ELECTROMAGNETIC FIELD IN AN OPTICAL RESONATOR WITH A MOVABLE MIRROR

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Submitted June 19, 1967

Zh. Eksp. Teor. Fiz. 53, 2123–2130 (December, 1967)

An ideal optical resonator with infinite plane mirrors is considered; one of the mirrors is fixed and other moves with a constant velocity. An exact one-dimensional solution is obtained. The solution shows that despite the dependence of the boundary conditions on time, "dynamic modes" characterized by a closed frequency cycle appear in such a system. An explanation of the modulation of the radiation from a laser with a movable mirror is proposed. The validity of the formulas derived is assessed for the case of a He-Ne laser. It is noted that the cessation of generation which the authors observed in a movable mirror laser can be explained on the basis of the theory.

1. We consider the behavior of an electromagnetic field propagating in the direction of the axis of a one-dimensional ideal resonator with infinite flat mirrors, of which one mirror is fixed and the other moves at a constant velocity $v = \beta c$ along the same axis (Fig. 1).

This behavior is not obvious beforehand. Indeed, on the one hand, owing to the Doppler effect, the frequency of the light will change following each reflection of the light from the moving mirror and the system will go out of the state of ordinary static resonance. On the other hand, owing to the nonstationary nature of the picture, the process will not be monochromatic so that the very concept of resonance can either be lost or changed. Getting ahead of ourselves, we state that it is the second case which takes place: unique quasistationary "dynamic" modes are produced in the system.

An investigation of the behavior of a resonator with a moving mirror is important for the question of the stability of laser operation in the case of longitudinal mirror vibrations, and for a clarification of the processes that occur a laser during the course of generation-frequency tuning with the aid of a moving mirror.

2. The electric field $E(x,t)$ in the resonator satisfies the d'Alambert equation with boundary conditions

$$E(0, t) = 0 \tag{1}$$

on the fixed mirror and

$$E\{(L_0 + vt), t\} - \beta H\{(L_0 + vt), t\} = 0 \tag{2}$$

on the moving mirror, where L_0 is the coordinate of the moving mirror at the instant of time $t = 0$. H is the magnetic field in the resonator, perpendicular to the field E; $\beta > 0$ for mirrors that move apart and $\beta < 0$ for mirrors that move towards each other.

The condition (2) is easiest to derive by first writing

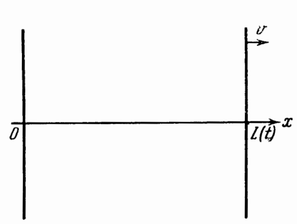

FIG. 1

it in a coordinate system that moves together with a mirror, where it has the same form as condition (1), and then using the Lorentz transformation in the laboratory system.

We seek the solution in the form of a superposition of plane-parallel waves propagating along the resonator axis. Taking into account the boundary condition on the stationary mirror only, the general solution for the fields E and H is

$$E(x, t) = f(x - ct) - f(-x - ct),$$
$$H(x, t) = f(x - ct) + f(-x - ct), \tag{3}$$

where f is an arbitrary function of its argument.

Substituting now (3) into the boundary condition (2) we obtain for f the functional equation

$$(1 - \beta) f\{L_0 - ct(1 - \beta)\} = (1 + \beta) f\{-L_0 - ct(1 + \beta)\}. \tag{4}$$

3. Let us find the exact solution of (4). To this end, we make a change of variables

$$ct = e^{\alpha} - L_0 / \beta. \tag{5}$$

We change over from the function $f(Z)$ to a new function F, defined by the relation

$$f\left(-Z + \frac{L_0}{\beta}\right) = \frac{F(\ln Z)}{Z}. \tag{6}$$

Substituting (5) and (6) in (4), we obtain for the function F the simple equation

$$F(\alpha - a) = F(\alpha + b), \tag{7}$$

where $a = -\ln(1 - |\beta|)$ and $b = \ln(1 + |\beta|)$. The constants a and b are positive. The variable α runs through the values $-\infty < \alpha < +\infty$. According to (5), the time t varies here between the limits

$$+\infty > t > -L_0/c\beta \text{ for separating mirrors}$$

$$-L_0/c\beta > t > -\infty \text{ for approaching mirrors}$$

Obviously, any periodic function with a period $(a + b)$ is a solution (7). Therefore the general solution of this equation can be written in the form

$$F(\alpha) = \sum_{n=-\infty}^{+\infty} a_n e^{ik_n \alpha}, \tag{8}$$

where

$$k_n = \frac{2\pi n}{a+b} = 2\pi n \Big/ \ln\frac{1+|\beta|}{1-|\beta|}, \quad a_n = \text{const}; \quad n = \ldots -2, -1, 0, 1, 2, \ldots . \tag{9}$$

Going over in (9) back to the electric field E(x), t we obtain the exact solution

$$E(x,t) = \sum_{n=-\infty}^{+\infty} \frac{a_n}{-x+ct+L_0/\beta} \exp\left[ik_n \ln\left(-x+ct+\frac{L_0}{\beta}\right)\right] - \sum_{n=-\infty}^{+\infty} \frac{a_n}{x+ct+L_0/\beta} \exp\left[ik_n \ln\left(x+ct+\frac{L_0}{\beta}\right)\right]. \tag{10}$$

The solution (10) is valid for any constant velocity of the moving mirror.

Let us change now the normalization in (10) in such a way that as $\beta \to 0$ we obtain the usual solution for mirrors at rest, and that the argument of the logarithm is the dimensionless quantity

$$a_n = \frac{B_n L_0}{\beta} \exp\left[-ik_n \ln\frac{L_0}{\beta}\right], \tag{11}$$

where B_n is a constant. We then get finally for E(x, t)

$$E(x,t) = \sum_{n=-\infty}^{+\infty} \frac{B_n}{1+(-x+ct)\beta/L_0} \exp\left[ik_n \ln\left\{1+\frac{(-x+ct)}{L_0}\beta\right\}\right] - \sum_{n=-\infty}^{+\infty} \frac{B_n}{1+(x+ct)\beta/L_0} \exp\left[ik_n \ln\left\{1+\frac{(x+ct)}{L_0}\beta\right\}\right]. \tag{12}$$

It is seen from this expression that in spite of the dependence of the boundary conditions on the time the general solution breaks up into a superposition of unique, if we can use the expression, "dynamic modes" and when $\beta = 0$ the dynamic modes go over the ordinary static modes of the resonator with stationary mirrors. To each dynamic mode there corresponds a forward and a backward wave. The amplitudes of the forward and backward waves (for both approaching and separating mirrors) are functions of the velocity and of the time. At some fixed value of x inside the resonator, the increase of β (for t) leads to a decrease of the amplitude in the separating system and to an increase in the approaching system. This is natural, since in the former case the energy becomes spread out in space and in the latter it becomes concentrated. At the instant when the mirrors meet, the amplitude of the field becomes infinite.

4. Let us discuss the physical meaning of expression (12). Since the oscillations of the electric field have an anharmonic character, it is possible to use the concept of the instantaneous frequency as the derivative of the phase Φ with respect to time: $\omega(t) = \partial\Phi/\partial t$.

We then find from (12) for the n-th dynamic mode propagating along the x axis, in the case of separating mirrors, that the frequency $\omega_{n(+)}^{out}$ and $\omega_{n(-)}^{out}$ of the waves propagating respectively in the direction of the x axis and in the opposite direction are equal to

$$\omega_{n(+)}^{out}(x,t) = \frac{\partial\Phi_{out}^{+}(-x+ct)}{\partial t} = \frac{n\pi c}{L_0} \frac{2\beta}{1+(-x+ct)\beta/L_0} \Big/ \ln\frac{1+\beta}{1-\beta}, \tag{13}$$

$$\omega_{n(-)}^{out}(x,t) = \frac{\partial\Phi_{out}^{-}(x+ct)}{\partial t} = \frac{n\pi c}{L_0} \frac{2\beta}{1+(x+ct)\beta/L_0} \Big/ \ln\frac{1+\beta}{1-\beta}, \tag{14}$$

where

$$\Phi_{out}^{+}(-x+ct) = k_n \ln\left[1+\frac{(-x+ct)}{L_0}\beta\right],$$

$$\Phi_{out}^{-}(x+ct) = k_n \ln\left[1+\frac{(x+ct)}{L_0}\beta\right].$$

On the moving mirror, the frequency shift between the incident and reflected wave corresponds to the Doppler shift in normal reflection of the wave from the moving mirror:

$$\omega_{n(+)}^{out}[L(t), t]\frac{1-\beta}{1+\beta} = \omega_{n(-)}^{out}[L(t), t], \tag{15}$$

where L(t) is the coordinate of the moving mirror. On the stationary mirror, the frequencies of the forward and backward waves coincide.

The situation is similar also for the approaching mirrors:

$$\omega_{n(+)}^{in}[L(t), t]\frac{1+\beta}{1-\beta} = \omega_{n(-)}^{in}[L(t), t], \tag{16}$$

where $\beta > 0$. Thus, a spectrum of closed frequency cycles is realized in the system at each given instant of time. Figures 2 and 3 show the dependence of the frequency and of the amplitude of the coordinate at a fixed instant of time for separating and approaching mirrors, where the horizontal arrows indicate the direction of the motion of the wave. With increasing time, the frequency and the amplitude cycles move upward or downward in accordance with the direction of the vertical arrows. It is easy to show that if the frequency of the wave on the standing mirror at the instant of time t = 0 is

$$\omega_n^{in}(0,0) = \frac{n\pi c}{L_0}\cdot 2\beta \Big/ \ln\frac{1+|\beta|}{1-|\beta|},$$

then after the time

$$\Delta t = \sum_{m=1}^{N} \frac{2L_0}{c+|v|}\left[\frac{1-|\beta|}{1+|\beta|}\right]^{m-1},$$

needed for the wave to make N passages through the resonator, the frequency of the wave on the same mirror will be

$$\omega_n^{in}(0,\Delta t) = \omega_n^{in}(0,0)\left(\frac{1+|\beta|}{1-|\beta|}\right)^N. \tag{17}$$

this corresponds to the conclusion obtained by Kurilenko[1] for approaching mirrors.

Expression (12), which is valid for any velocity, will satisfy the adiabaticity condition ωL = inv if $|\beta| \ll 1$.

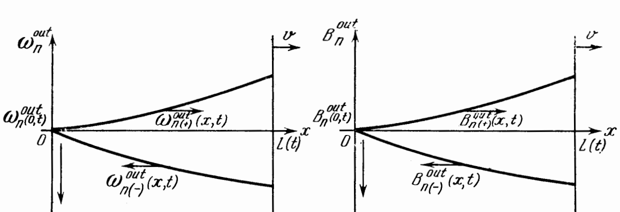

FIG. 2. Separating mirrors.

$$B_{n(+)}^{out}(x,t) = \frac{B_n}{1+(-x+ct)\beta/L_0}, \quad B_{n(-)}^{out}(x,t) = \frac{B_n}{1+(x+ct)\beta/L_0}$$

5. At small accelerations, the solution (12) is valid also for the case of non-uniform motion of the mirror. Let us estimate the region of applicability of such a solution if $v = v_0 \cos \Omega t$, so that the mirror oscillates about an equilibrium position. $x = L_0$ with amplitude.

$$L(t) = L_0 + l_0 \sin \Omega t. \tag{18}$$

We take into account the fact that in all the real cases $v \ll c$ and assume that $l_0 \ll L_0$ and $|vt|/L_0 \ll 1$, and then expression (12) is rewritten in the form

$$E(x,t) = \sum_{n=-\infty}^{+\infty} B_n \exp \frac{in\pi}{L_0}(-x+ct)\left[1 - \frac{(-x+ct)}{2L_0}\beta\right] - \sum_{n=-\infty}^{+\infty} B_n \exp \frac{in\pi}{L_0}(x+ct)\left[1 - \frac{(x+ct)}{2L_0}\beta\right]. \tag{19}$$

The influence of the acceleration on the form of the solution will be insignificant if

$$\frac{\partial E}{\partial t} \gg \frac{\partial E}{\partial v}\frac{\partial v}{\partial t}, \tag{20}$$

from which it is easy to show that our solution, which was obtained for the case $v = \text{const}$, is valid also for the case $v = v(t) = l_0\Omega \cos \Omega t$ under the condition that the number of passages of the radiation in the resonator satisfies the condition

$$N \lll cT / 2L, \tag{21}$$

where T is the period of oscillation of the mirror.

6. It was assumed in the entire foregoing analysis that there are no losses in the resonator and that the power of each given mode does not depend on its frequency. In a real laser resonator losses do occur and are determined principally by the energy radiated from the resonator and by the diffraction phenomena on the mirrors, owing to finite mirror aperture. These laser losses are replenished by the stimulated emission. The spectral region where the losses can be compensated for is small and is determined by the width of the generation line. In addition, the power of the generated mode depends on its position on the contour of the spectral line. This means that if at any instant of time there is excited in the laser only one dynamic mode with index n, then if the mode moves uniformly and sufficiently slowly over the contour of the spectral line, a change takes place in the laser power, from zero when the mode is in the region of the lasing threshold, through a maximum value (near the center of the line), and then back to zero when the mode falls into the extinction region. The mode with an index differing by

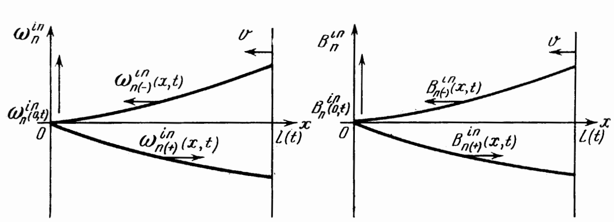

FIG. 3. Approaching mirrors.

$$B^{in}_{n(+)}(x,t) = \frac{B_n}{1 + (x - ct)\beta/L_0}$$
$$B^{in}_{n(-)}(x,t) = \frac{B_n}{1 + (-x - ct)\beta/L_0}, \quad \beta > 0$$

unity from the preceding mode, will experience a similar variation cycle.

Consequently, in the single-mode generation regime, the laser emission with wavelength λ is modulated with a period $T^1 = \lambda/2v$ and with a modulation depth $K = 100\%$ ($v = \text{const}$).

If several modes are excited simultaneously, the modulation depth is smaller since different modes reach their extremal values at different times. It is clear from the foregoing reasoning that power modulation occurs at external-mirror motion amplitudes $l_0 \gtrsim \lambda/2$. In the case of small amplitudes $l_0 \ll \lambda/2$ and a single-mode regime, no power modulation is observed in the general case.[2]

A rigorous analysis of the conditions for the existence of dynamic modes in a laser with a moving mirror calls for allowance for the inertia of the mechanism of mode establishment in the resonator. Let us illustrate the role of this process with the He-Ne laser ($\lambda = 328$ Å) as an example.

At the transition in question ($3S_2 - 2P_4$) the gain of the He-Ne laser is small and amounts to approximately 2% per meter.[3] The time of establishment of the mode in the resonator for a low-gain medium is given by the formula[4]

$$\tau_{\text{est}} = \frac{L\alpha}{c(\alpha - f'/2)f'/2}, \tag{22}$$

for L is the resonator length, f′ is the loss, and α is the gain per pass.

If the dynamic resonance passes through the entire generation frequency interval $\Delta\nu_g$ within a time shorter than τ_{est}, then the dynamic mode does not have time to develop. From this, using expression (22), we obtain the limiting mirror velocity

$$v \leqslant \frac{\Delta\nu_g \lambda f'(\alpha - f'/2)}{2\alpha}. \tag{23}$$

The equality sign in expression (23) gives the critical velocity of the mirror at which cessation of generation should be observed. The first to point out the existence of a critical velocity was Askar'yan.[5] Obviously, the limitation (23) is meaningful only for large mirror-motion amplitudes, which ensure complete passage of the mode through the entire interval $\Delta\nu_g$. If the moving mirror in the laser oscillates harmonically with an amplitude $l_0 \gg (L_0/\nu)\Delta\nu_g$ and frequency $\Omega \ll c/2L_0$, then each generated mode passes through the interval as a practically constant mirror velocity.

As seen from (23), the deterioration of the Q of the resonator, which leads simultaneously to a decrease of $\Delta\nu_g$ and $(\alpha - f'/2)$, should greatly decrease the value of v_{cr} when $\alpha \approx f'/2$.

It should be noted that the mechanism of generation of dynamic mode does not reduce to a constant attenuation of this mode at certain frequencies and to a build-up at neighboring frequencies. The change of the frequency in the resonator is the result of a frequency transformation of the mode itself, owing to the change of the energy of the light wave upon reflection of the moving mirror. Therefore, the phenomena connected with the damping of the mode in a large-Q resonator should not confine the mirror oscillation frequency Ω to the relation $\Omega \ll \nu/Q$, where Q is the quality factor of the resonator. The latter remark is valid whenever the

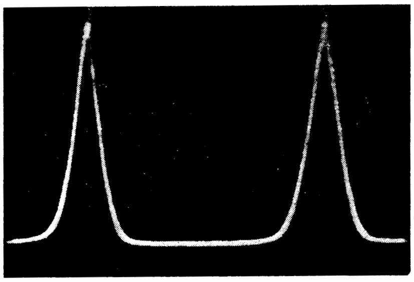

FIG. 4. Power oscillogram of a laser whose mirror oscillates with frequency $\Omega/2\pi = 50$ Hz and amplitude $l_0 = 0.4$ mm.

generated mode does not go outside the region $\Delta\nu_g$ during the entire period of the mirror oscillation.

A similar conclusion was arrived at by Yariv,[6] who considered the modulation of laser power with the aid of the linear electrooptical effect.

7. We found experimentally that for an He-Ne laser ($\lambda = 6328$Å) with a tube diameter 3 mm and with a distance between mirrors $L_0 = 50$ cm (the reflection coefficient of the plane stationary mirror was $R_1 = 0.98$, and that of the spherical moving mirror, with curvature radius 1.2 m, was $R_2 = 0.99$), the laser generation stops at the maximum Q of the resonator Q_{max} when the mirror velocity is $v_{cr} \approx 25$ cm/sec.

Polanyi and Watson[7] give for $\Delta\nu_g$ a value of 10^9 Hz at the emission maximum of a laser having similar parameters. From the experimentally determined value of v_c it is easy to find that the experimentally realized minimum loss per pass is $f'_e = 3.92 \times 10^{-2}$.

The loss calculated from the known mirror reflection coefficient is $f' = 3 \times 10^{-2}$. If we recognize that f' does not include the diffraction losses, then the agreement between the experiment and the theoretical can be regarded as good. A decrease of the resonator Q led to a sharp decrease of v_{cr}.

Figure 4 shows a typical power oscillogram of a laser whose mirror oscillates harmonically and with large amplitude. The horizontal port of the oscillogram corresponds to zero power level. The maximum power is obtained at mirror velocity $v = 0$ (points m and N). In this case, Q is smaller than Q_{max}.

8. On the basis of the foregoing, we can draw the following conclusions:

1) Dynamic modes are realized in an optical resonator with moving mirrors.

2) As $\beta \to 0$, the dynamic modes go over into the ordinary static modes of an optical resonator with two flat infinite mirrors.

3) Each dynamic mode is characterized by a closed frequency cycle: on the moving mirror the frequency shift between the incident and reflected waves corresponds to the Doppler shift at normal reflection of the wave from the moving mirror, and on the stationary mirror the frequencies of the forward and backward waves are equal.

4) Depending on the mirror motion direction, the frequency cycles shift either in the direction of increasing frequency (approaching mirrors) or decreasing frequency (separating mirrors).

5) When analyzing the conditions for the existence of dynamic modes in a laser with moving mirror, it is necessary to take into account the fact that if the time of passage of the dynamic mode through the entire generation interval $\Delta\nu_g$ is smaller than the time of establishment of the mode in the generator τ_{est}, then no laser generation at the dynamic mode takes place.

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Translated by J. G. Adashko
240