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COORDINATE TRANSFORMATIONS THAT ELIMINATE SINGULARITIES ON THE GRAVITATIONAL RADIUS IN THE SCHWARZSCHILD METRIC

P. I. FOMIN

Physico-technical Institute, Ukrainian Academy of Sciences

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It is shown that the transformations that eliminate the singularities in the Schwarzschild metric, being represented in the form of local Lorentz transformations, have the following common feature: the transformation parameter V, which has the meaning of the local velocity of the new system of coordinates relative to the initial system, reaches the velocity of light on the Schwarzschild sphere. A general prescription is formulated for the construction of coordinates in terms of which the Schwarzschild metric has no singularities on the gravitational radius.

1. The question of singularities on the gravitational radius in the well known Schwarzschild metric

$$ds^2 = \varphi(r)\,dt^2 - \varphi^{-1}(r)\,dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\Phi^2), \tag{1}$$

where $\varphi(r) = 1 - a/r$ and a is the "gravitational radius," has recently again attracted attention. An important aspect is the clarification of the invariant character of these singularities, i.e., the possibility of their elimination with the aid of suitable coordinate transformations [1,2]. Concrete examples of such transformations were obtained in the papers of Kruskal [3] 1)

$$T(r,t) = a\sqrt{\frac{r}{a} - 1}\,\exp\left(\frac{r}{2a}\right)\mathrm{sh}\,\frac{t}{2a}, \tag{2}$$

$$R(r,t) = a\sqrt{\frac{r}{a} - 1}\,\exp\left(\frac{r}{2a}\right)\mathrm{ch}\,\frac{t}{2a},$$

and Rylov [4]:

$$T(r,t) = t - \int dr/\varphi(r)f(r), \tag{3}$$

$$R(r,t) = \int f(r)\,dr/\varphi(r) - t$$

($f(r)$ is an arbitrary function with properties $f^2(a) = 1$, $f^2(r) > 1$ for $r > a$, and $f^2(r) < 1$ for $r < a$).

In the present paper we discuss from a unified point of view the entire class of transformations eliminating the Schwarzschild singularities. This is effected by representing the general coordinate transformations in the form of local Lorentz transformations. This results in the following common characteristic of the transformations that eliminate the Schwarzschild singularities: they correspond to local Lorentz transformations with a parameter V (local velocity in the new coordinate system relative to the initial one), reaching a value equal to unity (velocity of light) on the Schwarzschild sphere.

The existence of the indicated characteristic makes it possible to formulate a general prescription for finding the coordinates in which there are no Schwarzschild singularities.

1) We present here the form of the Kruskal transformations only for the region $r \geqslant a$, $R > |T| > 0$.

2. We perform in (1) the following coordinate transformation

$$r = r(R, T), \quad t = t(R, T), \tag{4}$$

$$dr = r_R dR + r_T dT, \quad dt = t_R dR + t_T dT, \tag{4'}$$

and require that the property of orthogonality of the metric be conserved. In the new coordinates we have

$$ds^2 = F dT^2 - G dR^2 - r^2(R, T)(d\theta^2 + \sin^2\theta\, d\Phi^2), \tag{5}$$

where

$$F = t_T^2\varphi - r_T^2\varphi^{-1}, \quad G = r_R^2\varphi^{-1} - t_R^2\varphi,$$
$$t_R t_T \varphi = r_R r_T \varphi^{-1}. \tag{5'}$$

The meaning of the symbols is obvious from the context.

We seek a transformation to coordinates R and T such that the form (5) has no singularities in the entire region $r > 0$, i.e., in this region we have

$$0 < F < \infty, \quad 0 < G < \infty, \tag{6}$$

and the coordinates T and R retain, respectively, the temporal and spatial character.

We introduce the local Lorentz transformation relating (1) and (5):

$$\varphi^{1/2}dt = (F^{1/2}dT + VG^{1/2}dR)(1 - V^2)^{-1/2},$$
$$\varphi^{-1/2}dr = (G^{1/2}dR + VF^{1/2}dT)(1 - V^2)^{-1/2}. \tag{7}$$

The transformation parameter V has the meaning of the local relative velocity of the two reference systems, but obviously only in the region $r > a$, where $\varphi^{1/2}$ is real. In the region $r < a$, where φ is negative, the coordinate t acquires a space-like character, and r becomes time-like [5]. By virtue of this, the Lorentz transformation for $r < a$ should be of the form

$$(-\varphi)^{1/2}dt = (G^{1/2}dR + UF^{1/2}dT)(1 - U^2)^{-1/2},$$
$$(-\varphi)^{-1/2}dr = (F^{1/2}dT + UG^{1/2}dR)(1 - U^2)^{-1/2} \tag{7'}$$

with U serving as the relative velocity of the two coordinate systems. It is easy to see, however, the formulas (7′) can be transformed into (7) by substituting $U = V^{-1}$. We can therefore use the single form (7) in the entire region $r > 0$, bearing in mind that in the region $r < a$ the meaning of the relative velocity of the two systems is assumed by V^{-1}.

Comparing (7) with (4) we get

$$V = \frac{r_T}{\varphi t_T} = \frac{\varphi t_R}{r_R} = \frac{t_R}{t_T}\left(\frac{F}{G}\right)^{1/2} = \frac{r_T}{r_R}\left(\frac{G}{F}\right)^{1/2},$$

$$V^2 = t_R r_T / t_T r_R, \tag{9}$$

$$F = t_T^2 \varphi(r)(1-V^2) = r_T^2(1-V^2)/\varphi(r)V^2,$$
$$G = t_R^2\varphi(r)(1-V^2)/V^2 = r_R^2(1-V^2)/\varphi(r), \tag{10}$$

Equivalent formulas are produced by using the inverse transformation

$$T = T(r, t), \quad R = R(r, t), \tag{11}$$

namely

$$-V = \frac{R_t}{\varphi R_r} = \frac{\varphi T_r}{T_t} = \frac{T_r}{R_r}\left(\frac{F}{G}\right)^{1/2} = \frac{R_t}{T_t}\left(\frac{G}{F}\right)^{1/2}. \tag{12}$$

$$V^2 = T_r R_t / T_t R_r \tag{13}$$

$$F = \varphi(r)/T_t^2(1-V^2) = V^2/T_r^2\varphi(r)(1-V^2),$$
$$G = V^2\varphi(r)/R_t^2(1-V^2) = 1/R_r^2\varphi(r)(1-V^2). \tag{14}$$

Conditions (6) for the absence of singularities in the metric (5) impose obvious requirements with respect to the first derivatives of the transformation functions (4) and (11), in terms of which F, G, and V are expressed. The most definite and interesting consequence of these requirements is the condition

$$\varphi(r)/(1-V^2) > 0. \tag{15}$$

Since $\varphi(r)$ reverses sign when $r = a$, V^2 should simultaneously pass through unity.

We emphasize that the condition $V^2 > 1$ when $r < a$ does not denote actual realization of velocities exceeding the speed light, since in this region the relative velocity of the two reference systems is represented not by V but by V^{-1}. The fact that V and V^{-1} coincide on the Schwarzschild sphere ensures continuity of the transition.

3. Integration of the differential relations (8) or (12) with allowance for condition (15) gives the general prescription for constructing coordinate systems (R, T) in which there are no Schwarzschild singularities. Thus, on the basis of (12), we have

$$V(r,t)\varphi(r)\frac{\partial R}{\partial r} + \frac{\partial R}{\partial t} = 0,$$
$$V(r,t)\frac{\partial T}{\partial t} + \varphi(r)\frac{\partial T}{\partial r} = 0. \tag{16}$$

Specifying arbitrarily the function $V(r, t)$ subject to condition (15), and integrating (16), we get the corresponding $R(r, t)$ and $T(r, t)$. It is necessary here, of course, to choose such solutions of (10) that ensure satisfaction of conditions (6) in the entire region $r > 0$, since the condition (15) is necessary but in itself not sufficient for satisfaction of (6).

Condition (15) is easiest to satisfy in the case $V = V(r)$. The solution of (16) then takes the form

$$R = \tilde{R}(\int dr/\varphi(r)V(r) - t),$$
$$T = \tilde{T}(t - \int dr V(r)/\varphi(r)), \tag{17}$$

where $\tilde{R}$ and $\tilde{T}$ are arbitrary functions of their arguments. In the particular case of linear functions we arrive at the Rylov transformations (3).

How is the condition (15) to be satisfied in the general case $V = V(r, t)$? To answer this question, we note, first, that (in the region $r > a$) V is a velocity (referred to a local-Galilean coordinate system), with which a point with fixed value of R moves relative to the Schwarzschild coordinate system. Indeed, for the metric (1) the local-Galilean velocity is determined by the expression $V = \varphi^{-1/2}dr/\varphi^{1/2}dt$. Substituting here dr/dt determined from the condition $R(r, t) = \text{const}$, we get $V = -R_t/\varphi R_r$, which coincides with the first equation of (12). Thus, at fixed R the variable t is a function of r, satisfying the equation

$$V(r, t)\varphi(r)\,dt/dr = 1. \tag{18}$$

The solution $t = t(r)$, when substituted in $V(r, t(r))$, transforms it into a function of r. The condition (15) denotes in the general case that the function $V(r, t(r)) = (\varphi(r)\,dt/dr)^{-1}$ defined in precisely this manner should go through ± 1 when r goes through a.

The asymptotic form of (18) near[2)] the Schwarzschild sphere has consequently the form

$$\pm\varphi(r)\,dt/dr = 1, \tag{19}$$

which coincides with the asymptotic form of the equation for the radial geodesic in the Schwarzschild space. Indeed, for such a geodesic there follows from (1) the relation

$$\left(\frac{dr}{dt}\right)^2 = \varphi^2(r)[1 - \text{const}\cdot\varphi(r)], \tag{20}$$

which is equivalent to (19) when $r \to a$.

Integrating (19) in the region $r > a$, we find that near the Schwarzschild sphere a point with fixed R moves in accordance with the law

$$\pm(t - t_0(R)) = r + a\ln((r-a)/a). \tag{21}$$

It follows therefore that $t \to \mp\infty$ when $r \to a + 0$ and R is fixed and consequently the behavior of $V(r, t)$ when $r \approx a$ and $|\hat{t}|$ is large should have the form

$$V(r, t) = \pm 1 + A(r-a) + Be^{\pm t/a} + \ldots, \tag{22}$$

where

$$A = \left[\frac{\partial}{\partial r}V(r,t)\right]_{\substack{r=a\\ t=\mp\infty}}, \quad B = \pm a\left[e^{\mp t/a}\frac{\partial}{\partial t}V(r,t)\right]_{\substack{r=a\\ t=\mp\infty}}.$$

By way of illustration let us examine the example $V = V(t) = -\tanh(t/2a)$, which obviously satisfies the condition (22). The solution of Eqs. (16) in the region $r > a$ with this V is

$$R = \bar{R}(x), \quad T = \bar{T}(y), \tag{23}$$

where

$$x = r + a\ln\frac{r-a}{a} + 2a\ln\operatorname{ch}\frac{t}{2a},$$
$$y = r + a\ln\frac{r-a}{a} + 2a\ln\operatorname{sh}\frac{t}{2a}, \tag{24}$$

and $\bar{R}$ and $\bar{T}$ are arbitrary functions that satisfy the requirements (6). The choice of $R(x) = \exp(x/2a)$ and $T(y) = \exp(y/2a)$ leads to the Kruskal metric (2).

We note that Eqs. (16) can be easily integrated also in the more general case $V = V_1(r)V_2(t)$ which admits of separation of the variables. Here again the solutions take the form (23), but now

$$x = \int\frac{dr}{V_1(r)\varphi(r)} - \int dt V_2(t),$$
$$y = \int\frac{dt}{V_2(t)} - \int dr\frac{V_1(r)}{\varphi(r)}. \tag{25}$$

In accordance with the foregoing, $V_1(r)$ and $V_2(t)$

2)The exact meaning of the concept "near the Schwarzschild sphere" is made clear below and in general includes, besides $(r - a) \ll a$, also the condition $|t| \gg a$.

should be such that their product satisfies the condition (22).

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