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ANGULAR DISTRIBUTION OF CHARGED PARTICLES IN PAIR PRODUCTION BY PHOTONS AND THE EFFECT OF MULTIPLE SCATTERING

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The effect of multiple scattering on the angular and energy distributions of charged particles produced by photons in a condensed medium is considered. It is shown that multiple scattering lowers the rate of the process at small angles and leaves it unchanged in the large-angle range. The formula obtained by integration over the angles leads to Migdal's result[1].

1. As shown by Migdal[1], the energy distribution of charged particles during the process of pair production by high-energy photons in condensed media is strongly distorted by the strong influence of multiple scattering. The existence of such an effect is connected with the fact that at high photon energies the pair production takes place at large longitudinal distances from the nucleus, on the order of $E_+E_-/m^2\omega$, where ω, E_+, and E_- are respectively the energies of the quantum, the positron, and the electron ($\hbar = c = 1$).

The analysis of the energy spectrum is greatly facilitated by the fact that the form of the energy spectrum depends only on the multiple scattering over the effective length of the pair production process, and is completely independent of the multiple scattering of the charged particles after the end of the process. The difficulty in calculating the angular distribution of the pair particles lies in the fact that it is impossible to separate beforehand the nontrivial influence of the scattering over the effective length from the trivial effect of the succeeding scattering.

The analysis of the angular distribution of the charged particles during the pair production process is best carried out by describing the multiple scattering quantum-mechanically, i.e., by characterizing the state of the charged particle in the medium by a wave function, as is done in[2,3]. Then the pair-production process can be regarded as a first-order process with the photon.

2. We write down the probability of the pair production process without allowance for multiple scattering (averaged over the polarization of the quantum and summed over the spins of the charged particles) in the form

$$dW_0(\mathbf{k}, \mathbf{p}_+, \mathbf{p}_-) = d^3\mathbf{p}_+ d^3\mathbf{p}_- \int d^3\mathbf{q}\, w_0(\mathbf{k}, \mathbf{p}_+, \mathbf{p}_-, \mathbf{q})\, \delta(\mathbf{k} - \mathbf{p}_+ - \mathbf{p}_- - \mathbf{q})\, \delta(E_+ + E_- - \omega). \tag{2.1}$$

using the expression for the wave function of the charged particle in matter in the form[2,3]

$$\psi(\mathbf{r}, t) = \exp(i\mathbf{p}_0\mathbf{r} - iE_0 t)[1 - i(2E_0)^{-1}(\boldsymbol{\alpha}\nabla)]u_0 \times \prod_{(a)} \left\{1 + \frac{1}{2\pi^2} \int \frac{d^3\mathbf{l} f(\mathbf{l})}{l^2 + 2\mathbf{pl} - i\delta} e^{i\mathbf{l}(\mathbf{r} - \mathbf{R}_a)}\right\} \tag{2.2}$$

(here $f(\mathbf{l})$ is the amplitude of the single-center scattering as a function of the momentum transfer, $\mathbf{R}_a$ is the radius vector of the atom, and the product is taken over all the atoms of the substance), we can obtain an expression for the probability of the pair-production process, averaged over the coordinates of the atoms of an amorphous medium, in the form

$$dW(\mathbf{k}, \mathbf{p}_+, \mathbf{p}_-) = d^3\mathbf{p}_+ d^3\mathbf{p}_- \int d^3\mathbf{q}\, w_0(\mathbf{k}, \mathbf{p}_+; \mathbf{p}_-, \mathbf{q})\, \delta(E_+ + E_- - \omega) \cdot \Delta(\mathbf{p}_+, \mathbf{p}_-; \mathbf{k} - \mathbf{p}_+ - \mathbf{p}_- - \mathbf{q}), \tag{2.3}$$

where

$$\Delta(\mathbf{p}_+, \mathbf{p}_-, \mathbf{s}) = \int \frac{d^3\mathbf{r}}{(2\pi)^3} e^{i\mathbf{sr}} \int \frac{d^3\mathbf{R}}{V} \exp \Phi(\mathbf{R}, \mathbf{r}). \tag{2.4}$$

In the case when the medium is homogeneous and fills the layer between the planes $x = 0$ and $x = L$, we have $\Phi(\mathbf{R}, \mathbf{r}) = \Phi(R_x, \mathbf{r})$. If the dimensions of the effective region of the process are small compared with L, we can obtain

$$\Phi(R_x, \mathbf{r}) = -n_0 \int d^2\mathbf{l}_\perp |f(\mathbf{l}_\perp)|^2 (\exp i\mathbf{l}_\perp \mathbf{r}_\perp - 1) \left(\frac{p_-^2 + p_+^2}{p_-^2 p_+^2}\right) x + 2n_0 i\pi \left[f(0) R_x + f^*(0)(L - R_x) \left(\frac{p_- + p_+}{p_- p_+}\right) - \frac{n_0}{p_+ p_-} \int d^2\mathbf{l}_\perp |f(\mathbf{l}_\perp)|^2 \left[(L - x) - i\frac{p_+ + p_-}{4p_+p_-} \mathbf{l}_\perp^2 \left(R_x^2 + (L - R_x)^2 + Lx + \frac{x^2}{2}\right)\right]\right]. \tag{2.5}$$

3. The scattering in the effective region can be separated from the trivial influence of the subsequent scattering by introducing the probability $W_L(\mathbf{p}_-, \mathbf{p}_+; \mathbf{p}_-, \mathbf{p}_+)$ that multiple scattering will cause an electron with initial momentum $\mathbf{p}_-$ or a positron with initial momentum $\mathbf{p}_+$ will acquire in a layer of thickness L momenta $\mathbf{p}_-$ and $\mathbf{p}_+$. By calculating such a probability with the aid of the same wave functions as in (2.2), it is easy to see that the probability of pair production is connected with the scattering probability by the formula

$$\frac{dW(\mathbf{k}, \mathbf{p}_+, \mathbf{p}_-)}{d^3\mathbf{p}_+ d^3\mathbf{p}_-} = \int \frac{dR_x}{L} \int \frac{d\widetilde{W}(\mathbf{k}, \mathbf{p}_+', \mathbf{p}_-')}{d^3\mathbf{p}_+' d^3\mathbf{p}_-'} w_{L-R_x}(\mathbf{p}_-', \mathbf{p}_+'; \mathbf{p}_-, \mathbf{p}_+) d^3\mathbf{p}_-' d^3\mathbf{p}_+'. \tag{3.1}$$

Expression (3.1) can be interpreted as follows: the process of pair-production in a layer of thickness L is the result of two successive processes—the elementary act of pair production over an effected length $E_+E_-/m^2\omega$ near the point R_x, and the succeeding multiple scattering of the particles of the pair over the length $L - R_x$. The quantity $dW(\mathbf{k}, \mathbf{p}_+', \mathbf{p}_-')$ can consequently be called the effective probability of the elementary pair-production act. The importance of this quantity lies in the fact that in order to calculate the cascade processes it is necessary to have just the probability of the elementary act of pair production with allowance for the influence of scattering on the effective length.

Using the foregoing, we can easily obtain an explicit expression for this quantity:

$$\frac{d\tilde{W}(\mathbf{k},\mathbf{p}_+',\mathbf{p}_-')}{d^3\mathbf{p}_+\,d^3\mathbf{p}_-}=\int d^3\mathbf{q}\,w_0(\mathbf{k},\mathbf{p}_+,\mathbf{p}_-,\mathbf{q})\,\delta(\omega-E_+-E_-)\,\Delta(\mathbf{k}-\mathbf{p}_+-\mathbf{p}_--\mathbf{q}). \tag{3.2}$$

From (3.2) we see that the difference between this quantity and the expression for the pair-production probability in the Born approximation (2.1) lies in replacing the momentum δ-function by a certain "smeared" function

$$\Delta(\mathbf{s})=(2\pi)^{-3}\int d^3\mathbf{r}\exp\left\{i\mathbf{s}\mathbf{r}-ix|x|\frac{p_{+x}+p_{-x}}{2p_{+x}p_{-x}}\varkappa\right\}$$
$$=\delta(\mathbf{s}_\perp)\frac{1}{2\sqrt{2\pi}}\sqrt{\frac{p}{\varkappa}}\left[\cos\left(\frac{p}{4\varkappa}s_x^2\right)+\sin\left(\frac{p}{4\varkappa}s_x^2\right)\right],$$
$$\varkappa\cong E_s^2L_{\text{rad}}^{-1},\quad p=\frac{2p_{+x}p_{-x}}{p_{+x}+p_{-x}}. \tag{3.3}$$

Substitution of the explicit form of w_0 in (3.2) yields

$$d\tilde{W}(\mathbf{k},\mathbf{p}_+',\mathbf{p}_-')=\frac{Z^2\alpha^3}{\pi^3\omega q^4}\Big\{[(E_-^2+E_+^2)(\mathbf{p}_{\perp-}'+\mathbf{p}_{\perp+}')^2+2E_-E_+(\mathbf{p}_{\perp-}'^2+\mathbf{p}_{\perp+}'^2)]$$
$$\times\frac{2\pi^3}{b}\left(\cos\frac{a_+^2}{4b}+\sin\frac{a_+^2}{4b}\right)\left(\cos\frac{a_-^2}{4b}+\sin\frac{a_-^2}{4b}\right)$$
$$-\frac{\sqrt{2}E_-E_+\pi^{3/2}}{b^{3/2}}\left[a_+p_{\perp+}'^2\left(\cos\frac{a_+^2}{4b}+\sin\frac{a_+^2}{4b}\right)\right.$$
$$\left.+a_-p_{\perp-}'^2\left(\cos\frac{a_-^2}{4b}+\sin\frac{a_-^2}{4b}\right)\right]\Big\}Q\theta_-\theta_+d\theta_-d\theta_+d\varphi_+, \tag{3.4}$$

where

$$a_\pm=\frac{\omega m^2}{2E_\pm^2}+\frac{\omega\theta_\pm^2}{2},\quad b=n_0\varkappa\frac{m^2\omega}{2E_-^2E_+^2},\quad Q=|1-Z^{-1}F(q)|^2.$$

Formula (3.4) is the final expression for the distribution over the angles and the frequencies.

Let us consider two limiting cases. In the low-density limit, when the influence of the multiple scattering of the particles in the effective region can be neglected, i.e., $a_\pm^2\gg b$ or

$$\left(\frac{\omega m^2}{2E_\pm^2}+\frac{\omega\theta_\pm^2}{2}\right)^2\gg\varkappa n_0\frac{m^2\omega}{2E_-^2E_+^2},$$

we get

$$d\tilde{W}(\mathbf{k},\mathbf{p}_+',\mathbf{p}_-')=\frac{Z^2\alpha^3}{\pi^3\omega q^4}Q\Big\{4\pi^3[(E_-^2+E_+^2)(p_+'^2\theta_+^2+p_-'^2\theta_-^2$$
$$+2p_-'\theta_-p_+'\theta_+\cos\varphi_+)+2E_-E_+(p_-'^2\theta_-^2+p_+'^2\theta_+^2)]$$
$$\times\left[\frac{\omega m^2}{2E_+^2}+\frac{\omega\theta_+^2}{2}\right]^{-1}\left[\frac{\omega m^2}{2E_-^2}+\frac{\omega\theta_-^2}{2}\right]^{-1}$$
$$-2E_-E_+\left[\frac{p_+'^2\theta_+^2}{(\omega m^2/2E_+^2+\omega\theta_+^2/2)^2}+\frac{p_-'^2\theta_-^2}{(\omega m^2/2E_-^2+\omega\theta_-^2/2)^2}\right]\Big\}\theta_+\theta_-d\theta_+d\theta_-d\varphi_+. \tag{3.5}$$

this expression coincides with the Bethe-Heitler angular distribution [4].

For the second limiting case, when the multiple scattering is significant in the effective region, i.e., $a_\pm^2\ll b$ or

$$\left(\frac{\omega m^2}{2E_\pm^2}+\frac{\omega\theta_\pm^2}{2}\right)^2\ll\varkappa n_0\frac{m^2\omega}{2E_-^2E_+^2}, \tag{3.6}$$

we obtain

$$d\tilde{W}(\mathbf{k},\mathbf{p}_+',\mathbf{p}_-')=\frac{Z^2\alpha^3}{\pi^3\omega q^4}Q\Big\{4\pi^3[(E_-^2+E_+^2)(p_-'^2\theta_-^2+p_+'^2\theta_+^2$$
$$+2p_-'\theta_-p_+'\theta_+\cos\varphi_+)+2E_-E_+(p_-'^2\theta_-^2+p_+'^2\theta_+^2)]\left[\varkappa n_0\frac{m^2\omega}{2E_-^2E_+^2}\right]^{-1}$$
$$-\sqrt{2}\pi^{3/2}E_-E_+\left[\frac{p_+'^2\theta_+^2(\omega m^2/2E_+^2+\omega\theta_+^2/2)}{(\varkappa n_0 w^2\omega/2E_-^2E_+^2)^{3/2}}+\frac{p_-'^2\theta_-^2(\omega m^2/2E_-^2+\omega\theta_-^2/2)}{(\varkappa n_0 m^2\omega/2E_-^2E_+^2)^{3/2}}\right]\Big\}$$
$$\times\theta_+d\theta_+\theta_-d\theta_-d\varphi_+. \tag{3.7}$$

An analysis of (3.6) and (3.7) shows that multiple scattering suppresses the probability of pair production in the region of small particle emission angles, but does not change the probability in the region of large angles. A consequence of this is the increase of the effective electron and positron emission angles. The maximum of the angular distribution of the emitted particles corresponds to

$$\theta_{max\pm}\sim\sqrt{E_s/E_\pm}(\omega L_{\text{rad}})^{-1/4}. \tag{3.8}$$

It is interesting to note that this conclusion can be drawn from a qualitative analysis, just as was done by Galitskiĭ and Gurevich for bremsstrahlung [5]. Indeed, the probability of pair production can be represented in the form

$$w\sim Al^2, \tag{3.9}$$

where l is the coherent length of the process. Let us find this quantity in the medium. For a positron or for an electron emitted at an angle $\theta_\pm$ to the trajectory of the gamma quantum, with account of the multiple-scattering angle, we obtain the coherent length from the condition

$$l/v\cos\overline{(\theta_\pm+\theta_s)}-l=\lambda\!\!\!^{-}/2. \tag{3.10}$$

For ultrarelativistic velocities we get in the small-angle approximation from (3.10)

$$l(\omega,\theta_\pm)=\frac{\lambda\!\!\!^{-}/2}{1-v-\overline{\theta}_s^2/2+\theta_\pm^2/2}.$$

Recognizing that the coherent length in the absence of a medium at 0° angle is equal to $l_0(\omega,0)=\lambda\!\!\!^{-}(E/m)^2$ and $\theta_S^2=(E_S/E)^2 l/L_{\text{rad}}$, we get

$$l(\omega,\theta_\pm)=l_0\left[\frac{E_sE_\pm}{\sqrt{L_{\text{rad}}\omega}\,m^2}+\frac{E_\pm^2}{m^2}\theta_\pm^2\right]^{-1}. \tag{3.11}$$

Since the probability is determined by the coherent length l, we see from (3.11) that the angular width coincides with that obtained above in (3.8). Integration of (3.4) over the particle emission angles gives the energy distribution. In the low density limit we have

$$\frac{d\sigma}{dE_+}=\frac{\text{const}}{\omega^3}\left(E_+^2+E_-^2+\frac{2}{3}E_+E_-\right)\left(\ln^2\frac{E_+E_-}{m\omega}-\frac{1}{2}\right), \tag{3.12}$$

which corresponds to the Bethe-Heitler spectrum. In the second limiting case we have

$$\frac{d\sigma}{dE_+}=\text{const}\frac{\sqrt{\omega}}{E_+}\left(\frac{E_s^2}{E_+^2}\frac{1}{2L_{\text{rad}}}\right)^{-3/2}, \tag{3.13}$$

which coincides with Migdal's results [1].

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25