QUANTUM STATISTICS OF MULTI-MODE RADIATION FROM AN ENSEMBLE OF ATOMS

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The statistics of the quantum fluctuations of multimode radiation from an ensemble of two-level atoms is studied theoretically. A master equation is derived which describes the evolution of the probability $P_m^{N,n}$ for states with a prescribed number m of atoms in the lower level, n_l photons in the lth mode, and total number N of photons in the whole set of L modes. This equation is used to determine the distribution functions of the number of photons in a single mode P^{n_l} and of the total number of photons in all modes P^N for equilibrium radiation of the ensemble of atoms and for non-equilibrium radiation at negative temperature (below the threshold for self-excitation). The nonlinear coupling between the radiation and the atoms (saturation effect) must be taken into account in the case of non-equilibrium radiation. It is shown that in the case of non-equilibrium radiation the distribution function of the number of photons in a single mode coincides with the Bose-Einstein distribution function for equilibrium radiation, but the dispersion of the fluctuations of the total number of photons N in all L modes is less than for equilibrium radiation with the same number of modes.

1. INTRODUCTION

In recent times the study of the statistics of photons and the coherent properties of light has been intensely developed.[1,2] To a considerable degree, these investigations have been stimulated by the appearance of laser light sources whose statistical properties are essentially different from the statistical properties of thermal or luminescent light sources.[3-7] In numerous papers the coherent properties of free optical fields and the methods of their measurement have been studied in considerable detail (cf. the reviews[1,2]).

In the investigation of the statistical properties of coherent laser light it is necessary to consider the interaction of radiation with an active medium. The statistics of the photons in a single mode of laser radiation has been investigated theoretically in a number of papers.[8-10] One of the most consistent treatments of the statistics of single-mode laser radiation is due to Fleck.[11,12] The single-mode model is very useful for the description of the statistical properties of single-mode lasers or multi-mode lasers with a Fabry-Perot resonator if neighboring modes interact relatively weakly with one another.

There exist light sources which emit a whole set of modes or field degrees of freedom. The simplest and best known example is the equilibrium radiation of a black body. Another example is the non-equilibrium radiation from a system of atoms with "negative" temperature without feedback (super-radiance). Finally, even in a laser above the threshold, if only the feedback does not single out several isolated high-Q modes, the generation of radiation can lead at once to a set of interacting modes. Here we have in mind a laser with nonresonant feedback[13,14] in which a set of modes is generated which differ in the direction of the wave vector and which interact strongly with one another via scattering. In such a laser there are no isolated modes, and it is therefore necessary to consider the entire set of modes coupled through scattering as a whole.

In all such cases a statistical description of the photons requires the consideration of multi-mode radiation.

In the present paper we investigate the statistical properties of multi-mode radiation from an ensemble of two-level atoms. Account is taken of the change in the probabilities for finding the atoms in the lower or upper levels owing to the influence of the radiation (saturation effect). A master equation is obtained for the joint distribution function for the number of photons in a single mode out of the total set and for the total number of photons in the entire set of modes. In the particular case of heat equilibrium the equations lead to the known results for equilibrium radiation. In the case when the atoms have "negative" temperature but the threshold for self-excitation is not reached, the equations describe the statistics of "super-radiance" with account of saturation.[1)]

The method of solution is analogous to that used by Fleck[11] in the investigation of the statistics of the radiation of a laser with a single mode. We consider the Hamiltonian of a system of M two-level atoms which are in resonant interaction with a quantized field with L degrees of freedom which differ only in the direction of the wave vector. Equations are obtained which describe the behavior of the diagonal elements of the density matrix for the total system of M atoms and L modes of radiation. The effect of radiation damping, pumping, and de-excitation of the atoms are introduced phenomenologically, following Fleck.[11] From the master equation for the probability $P_m^{n_1,\ldots,n_L}$ of finding a state with m atoms in the lower level and n_i photons in the i-th mode, a master equation is derived for the probability $P_m^{n_l,N}$ for finding a state with n_l photons in the l-th mode and N photons in all M modes. This equation is employed in

[1)]In the equations of[15] an interaction of the modes on account of photon exchange is introduced. These equations are then useful for the description of the statistical properties of the radiation of a laser with nonresonant feedback above threshold.

the stationary case to investigate the statistical properties of equilibrium and non-equilibrium radiation.

2. DERIVATION OF THE MASTER EQUATION FOR $P_m^{n_l N}$

Let us consider the resonance interaction of an ensemble of M two-level atoms with quantized radiation in L modes. We assume that the atoms do not interact with one another, and the energy of each atom in the upper or lower states is equal to $E_a = -\hbar\omega_a$ and $E_b = -\hbar\omega_b$, respectively. We assume further that the radiation in all L modes has the same frequency ω, i.e., differs only in the direction of the wave vector. Then the total Hamiltonian of the system H can be written in the form (cf. Fleck[11])

$$H = \hbar\omega \sum_{l=1}^{L} a_l^+ a_l - \hbar\omega_a \sum_{j=1}^{M} \sigma_j^+ \sigma_j^- - \hbar\omega_b \sum_{j=1}^{M} \sigma_j^- \sigma_j^+ + \sum_{l=1}^{L} (a_l + a_l^+) \sum_{j=1}^{M} \hbar\alpha_{lj} (\sigma_j^+ + \sigma_j^-), \tag{1}$$

where $a_l^\dagger$ and a_l are the creation and annihilation operators for a photon in the lth mode; σ_j^+ is the operator for raising the j th atom from the lower to the upper level (transition $b \to a$); σ_j^- is the operator for lowering the j-th atom from the upper to the lower level (transition $a \to b$).

The quantity $\alpha_{lj} = (2\pi\omega/\hbar)^{1/2} \mu_l E_l(X_j)$ determines the interaction of the l-the mode of the field with the j-th atom. If the interaction is of the electric dipole type, then μ_l is the projection of the matrix element of the dipole moment on the direction of the polarization vector of the l-th mode of the field, E_l is the normalized eigenfunction of the l-th mode, and X_j is the coordinate of the j-th atom. We restrict ourselves to the case of identical modes and neglect the dependence of the interaction constant on the position of the atom ($\alpha_{lj} = \alpha$). The latter assumption is not essential if the atoms are distributed uniformly in the radiation field.[11] The first sum in (1) is the Hamiltonian of free radiation with L modes, the second and third sums are the Hamiltonian of an ensemble of M two-level atoms, and the last sum in (1) is the interaction between the radiation and the atoms.

A state of the total system will be described by the number of atoms in the lower level m and the number of photons in the l-th mode n_l ($l = 1, 2, \ldots, L$). The wave function for such a state will be denoted by

$$|\psi\rangle = |\{m\} n_1, n_2, \ldots, n_L\rangle, \tag{2}$$

where $\{m\}$ is some particular configuration of the ensemble of atoms with m atoms in the lower level. The wave function of an arbitrary state can be expanded in terms of the orthonormal eigenfunctions (2):

$$\left|\psi(t)\right\rangle = \sum_{n_1} \sum_{n_2} \ldots \sum_{n_L} \sum_{\{m\}} C_{\{m\}}^{n_1, n_2, \ldots, n_L} \left| \{m\} n_1, n_2, \ldots, n_L \right\rangle, \tag{3}$$

where the sum over $\{m\}$ denotes the summation over all configurations with given m and over all values of m.

The wave function of the system $\psi(t)$ satisfies the Schrödinger equation

$$i\hbar\partial\psi / \partial t = H\psi \tag{4}$$

with the Hamiltonian (1). Substituting the wave function (3), we find after standard transformations[2)] the following equation for the amplitudes:[3)]

$$\dot{C}_{\{m\}}^{n_1 \ldots n_l \ldots n_L} = -i\left[\omega \sum_l n_l - \omega_a (M - m) - \omega_b m\right] C_{\{m\}}^{n_1 \ldots n_l \ldots n_L} - i\alpha \sum_{\{m+1\}}^{\{m\}} \sum_l \sqrt{n_l + 1}\, C_{\{m+1\}}^{n_1 \ldots n_l + 1 \ldots n_L} - i\alpha \sum_{\{m-1\}}^{\{m\}} \sum_l \sqrt{n_l}\, C_{\{m-1\}}^{n_1 \ldots n_l - 1 \ldots n_L}. \tag{5}$$

In deriving (5) we have neglected the nonresonant terms in the interaction Hamiltonian, $a_l^+ \sigma_j^+$ and $a_l \sigma_j^-$. The sum $\sum_{\{m'\}}^{\{m\}}$ denotes the summation over all configurations $\{m'\} = \{m \pm 1\}$ which can be obtained from a given configuration $\{m\}$ by changing the state of one of the atoms.

With the help of (5) one can obtain the following equation:

$$\frac{d}{dt} |C_{\{m\}}^{\ldots n_l \ldots}|^2 = i\alpha \sum_{\{m+1\}}^{\{m\}} \sum_l \sqrt{n_l + 1}\, C_{\{m+1\}}^{\ldots n_l + 1 \ldots *} C_{\{m\}}^{\ldots n_l \ldots} + i\alpha \sum_{\{m-1\}}^{\{m\}} \sum_l \sqrt{n_l}\, C_{\{m-1\}}^{\ldots n_l - 1 \ldots *} C_{\{m\}}^{\ldots n_l \ldots} + \text{c.c.} \tag{6}$$

The problem then consists in determining the terms on the right-hand side of (6). With the help of (5) we find

$$\frac{d}{dt} C_{\{m+1\}}^{\ldots n_l + 1 \ldots *} C_{\{m\}}^{\ldots n_l \ldots} = \left[i(\omega - \omega_0) - \frac{1}{T_2}\right] C_{\{m+1\}}^{\ldots n_l + 1 \ldots *} C_{\{m\}}^{\ldots n_l \ldots} + i\alpha \sum_{\{m\}'}^{\{m+1\}} \sum_{l'} \sqrt{n_{l'} + \delta_{ll'}}\, C_{\{m\}'}^{\ldots n_{l'} - 1 \ldots n_l + 1 \ldots *} C_{\{m\}}^{\ldots n_{l'} \ldots n_l \ldots} - i\alpha \sum_{\{m+1\}'}^{\{m\}} \sum_{l'} \sqrt{n_{l'} + 1}\, C_{\{m+1\}}^{\ldots n_{l'} \ldots n_l + 1 \ldots *} C_{\{m+1\}'}^{\ldots n_{l'} + 1 \ldots n_l \ldots}, \tag{7}$$

where $\omega_0 = \omega_b - \omega_a$, and where we have introduced the phenomenological damping constant $1/T_2$ which takes account of the stochastic dephasing of the amplitudes $C_{\{m+1\}}^{\ldots n_l+1 \ldots *}$ and $C_{\{m\}}^{\ldots n_l \ldots}$ owing to relaxation. The time T_2 is the time of transverse relaxation or the damping time for the nondiagonal elements of the density matrix for the quantum system.[16,17] In deriving (7) we have neglected terms of the type $C_{\{m+2\}}^{\ldots n_{l'}+1 \ldots n_l+1 \ldots *} C_{\{m\}}^{\ldots n_{l'} \ldots n_l \ldots}$. This corresponds to the first approximation in the small dimensionless parameter αT_2.[11]

In the following we shall consider the case of normal radiation from an ensemble of atoms, where the probability for the emission of a photon is proportional to the number of atoms M (in contrast to the case of

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2) The operators a_l^+, a_l, σ_j^+, and σ_j^- act on the basis functions in the following way:

$$a_l^+ \mid \{m\} n_1, \ldots, n_l \ldots n_L\rangle = \sqrt{n_l + 1} \mid \{m\} n_1, \ldots, n_l + 1, \ldots n_L\rangle,$$
$$a_l \mid \{m\} n_1, \ldots, n_l \ldots n_L\rangle = \sqrt{n_l} \mid \{m\} n_1, \ldots, n_l - 1, \ldots, n_L\rangle,$$
$$\sigma_j^+ \mid \{m\} n_1, \ldots, n_L\rangle = \mid \{m - 1\} n_1, \ldots, n_L\rangle,$$
$$\sigma_j^- \mid \{m\} n_1 \ldots n_L\rangle = \mid \{m + 1\} n_1 \ldots n_L\rangle.$$

3) The first and the last indices n_1 and n_L in the amplitudes will be omitted in the following, for brevity.

"super-radiance," where the probability is proportional to M^2[17,18]). In this case the phases of states with different configurations $\{m\}$ for given m are random, and the sum over the configurations $\{m\}'$ reduces to the single configuration $\{m\}$. Analogously, in summing over the configurations $\{m+1\}'$, only the configuration $\{m+1\}$ gives a nonvanishing contribution to the sum. The terms (A) in the sum (7) for $l' \neq l$ are connected with the reradiation of a photon from one mode to another and are hence are proportional to $(\alpha T_2)^2$. They can be neglected in the first approximation in the parameter αT_2, which we are considering. Then Eq. (7) reduces to the following:

$$\frac{d}{dt} C_{\{m+1\}}^{\dots n_l+1\dots *} C_{\{m\}}^{\dots n_l\dots} = \left[i(\omega-\omega_0) - \frac{1}{T_2}\right] C_{\{m+1\}}^{\dots n_l+1\dots *} C_{\{m\}}^{\dots n_l\dots} + i\alpha\sqrt{n_l+1}\,(|C_{\{m\}}^{\dots n_l\dots}|^2 - |C_{\{m+1\}}^{\dots n_l+1\dots}|^2). \quad (8)$$

If we consider a system near equilibrium, then the derivative in (8) can be neglected compared to the resonance term and the rate of the losses. Thus finally,

$$C_{\{m+1\}}^{\dots n_l+1\dots *} C_{\{m\}}^{\dots n_l\dots} \cong \frac{i\alpha}{-i(\omega-\omega_0)+T_2^{-1}}\sqrt{n_l+1}\,(|C_{\{m\}}^{\dots n_l\dots}|^2 - |C_{\{m+1\}}^{\dots n_l+1\dots}|^2). \quad (9)$$

Analogously, we find for the second term in (6)

$$C_{\{m-1\}}^{\dots n_l-1\dots *} C_{\{m\}}^{\dots n_l\dots} \cong \frac{i\alpha}{i(\omega-\omega_0)+T_2^{-1}}\sqrt{n_l}\,(|C_{\{m\}}^{\dots n_l\dots}|^2 - |C_{\{m-1\}}^{\dots n_l-1\dots}|^2). \quad (10)$$

Let us substitute (9) and (10) in (6) and go over to the total probability for all states with given m and given occupation numbers $n_1, \dots, n_l, \dots, n_L$, using the relation

$$P_m^{\dots n_l\dots} = \frac{M!}{m!(M-m)!}|C_{\{m\}}^{\dots n_l\dots}|^2, \quad (11)$$

where $M!/m!(M-m)!$ is the number of possible configurations $\{m\}$ with given value of m. As a result we obtain a master equation for the probabilities

$$\dot P_m^{\dots n_l\dots} = -k\sum_{l=1}^{L}(n_l+1)[(M-m)P_m^{\dots n_l\dots} - (m+1)P_{m+1}^{\dots n_l+1\dots}] - k\sum_{l=1}^{L} n_l[mP_m^{\dots n_l\dots} - (M-m+1)P_{m-1}^{\dots n_l-1\dots}],$$
$$k = 2\alpha^2 T_2^{-1}/[(\omega-\omega_0)^2 + T_2^{-2}], \quad (12)$$

which generalizes the equation of Fleck to the case of multimode radiation.

For a complete description of the radiation from an ensemble of atoms, including the stationary state, Eq. (12) must be supplemented by terms describing the radiation losses as well as the excitation (pumping) and de-excitation of atoms without the participation of photons. If γ_l is the rate of damping for photons of the l-th mode, then the change of the probabilities due to radiation losses is described by the equation

$$\dot P_m^{\dots n_l\dots} = \sum_{l=1}^{L}\gamma_l(n_l+1)P_m^{\dots n_l+1\dots} - \sum_{l=1}\gamma_l n_l P_m^{\dots n_l\dots}. \quad (13)$$

If $\mathcal{P}$ is the rate of transition from the lower to the upper atomic level owing to pumping, and S is the rate of transition from the upper to the lower atomic level owing to de-excitation effects, then the change of the probability due to pumping and de-excitation is described by the expression[11]

$$\dot P_m^{\dots n_l\dots} = S(M-m+1)P_{m-1}^{\dots n_l\dots} - S(M-m)P_m^{\dots n_l\dots} + \mathcal{P}(m+1)P_{m+1}^{\dots n_l\dots} - \mathcal{P}mP_m^{\dots n_l\dots} \quad (14)$$

In the following we consider the case of identical damping of all modes ($\gamma_l = \gamma$; $l = 1, 2, \dots, L$). Then the evolution is on the average the same for all modes, and we can go over to conditional probability for states with n_l photons in the l-th mode and N photons in all L modes. This probability is defined by

$$P_m^{n_l,N} = \sum_{\{n_{l'}\}}^{l'\neq l} P_m^{\dots n_{l'}\dots n_l\dots}, \quad (15)$$

where $\{n_{l'}\}$ with $l' \neq l$ denotes the summation over all states $\{n_1, \dots, n_{l'}, \dots, n_l, \dots, n_L\}$ with fixed value of n_l, such that

$$\sum_{l'=1}^{L} n_{l'} = N.$$

Carrying out this summation in (12) together with the additional terms (13) and (14), we find the following equation:

$$\begin{aligned}\dot P_m^{n_l,N} = &-k(N+L)(M-m)P_m^{n_l,N} + k(m+1)(N-n_l+1)P_{m+1}^{n_l,N+1} \\ &+ k(m+1)(n_l+1)P_{m+1}^{n_l+1,N+1} - kNmP_m^{n_l,N} \\ &+ k(M-m+1)(N-n_l+L-2)P_{m-1}^{n_l,N-1} + k(M-m+1)n_lP_{m-1}^{n_l-1,N-1} \\ &+ S(M-m+1)P_{m-1}^{n_l,N} - S(M-m)P_m^{n_l,N} - \mathcal{P}mP_m^{n_l,N} + \mathcal{P}(m+1)P_{m+1}^{n_l,N} \\ &+ \gamma(N+1-n_l)P_m^{n_l,N+1} + \gamma(n_l+1)P_m^{n_l+1,N+1} - \gamma NP_m^{n_l,N}. \quad (16)\end{aligned}$$

In deriving (16) we have taken account of the fact that the number of terms in the sum (15) is equal to the number of ways in which $N - n_l$ elements can be distributed over $L - 1$ cells, where a cell can contain any number of elements. This number is equal to[19,20]

$${}^{(r)}C_{N-n_l}^{L-1} = (N-n_l+L-2)!/(N-n_l)!(L-2)!$$

3. DISTRIBUTION OF THE NUMBER OF PHOTONS IN L MODES

Let us determine the distribution function for the probability P^N of states with N photons in all modes. Summing over all values of n_l in (16) and using

$$P_m{}^N = \sum_{n_l} P_m^{n_l,N},$$

we find the master equation for the probability P_m^N:

$$\begin{aligned}\dot P_m{}^N = &-k(N+L)(M-m)P_m{}^N + k(N+1)(m+1)P_{m+1}^{N+1} \\ &- kNmP_m{}^N + k(N+L-1)(M-m+1)P_{m-1}^{N-1} \\ &+ S(M-m+1)P_{m-1}^{N} - S(M-m)P_m{}^N + \mathcal{P}(m-1)P_{m+1}^{N} - \mathcal{P}mP_m{}^N \\ &+ \gamma(N+1)P_m^{N+1} - \gamma NP_m{}^N. \quad (17)\end{aligned}$$

In order to understand better the master equation (17), we derive equations for the average values:

$$\langle N\rangle = \sum_{N,m} NP_m{}^N, \quad \langle m\rangle = \sum_{N,m} mP_m{}^N.$$

After the corresponding transformations of (17) we find

$$\begin{aligned}d\langle N\rangle/dt &= k\langle(M-2m)N\rangle + kL\langle M-m\rangle - \gamma\langle N\rangle, \\ d\langle m\rangle/dt &= k\langle(M-2m)N\rangle + kL\langle M-m\rangle + S\langle M-m\rangle - \mathcal{P}\langle m\rangle. \quad (18)\end{aligned}$$

We sum over m in (17) and introduce, following Fleck,[11] the conditional probabilities for the occupation of the upper and lower levels of the atoms for N photons in all L modes:

$$\eta_a(N)=\frac{1}{P^N}\sum_m P_m{}^N\frac{M-m}{M},\quad \eta_b(N)=\frac{1}{P^N}\sum_m P_m{}^N\frac{m}{M},\quad P^N=\sum_m P_m{}^N, \tag{19}$$

we then obtain the master equation for the total probability for states with N photons in all modes:

$$\dot P^N=-kM(N+L)\eta_a(N)P^N+kM(N+1)\eta_b(N+1)P^{N+1} - kMN\eta_b(N)P^N+kM(N+L-1)\eta_a(N-1)P^{N-1} + \gamma(N+1)P^{N+1}-\gamma NP^N. \tag{20}$$

The solution of (20) in the stationary case is

$$(N+1)[\gamma+kM\eta_b(N+1)]P^{N+1}-(N+L)kM\eta_a(N)P^N=0, \tag{21}$$

which leads to the recurrence relation

$$P^{N+1}=\frac{N+L}{N+1}\frac{kM\eta_a(N)}{\gamma+kM\eta_b(N+1)}P^N. \tag{22}$$

The general expression for the distribution function P^N has the form

$$P^N=\frac{(N+L-1)!}{N!(L-1)!}\prod_{N'=0}^{N-1}\frac{kM\eta_a(N')}{\gamma+kM\eta_b(N'+1)}P^0. \tag{23}$$

Expression (23) is rather general, since it describes the equilibrium and non-equilibrium radiation from an ensemble of atoms. Let us find the form of P^N in the different cases.

Equilibrium radiation. Let us assume that the ratio η_a/η_b is determined by the Boltzmann factor $\exp(-\hbar\omega/kT)$ and is independent of N. Then the distribution function for $\gamma = 0$ has the form

$$P^N=\frac{(N+L-1)!}{N!(L-1)!}e^{-N\hbar\omega/kT}P^0. \tag{24}$$

Introducing the average number of photons in a single mode,

$$\langle n\rangle=\sum_N NP^N(L=1)=(e^{\hbar\omega/kT}-1)^{-1}. \tag{25}$$

and normalizing, we can reduce the distribution (24) to the final form:

$$P^N=\frac{(N+L-1)!}{N!(L-1)!}\frac{\langle n\rangle^N}{(1+\langle n\rangle)^{N+L}}. \tag{26}$$

This expression agrees with the results of Mandel[21] and Kano.[22] The above-given expression for the distribution function remains also true for $\gamma \neq 0$. In this case it suffices to introduce in the expression (25) for $\langle n\rangle$ the effective temperature $T_{eff} < T$, which is defined by

$$\frac{kM\eta_a}{\gamma+kM\eta_b}=\exp\left\{-\frac{\hbar\omega}{kT_{\text{eff}}}\right\}. \tag{27}$$

The distribution function P^N can be written in another more perspicuous form. Although this can be done by transforming (26), we use another method, which we shall need in the following. We rewrite the recurrence relation (22) in the form

$$P^{N+1}-P^N=f(N)P^N,\quad f(N)=\frac{N+L}{N+1}\frac{\langle n\rangle}{1+\langle n\rangle}-1, \tag{28}$$

and, since we are interested in the region of large $N \gg 1$, go over to the differential equation

$$dP^N/dN=f(N)P^N. \tag{29}$$

The distribution function P^N evidently has a maximum at the point $N = \bar N$, where $\bar N$ is determined by the condition $F(\bar N) = 0$:

$$\bar N=(L-1)\langle n\rangle-1=\frac{L-1}{L}\langle N\rangle-1. \tag{30}$$

The most probable value $\bar N$ is in general smaller than the average value $\langle N\rangle$, which is connected with the asymmetry of the distribution function P^N with respect to the point $N = \bar N$. In the following we consider the case of a large number of modes ($L \gg 1$), where the asymmetry is negligibly small so that we can set $\bar N = \langle N\rangle$. We expand $f(N)$ in a series about the point $N = \langle N\rangle$ and retain only the first term, assuming $\langle N\rangle \gg 1$. Integrating (29), we find

$$P^N=P^{\langle N\rangle}\exp\left\{-\frac{1}{2}\frac{\partial f}{\partial N}\bigg|_{N=\langle N\rangle}(N-\langle N\rangle)^2\right\} \tag{31}$$

or finally

$$P^N=\frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{(N-\langle N\rangle)^2}{2\sigma^2}\right\},\quad \sigma^2=\frac{\langle N\rangle(\langle N\rangle+L)}{L-1}. \tag{32}$$

It is seen from this that the distribution function has δ-function like character with a maximum at the point $N = \langle N\rangle$ and the width (dispersion) $\sigma \approx \langle N\rangle/\sqrt{L}$.

Non-equilibrium radiation. Let us assume that the ensemble of atoms has "negative" temperature,

$$\eta_a(0)/\eta_b(1)=e^{-\hbar\omega/kT}>1, \tag{33}$$

but that the radiation losses γ are such that the threshold for self-excitation is not reached, i.e.,

$$\frac{kM\eta_a(0)}{\gamma+kM\eta_b(1)}<1. \tag{34}$$

The radiation from such an ensemble of atoms is non-equilibrium radiation reminiscent of enhanced spontaneous radiation which is now commonly called "super-radiance." However, there is no complete correspondence, since in our case we do not consider the propagation of the photons and the corresponding dependence of P^N on the spatial coordinates.[4)]

To determine the distribution function P^N in this case we must find the explicit form of the functions $\eta_a(N)$ and $\eta_b(N)$. To this end we simplify the original master equation (17). We include only the effects of changing m and correspondingly, replace

$(N+1)P_{m+1}^{N+1}$ by NP_{m+1}^N and $(N+L-1)P_{m-1}^{N-1}$ by $(N+L)P_{m-1}^N$.

Moreover, we discard the last two terms proportional to γ, since the effect of radiation damping is already included in the distribution function in virtue of Eq. (22). In the stationary case we have then

$$(m+1)(\mathcal{P}+kN)P_{m+1}^N-(M-m)[S+k(N+L)]P_m{}^N - m(\mathcal{P}+kN)P_m{}^N+(M-m+1)[S+k(N+L)]P_{m-1}^N=0. \tag{35}$$

The solution of (35) is

$$P_{m+1}^N=\frac{M-m}{m+1}\frac{S+k(N+L)}{\mathcal{P}+kN}P_m{}^N. \tag{36}$$

Then the distribution function P_m^N has the form

$$P_m{}^N=\left[\frac{S+k(N+L)}{\mathcal{P}+kN}\right]^m\frac{M!}{m!(M-m)!}P_0{}^N. \tag{37}$$

4)Our method can be generalized to the case of the spatial evolution of multi-mode radiation; however, this will be the subject of a separate investigation.

Substituting P_m^N in (19) and summing, we find

$$\eta_a(N)=\frac{\mathcal{P}+kN}{\mathcal{P}+S+k(2N+L)},\quad \eta_b(N)=\frac{S+k(N+L)}{\mathcal{P}+S+k(2N+L)}. \tag{38}$$

For values of $N \gg 1$, we reduce the recurrence relation (22) to the differential equation (29) with the function

$$f(N)=\frac{N+L}{N+1}\frac{kM\eta_a(N)}{\gamma+kM\eta_b(N)}-1, \tag{39}$$

on the right-hand side, where $\eta_a(N)$ and $\eta_b(N)$ are the functions (38). The distribution function P^N has a maximum at the point $N = \bar{N}$ determined by the condition $f(\bar{N}) = 0$. In our approximation $\bar{N} = \langle N \rangle$. For

$$\langle N\rangle \gg L+\frac{\mathcal{P}+S}{k}+\frac{kM}{\gamma}$$

the expression for $\langle N \rangle$ has the form

$$\langle N\rangle=\frac{(L-1)\mathcal{P}}{L\gamma/M+k+\gamma(\mathcal{P}+S)/kM} \tag{40}$$

To find P^N we use the previous approach. We expand $f(N)$ in a series about the point $N = \langle N \rangle$ and, retaining only the first term ($\langle N \rangle \gg 1$), integrate the differential equation. As a result we have

$$P^N=P^{\langle N\rangle}\exp\left\{-\frac{(L-1)}{2\langle N\rangle(\langle N\rangle+L)}(N-\langle N\rangle)^2\right\}$$
$$\times\exp\left\{-\frac{k(\mathcal{P}-S-kL)(N-\langle N\rangle)^2}{(\mathcal{P}+k\langle N\rangle)[\mathcal{P}+S+k(2\langle N\rangle+L)]}\right\}. \tag{41}$$

The distribution function P^N for non-equilibrium radiation from an ensemble of atoms is similar to the distribution for the equilibrium distribution (32). The difference consists in the second exponential factor in (41), which reduces the dispersion of the quantity N as compared to the dispersion in the case of equilibrium radiation. Indeed, according to (38) the condition of negative temperature of the atoms (33) leads to the inequality $\mathcal{P} > s + kL$, and hence the second factor reduces the fluctuations of N. Physically, this is explained by the nonlinear nature of the system owing to the saturation effect, which "stabilizes" the total number of photons in all modes, N.

4. DISTRIBUTION OF THE NUMBER OF PHOTONS IN A SINGLE MODE

Let us find the distribution function P^{n_l} for the number of photons in the l-the mode, n_l, for an arbitrary number of photons in the remaining $L - 1$ modes. We define $P_m^{n_l}$ by

$$P_m^{n_l}=\sum_N P_m^{n_l,N}, \tag{42}$$

where the probabilities $P_m^{n_l,N}$ satisfy the master equation (16).

In order to obtain the master equation for the probabilities $P_m^{n_l}$ we must sum Eq. (16) over N. In this summation we encounter the sum

$$\sum_N NP_m^{n_l,N},$$

which is simplified in a natural way by using the properties of the total number of photons N found above. For $L \gg 1$ the total number of photons N is subject to relatively few fluctuations about the average value $\langle N \rangle$, which practically coincides with the most probable value $\bar{N}$ for $\bar{N} \gg 1$. Therefore the above-mentioned sum can be transformed in the following way:

$$\sum NP_m^{n_l,N}=\langle N\rangle\sum P_m^{n_l,N}=\langle N\rangle P_m^{n_l} \tag{43}$$

Then the master equation for $P_m^{n_l}$ can be reduced to the form

$$\dot{P}_m^{n_l}=-k(M-m)(\langle N\rangle+L)P_m^{n_l}+k(m+1)(\langle N\rangle-n_l)P_{m+1}^{n_l}$$
$$+k(m+1)(n_l+1)P_{m+1}^{n_l+1}-km\langle N\rangle P_m^{n_l}+k(M-m+1)$$
$$\times(\langle N\rangle+L-n_l-1)P_{m-1}^{n_l}+k(M-m+1)n_lP_{m-1}^{n_l-1}+S(M-m+1)P_{m-1}^{n_l}$$
$$-S(M-m)P_m^{n_l}+\mathcal{P}(m+1)P_{m+1}^{n_l}-\mathcal{P}mP_m^{n_l}+\gamma(n_l+1)P_m^{n_l+1}-\gamma n_lP_m^{n_l} \tag{44}$$

In order to understand the character of the approximation (43), we derive the equations for the average values

$$\langle n_l\rangle=\sum_{m,n_l}n_lP_m^{n_l},\qquad \langle m\rangle=\sum_{m,n_l}mP_m^{n_l}.$$

We have

$$d\langle n_l\rangle/dt=k\langle(M-2m)n_l\rangle+k\langle M-m\rangle-\gamma\langle n_l\rangle,$$
$$d\langle m\rangle/dt=k\langle N\rangle\langle M-2m\rangle+kL\langle M-m\rangle-\mathcal{P}\langle m\rangle+S\langle M-m\rangle. \tag{45}$$

As is seen from (45), the approximation (43) corresponds to the replacement of $\langle N(M - 2m)\rangle$ by $\langle N\rangle\langle M - 2m\rangle$, but correlations of the type $\langle(M - 2m)n_l\rangle$ are taken into account.

Summing (44) over m, we obtain the following master equation:

$$\dot{P}^{n_l}=-kM(n_l+1)\eta_a'(n_l)P^{n_l}+kM(n_l+1)\eta_b'(n_l+1)P^{n_l+1}$$
$$-kMn_l\eta_b'(n_l)P^{n_l}+kMn_l\eta_a'(n_l-1)P^{n_l-1}$$
$$+\gamma(n_l+1)P^{n_l+1}-\gamma n_lP^{n_l}, \tag{46}$$

where, in analogy to (19), we have introduced the conditional probabilities for the occupation of the upper and lower levels of the atoms:

$$\eta_a'(n_l)=\frac{1}{P^{n_l}}\sum_m\frac{M-m}{M}P_m^{n_l},\qquad \eta_b'(n_l)=\frac{1}{P^{n_l}}\sum_m\frac{m}{M}P_m^{n_l},$$
$$P^{n_l}=\sum_m P_m^{n_l}. \tag{47}$$

The solution of (46) in the stationary case is

$$P^{n_l+1}=\frac{kM\eta_a'(n_l)}{\gamma+kM\eta_b'(n_l+1)}P^{n_l}. \tag{48}$$

The general expression for the distribution function has the form

$$P^{n_l}=\prod_{n'=0}^{n_l-1}\frac{kM\eta_a'(n')}{\gamma+kM\eta_b'(n'+1)}P^0. \tag{49}$$

As before, let us find the distribution function P^{n_l} for the cases of equilibrium and non-equilibrium radiation from the ensemble of atoms.

Equilibrium radiation. If the ratio η'_a/η'_b is determined by the Boltzmann factor, then we obtain at once the known expression for the Bose-Einstein distribution function for the number of photons in a single quantum state:[3]

$$P^{n_l}=\frac{\langle n_l\rangle^{n_l}}{(1+\langle n_l\rangle)^{1+n_l}}, \tag{50}$$

which agrees with (26) for $L = 1$.

Non-equilibrium radiation. The functions $\eta'_a(n_l)$ and $\eta'_b(n_l)$ are found, as before, by summation over m in the original master equation (44). The master equation (44) is simplified in the previous way by replacing $(n_l + 1) P^{n_l+1}_{m+1}$ by $n_l P^{n_l}_{m+1}$ and $n_l P^{n_l-1}_{m-1}$ by $(n_l - 1) P^{n_l}_{m-1}$ and discarding terms proportional to γ. As a result we find the following recurrence relation in the stationary case:

$$P^{n_l}_{m+1} = \frac{M-m}{m+1} \frac{S + k(\langle N\rangle + L)}{\mathscr{P} + k\langle N\rangle} P_m^{n_l}, \tag{51}$$

which leads to the following distribution in m:

$$P_m^{n_l} = \frac{M!}{m!(M-m)!} \left[\frac{S + k(\langle N\rangle + L)}{\mathscr{P} + k\langle N\rangle}\right]^m P_0^{n_l}. \tag{52}$$

Substituting the distribution (52) in (47) and summing, we find the functions $\eta'_a(n_l)$ and $\eta'_b(n_l)$:

$$\eta_a'(n_l) = \frac{\mathscr{P} + k\langle N\rangle}{\mathscr{P} + S + k(2\langle N\rangle + L)} \quad \eta_b'(n_l) = \frac{S + k(\langle N\rangle + L)}{\mathscr{P} + S + k(2\langle N\rangle + L)} \tag{53}$$

It is seen that the probabilities η'_a and η'_b do not depend on n_l, but are determined by the average value of the total number of photons in all modes, $\langle N\rangle$. This is a natural result for the case $L \gg 1$ under consideration, where the approximation (43) is valid.

To find P^{n_l} it suffices to substitute the values of η'_a and η'_b in (49). However, the calculations can be simplified by noting that $\eta'_a = \eta_a(\langle N\rangle)$ and $\eta'_b = \eta_b(\langle N\rangle)$, where the functions η_a and η_b are given by (38). Indeed, then the factor in the product (49) can successively be transformed in the following way:

$$\frac{kM\eta_a'}{\gamma + kM\eta_b'} = \frac{kM\eta_a(\langle N\rangle)}{\gamma + kM\eta_b(\langle N\rangle)} = \frac{\langle N\rangle + 1}{\langle N\rangle + L} = \frac{\langle n_l\rangle + 1/L}{\langle n_l\rangle + 1}, \tag{54}$$

where we have used the fact that $\langle N\rangle = \bar{N}$ and the function $f(N)$ of (39) vanishes for $N = \langle N\rangle$. Substituting (54) in (49) and using $\langle N\rangle = \langle n_l\rangle L \gg 1$, we obtain, after normalization, the following expression for the distribution function:

$$P^{n_l} = \frac{\langle n_l\rangle^{n_l}}{(1 + \langle n_l\rangle)^{1+n_l}}. \tag{55}$$

Thus the distribution function for the number of photons in a single quantum state in the case of nonequilibrium radiation of the ensemble of atoms in the stationary state, is the same as for equilibrium radiation. The nonlinear nature of the system (saturation) reduces only the effective temperature of the radiation, which becomes equal to

$$T_{\text{eff}} = \frac{\hbar\omega}{k} \Big/ \ln\frac{\gamma + kM\eta_b(\langle N\rangle)}{kM\eta_a(\langle N\rangle)}. \tag{56}$$

5. CONCLUSION

In the present paper we have investigated the statistics of multi-mode equilibrium and non-equilibrium radiation by the method of master equations for the probabilities. The equations obtained are generalizations to the case of multi-mode radiation of the equations obtained by Fleck[11] for a single mode. In the particular case of heat equilibrium our theory leads to the known results for equilibrium radiation from a "black body." We have also considered the case of non-equilibrium radiation from an ensemble of atoms with negative temperature below the threshold for generation. It was shown that the saturation effect leads to a certain reduction of the dispersion of the fluctuations of the total number of photons in all modes. However, the fluctuations of the radiation in a single mode are described by Bose-Einstein distribution functions for the number of photons in a single quantum state of equilibrium radiation, and the saturation effect leads only to a reduction of the effective temperature of the radiation.

Our method can be used for the investigation of the statistical properties of other sources of multi-mode radiation, for example, of a laser with nonresonant feedback.[15]

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251