects (see below).

Equation (2.5) remains valid and for our purpose it is convenient to write it in the form

$$Z^{\text{int}}=\langle J(P/p)\exp(-U^{\text{int}}/\Theta)\rangle. \tag{A.1}$$

The pointed brackets indicate here averaging over the $N_e + N_i$ electrons and ions:

$$\langle(\ldots)\rangle=\int\prod_{i+e}\frac{d\mathbf{r}}{V}\prod_i\frac{d\mathbf{p}}{\Lambda_i}e^{-\varepsilon_i^0/\Theta}\prod_e\frac{d\mathbf{p}}{\Lambda_e}e^{-\varepsilon_e^0/\Theta}(\ldots), \tag{A.2}$$

and we assume here that the limit N_i, $N_e \to \infty$ has been taken.

Using the normal Ursell-Mayer method[8] we write the exponent with the binary interaction which occurs in (A.1) in the form

$$\exp(-U^{\text{int}}/\Theta)=\prod_{i<j}(1+f_{ij})=1+\sum_{i<j}f_{ij}+\sum ff+\sum fff+\ldots \tag{A.3}$$

We shall call the quantity

$$f_{ij}=\exp(-U_{ij}^{\text{int}}/\Theta)-1\cong-\frac{e_ie_j}{\Theta r_{ij}}\left(1+\frac{1}{2}\beta_i(\mathbf{p})\beta_j(\mathbf{p}):\tilde{\delta}_{ij}\right) \tag{A.4}$$

an "energy bond" of two particles. From the Darwin Lagrangian (1.1) we get

$$\mathbf{P}_i=\frac{\partial L}{\partial\mathbf{v}_i}=\mathbf{p}_i+\sum_{j\neq i}\frac{e_ie_j}{r_{ij}2c^2}\mathbf{v}_j(p)\cdot\tilde{\delta}_{ij}\quad\left(\mathbf{v}(p)=\frac{\mathbf{p}/m}{\sqrt{1+(p/mc)^2}}\right), \tag{A.5}$$

so that the Jacobian for the change of variables can be written in the form ($\alpha_{\lambda\lambda} = 0$)

$$J\left(\frac{P}{p}\right)=\left|\frac{\partial P_{i\alpha}}{\partial p_{j\beta}}\right|=\begin{vmatrix}1+\alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots\\ \alpha_{21} & 1+\alpha_{22} & \alpha_{23} & \cdots\\ \alpha_{31} & \alpha_{32} & 1+\alpha_{33} & \cdots\\ \cdots & \cdots & \cdots & \cdots\end{vmatrix}=|\delta_{\lambda\mu}+\alpha_{\lambda\mu}|. \tag{A.6}$$

The indices λ and μ go through the set (1_ex, 1_ey, 1_ez; $2_ex, \ldots, N_ez$; 1_ix, 1_iy, 1_iz; 2_ix, $2_iy, \ldots, N_iz$) so that the Jacobian (A.5) is a determinant of rank $3(N_e + N_i) \to \infty$. Such a determinant can be expanded in powers of the quantity α (summation over γ = x, y, z):

$$\alpha_{i\alpha,\,j\beta}=\frac{e_ie_j}{r_{ij}2m_jc^2}(\tilde{\delta}_{ij})_{\alpha\gamma}\left(\frac{\delta_{\beta\gamma}}{[1+(p_j/m_jc)^2]^{1/2}}-\frac{p_{j\beta}p_{j\gamma}/m_j^2c^2}{[1+(p_j/m_jc)^2]^{3/2}}\right) \tag{A.7}$$

which we shall call a Jacobian bond". Then we have

$$J=1+\Sigma\mu^{(1)}(\alpha)+\Sigma\mu^{(2)}(\alpha)+\Sigma\mu^{(3)}(\alpha)+\ldots \tag{A.8}$$

Here $\Sigma\mu^{(k)}(\alpha)$ is the sum of all principal minors of rank k of the matrix $\alpha_{\lambda\mu} = J_{\lambda\mu} - \delta_{\lambda\mu}$.

We can then write Eq. (A.1) in the form

$$Z^{\text{int}}=\langle J\exp(-U^{\text{int}}/\Theta)\rangle=1+\langle\Sigma\mu^{(1)}(\alpha)+\Sigma f\rangle+\langle\Sigma\mu^{(2)}(\alpha)+\Sigma\mu^{(1)}(\alpha)f+\Sigma ff\rangle+\langle\Sigma\mu^{(3)}(\alpha)+\Sigma\mu^{(2)}(\alpha)f+\Sigma\mu^{(1)}(\alpha)ff+\Sigma fff\rangle+\ldots \tag{A.9}$$

Each term of this series is formed from "energy" and "Jacobian" bonds of the particles combined in a group. If we depict an f bond by a full-drawn and an α bond by a dotted line we can assign diagrams such as the ones drawn in the figure for the group of third order in f or α:

a b c d

It is well known[9] that for the evaluation of the usual Debye corrections (without the relativistic terms) it is sufficient to limit oneself in the sum (A.3) to considering "connected loops" of the kind b. Unconnected chains of the kind a disappear when the quasi-neutrality of the plasma is taken into account. Considering the minors in the expansion (A.8) one can see that the bonds α_{ij} occur in them necessarily in the form of connected loops of the kind c. Terms with intersecting bonds of the kind d can be dropped as taking them into account exceeds our accuracy. In this approximation we can average independently the loops with "energy bonds" f and "Jacobian bonds" α. In other words, we can in Eq. (A.1) approximately assume that

$$Z^{\text{int}}=\langle J\exp(-U^{\text{int}}/\Theta)\rangle\cong\langle J\rangle\langle\exp(-U^{\text{int}}/\Theta)\rangle. \tag{A.10}$$

As we noted before the following formula holds for a Jacobian of the kind (A.6)

$$\det(\hat{1}+\hat{\alpha})=\exp[\operatorname{Sp}\ln(1+\hat{\alpha})]=\exp\left(\sum\alpha_{\lambda\lambda}-{}^1/_2\sum\alpha_{\lambda\mu}\alpha_{\mu\lambda}+{}^1/_3\sum\alpha_{\lambda\mu}\alpha_{\mu\nu}\alpha_{\nu\lambda}\ldots\right). \tag{A.11}$$

A similar expression is also obtained when we calculate the average value of the Jacobian. Since the Jacobian bonds in the expansion (A.8) are averaged independently the factor in Eq. (A.7) for the $\alpha_{\lambda\mu}$ depending on the momenta gives when we average over the momenta

$$\left\langle\frac{\delta_{\beta\gamma}}{[1+(p_j/m_jc)^2]^{1/2}}-\frac{p_{j\beta}p_{j\gamma}/m_j^2c^2}{[1+(p_j/m_jc)^2]^{3/2}}\right\rangle=\delta_{\beta\gamma}\xi_j,$$

$$\xi_j=\left\langle\left(1-\frac{1}{3}\beta^2\right)\sqrt{1-\beta^2}\right\rangle \tag{A.12}$$

and we see easily that we get for $\langle J\rangle$ the expression

$$\langle J\rangle=\exp\left(-\frac{\sigma_2}{2}+\frac{\sigma_3}{3}-\frac{\sigma_4}{4}+\ldots\right), \tag{A.13}$$

where

$$\sigma_m = \Big(\sum_{\alpha=i,e} \frac{n_\alpha e_\alpha^2 \xi_\alpha}{2m_\alpha c^2}\Big)^m \underbrace{\int\int\ldots\int}_{m} \frac{d\mathbf{r}_1\,d\mathbf{r}_2\ldots d\mathbf{r}_m}{r_{12}r_{23}\ldots r_{m1}} \mathrm{Sp}\,(\tilde{\delta}_{12}\cdot\tilde{\delta}_{23}\ldots\tilde{\delta}_{m1})$$
$$= \frac{V}{4\pi^3}\int\frac{d\mathbf{k}}{(k^2d_\xi^2)^m}, \quad \frac{1}{d_\xi^2} = \sum_{\alpha=i,e}\frac{4\pi n_\alpha e_\alpha^2\xi_\alpha}{m_\alpha c^2} \quad (A.14)$$

(cf. Eqs. (2.17) to (2.19) in the text). Collecting the series in the exponent in (A.8) we get by analogy with Eq. (2.20) in the text

$$\langle J\rangle = \exp(-V/3\pi d_\xi^3). \quad (A.15)$$

We now consider the second factor in Eq. (A.10) for $Z^{\rm int}$ (see (A.3)):

$$\langle \exp(-U^{\rm int}/\Theta)\rangle = \langle 1 + \Sigma f + \Sigma ff + \Sigma fff + \ldots\rangle. \quad (A.16)$$

It is well known [8] that in Mayer's theory one proves that for a gas consisting of one kind of molecules with binary interactions depending only on the coordinates this quantity reduces to an exponent with a sum over irreducible integrals:

$$\langle \exp(-U^{\rm int}/\Theta)\rangle = \exp\Big[V\sum_{s\geq 2}\frac{n^s}{s!\,V}\underbrace{\int\int\ldots\int}_{s}(\Sigma\Pi f)\,d\mathbf{r}_1\,d\mathbf{r}_2\ldots d\mathbf{r}_s\Big], \quad (A.17)$$

where $n = N/V$ is the gas density. In our case of a plasma containing electrons and ions where the interaction depends on coordinates and velocities we have

$$\Big\langle \exp\Big(-\frac{1}{\Theta}U^{\rm int}(\mathbf{r},\mathbf{p})\Big)\Big\rangle = \exp\Bigg[V\sum_{\substack{s_e\ s_i\\(s_e+s_i\geq 2)}}^{\infty\ \infty}\frac{n_e^{s_e}n_i^{s_i}}{s_e!\,s_i!\,V\Lambda_e^{s_e}\Lambda_i^{s_i}}$$
$$\times\underbrace{\int\int\ldots\int}_{s_e+s_i}\exp\Big\{-\Theta^{-1}\sum_{s_e+s_i}\varepsilon^0\Big\}(\Sigma\Pi f_{ij})\prod_{s_e+s_i}(d\mathbf{r}\,d\mathbf{p})\Bigg]. \quad (A.18)$$

A similar expression was also obtained in the paper by Krisan and Havas [3] but they determined the thermodynamic potential $\Omega(\Theta, V, \mu) = -pV$ and the fugacities z_e and z_i occurred therefore instead of the electronic and ionic densities n_e and n_i. The largest difference, however, consisted in the fact that in Eq. (A.18) the quantity f_{ij} is defined by Eq. (A.4) while in their case they had

$$f_{ij} = -\frac{e_i e_j}{\Theta r_{ij}}\Big(1 - \frac{\widehat{\mathbf{p}_i\mathbf{p}_j}}{2m_i m_j c^2}:\tilde{\delta}_{ij}\Big), \quad (A.19)$$

corresponding to the "simplified Darwin Hamiltonian" (1.9). We draw the attention of the reader to the difference in sign in front of the relativistic corrections in (A.19) and (A.4) (the difference between $\beta = v/c$ and p/mc can be considered to be unimportant). Just this difference in sign (a plus sign for us and a minus sign in [3]) made it impossible for Krisan and Havas to perform their calculation consistently. Indeed, using Eq. (A.4) for f_{ij} and considering only connected loops, we find[4)]

$$\frac{1}{\Lambda_e^{s_e}\Lambda_i^{s_i}V}\underbrace{\int\int\ldots}_{s_e+s_i}\exp\Big(-\Theta^{-1}\sum_{s_e+s_i}\varepsilon^0\Big)(\Sigma\Pi f_{ij})\prod_{s_e+s_i}(d\mathbf{r}\,d\mathbf{p})$$
$$= \Big(-\frac{e_e^2}{\Theta}\Big)^{s_e}\Big(-\frac{e_i^2}{\Theta}\Big)^{s_i}\frac{(s_e+s_i-1)!}{2V}\Bigg[\underbrace{\int\int\ldots\int}_{s_e+s_i}\frac{d\mathbf{r}_1\,d\mathbf{r}_2\ldots d\mathbf{r}_{s_e+s_i}}{r_{12}r_{23}\ldots r_{s_e+s_i,1}}$$
$$+\Big(\frac{1}{2}\Big)^{s_e+s_i}\Big(\frac{\langle\beta_e^2\rangle}{3}\Big)^{s_e}\Big(\frac{\langle\beta_i^2\rangle}{3}\Big)^{s_i}\underbrace{\int\int\ldots}_{s_e+s_i}\frac{d\mathbf{r}_1\,d\mathbf{r}_2\ldots d\mathbf{r}_{s_e+s_i}}{r_{12}r_{23}\ldots r_{s_e+s_i,1}}\mathrm{Sp}\,(\tilde{\delta}_{12}\cdot\tilde{\delta}_{23}\ldots\tilde{\delta}_{s_e+s_i,1})\Bigg]$$
$$= \frac{(s_e+s_i-1)!}{16\pi^3}\Big(-\frac{e_e^2}{\Theta}\Big)^{s_e}\Big(-\frac{e_i^2}{\Theta}\Big)^{s_i}$$
$$\times\Big[1 + 2\Big\langle\frac{\beta_e^2}{3}\Big\rangle^{s_e}\Big\langle\frac{\beta_i^2}{3}\Big\rangle^{s_i}\Big]\int d\mathbf{k}\Big(\frac{4\pi}{\mathbf{k}^2}\Big)^{s_e+s_i}. \quad (A.20)$$

Substituting this expression into (A.18) and evaluating the sum we get

$$\Big\langle\exp\Big(-\frac{1}{\Theta}U^{\rm int}\Big)\Big\rangle$$
$$= \exp\Big[\frac{V}{16\pi^3}\int dk\sum_{s=2}^{\infty}\frac{1}{s}\Big(\frac{-1}{k^2d^2}\Big)^s + \frac{V}{8\pi^3}\int dk\sum_{s=2}^{\infty}\frac{1}{s}\Big(\frac{-1}{k^2d_c^2}\Big)^s\Big]. \quad (A.21)$$

We introduced here the usual and the "relativistic" Debye radii:

$$d = \Big(\sum_{\alpha=e,i}\frac{4\pi n_\alpha e^2}{\Theta}\Big)^{-1/2}, \quad d_c = \Big(\sum_{\alpha=e,i}\frac{4\pi n_\alpha e_\alpha^2}{\Theta}\frac{\langle\beta_\alpha^2\rangle}{3}\Big)^{-1/2}. \quad (A.22)$$

For the single-type integrals of sums occurring in (A.21) we find

$$\int dk\sum_{s=2}^{\infty}\frac{1}{s}\Big(\frac{-1}{k^2d^2}\Big)^s = \int dk\Big[\frac{1}{k^2d^2} - \ln\Big(1+\frac{1}{k^2d^2}\Big)\Big] = \frac{4\pi^2}{3d^3}, \quad (A.23)$$

and we have thus for (A.21)

$$\Big\langle\exp\Big(-\frac{1}{\Theta}U^{\rm int}\Big)\Big\rangle = \exp\Big(\frac{V}{12\pi d^3} + \frac{V}{6\pi d_c^3}\Big). \quad (A.24)$$

It is further easy to show that when we average over the relativistic Maxwell distribution we get the relation (cf. (A.12))

$$\xi = \Big\langle\Big(1 - \frac{1}{3}\beta^2\Big)\sqrt{1-\beta^2}\Big\rangle = \frac{\nu}{3}\langle\beta^2\rangle = \frac{\nu^2}{K_2(\nu)}\int_\nu^\infty K_2(x)\frac{dx}{x^2}, \quad (A.25)$$

where $\nu = mc^2/\Theta$, while $K_2(x)$ is a MacDonald function so that therefore the length d_ξ is the same as the "relativistic Debye radius" d_c introduced in (A.22). Combining the results we find

$$Z^{\rm int} = \langle J\rangle\Big\langle\exp\Big(-\frac{1}{\Theta}U^{\rm int}\Big)\Big\rangle = \exp\Big(\frac{V}{12\pi d^3} - \frac{V}{6\pi d_c^3}\Big), \quad (A.26)$$

so that we find finally for the free energy

$$F = -\Theta\ln(Z^{\rm id}_{\rm rel}Z^{\rm int}) = F^{\rm id}_{\rm rel} + \Delta F^{\rm int}_{\rm Coul} + \Delta F^{\rm int}_{\rm rel}, \quad (A.27)$$

where

$$\Delta F^{\rm int}_{\rm rel} = V\Theta/6\pi d_c^3,$$

which in form is the same as Eq. (2.24) in the text, but now d_c is defined by Eq. (A.22) which only goes over into Eq. (2.19) of section 2 when $\Theta \ll mc^2$.

In contrast to our results Krisan and Havas [3] erroneously using the simplified Darwin Hamiltonian and Eq. (A.19) following from it for f_{ij}, where the relativistic correction differs in sign from the corresponding correction in our Eq. (A.4) obtain (cf. (A.21))

4) Because of the linearity in one of the vectors v (or p in [3]) the unconnected chains disappear from the relativistic terms and also the mixed type connected chains containing Coulomb and relativistic terms. If the quasi-neutrality of the plasma is taken into account the connected loops from the Coulomb terms disappear.

$$Z_{\mathrm{rel}}^{\mathrm{int}} = \exp\left[\frac{V}{8\pi^3}\int d\mathbf{k}\sum_{s=2}^{\infty}\frac{1}{s}\left(\frac{1}{k^2d_c^2}\right)^s\right]$$

$$= \exp\left\{\frac{V}{8\pi^3}\int d\mathbf{k}\left[-\frac{1}{k^2d_c^2}-\ln\left(1-\frac{1}{k^2d_c^2}\right)\right]\right\}. \qquad (A.28)$$

Here, in contrast to (A.21) – (A.23) the integrand has a singularity at $k^2d_c^2 = 1$ and to remove the divergence of the integral the authors needed to introduce arbitrary assumptions (they made the artificial substitution $k^2 \to k^2 + (1/d)^2$ after which the final equations of ref. [3] lose their independent interest).

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