SIMILARITY HYPOTHESIS FOR CORRELATIONS IN THE THEORY OF SECOND ORDER PHASE TRANSITIONS

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We assume that there exists a similarity for correlations of quantities which fluctuate strongly near a second-order phase transition and that it is possible to describe these correlations by a single correlation radiuc r_c. We study the dependence of r_c on the thermodynamic parameters and on the ratio of the powers occurring in the singularities of the thermodynamic quantities. The meaning of the constants occurring in the theory is discussed.

1. INTRODUCTION

IT is well known that in second-order phase transition (PT) points and in critical points thermodynamic quantities such as the heat capacity, the susceptibility, and the compressibility have a singular part which indicates the non-additivity of the thermodynamic quantities in a PT point. As an example, we consider the susceptibility of a system in an external field H when the extra term in the energy which is connected with the external field has the form

$$E_H = -H \int m(\mathbf{r})\, dV, \tag{1.1}$$

where $m(\mathbf{r})$ completely determines the configuration of the system. We determine the susceptibility χ:

$$\chi = T\frac{\partial^2}{\partial H^2} \ln \left\langle \exp\left(-\frac{E_H}{T}\right)\right\rangle. \tag{1.2}$$

Here and henceforth $\langle A\rangle \equiv \overline{A}$ denotes an (ensemble) average of the quantity A in the equilibrium state of the system:[1]

$$\langle A\rangle = \Bigl(\sum_{\{m\}} e^{-E/T}\Bigr)^{-1}\Bigl(\sum_{\{m\}} e^{-E/T} A\Bigr), \tag{1.3}$$

E is the energy of the system and depends on the configuration $\{m\}$.

If the values of $m(\mathbf{r})$ are finite the singularity in χ may appear by virtue of a slow decrease for large $|\mathbf{r}-\mathbf{r}'|$ of the correlation

$$\varphi_m(\mathbf{r}-\mathbf{r}') = \langle (m(\mathbf{r})-\overline{m})(m(\mathbf{r}')-\overline{m})\rangle, \tag{1.4}$$

$$\chi = \frac{V}{T}\int \varphi_m(\mathbf{r})\, dV. \tag{1.5}$$

By virtue of the principle of the reduction of correlations[2] Eq. (1.4) decreases for $|\mathbf{r}-\mathbf{r}'| \gg r_c$ with increasing $|\mathbf{r}-\mathbf{r}'|$ so fast that we can restrict ourselves in (1.5) to integrating over a region with linear dimensions $\sim r_c$. The correlation radius r_c introduced in this way is sufficiently well defined only if it is large and if (1.5) diverges for large $r \lesssim r_c$ so that r_c is the cut-off radius for the integral in (1.5). One can completely analogously introduce a correlation radius for the correlations of any quantity $b(\mathbf{r})$ while for a rigorous definition one uses the usual formula

$$r_{cb} = \int |\mathbf{r}|\varphi_b(\mathbf{r})\, dV \Big/ \int \varphi_b(\mathbf{r})\, dV,$$
$$\varphi_b(\mathbf{r}-\mathbf{r}') = \langle (b(\mathbf{r})-\overline{b})(b(\mathbf{r}')-\overline{b})\rangle. \tag{1.6}$$

For different quantities b the radii r_{cb} may turn out to be different. If all r_{cb} are finite, then for a system of volume $V \gg r_c^a$, where a is the dimensionality of the space, the thermodynamic quantities are additive. The manifold of points in which some $r_c = \infty$ is the manifold of points of second-order phase transitions (or critical points). In those points some of the thermodynamic quantities lose their additivity however large the volume of the system, so that their dependence on the volume and the other parameters is singular. It has been shown recently that there is a connection between the singularities of different quantities.[3-7] This connection is a consequence of the phenomenological theory based upon the similarity assumption for correlations at large distances near phase transition points. In[5,6] the thermodynamic relations of the theory were studied. The aim of the present paper is a more complete analysis of the behavior of the correlations required by the theory. It turns out that part of the assumptions of papers[5,6] are the consequence of the other assumptions.

2. SIMILARITY HYPOTHESIS FOR THE CORRELATIONS OF STRONGLY FLUCTUATING QUANTITIES

Near PT points the value of r_c is by definition arbitrarily large for strongly fluctuating quantities. We assume that the correlation radii are of the same order of magnitude for all strongly fluctuating quantities so that we shall talk about a single correlation radius of the system. Let $b_i(\mathbf{r})$ for i = 1, 2, ... be a set of strongly fluctuating quantities and let h_i be the corresponding external fields reckoned from the critical value. The energy of the system for a given configuration $\{m(\mathbf{r})\}$ has the form

$$E = E_0\{m(\mathbf{r})\} - \sum_i h_i \int b_i(\mathbf{r})\, dV. \tag{2.1}$$

The distance to the PT points is defined by the variables

$$\tau = (T-T_c)/T \tag{2.2}$$

and the values of h_i.

Let us consider an isolated PT point when $r_c \to \infty$ as

$\tau \to 0$ and $h_i \to 0$. The basic assumption which we shall use is an assumption that for $r_c^{-1} = 0$, $r \gg r_0$, where r_0 is a fixed length (interaction radius), it is possible to write

$$\varphi^{b_i}(\mathbf{r}-\mathbf{r}') = \langle (b_i(\mathbf{r})-\bar{b}_i)(b_i(\mathbf{r}')-\bar{b}_i)\rangle = C_{b_i}/|\mathbf{r}-\mathbf{r}'|^{\alpha_i}, \quad (2.3)$$

where $0 < \alpha_i \leq a$. Near the PT point the behavior (2.3) is assumed for $|\mathbf{r}-\mathbf{r}'| \ll r_c$ while for $|\mathbf{r}-\mathbf{r}'| \gg r_c$ the correlations decrease sufficiently rapidly (faster than $|\mathbf{r}-\mathbf{r}'|^a$). For the cases which at the present time are rigorously or approximately solved (Ising model, perfect Bose gas[1], ...) the correlations have indeed the form (2.3). For other cases the behavior (2.3) is a plausible assumption leading to consequences which are reasonable and do not contradict experiments. Also possible are cases (the magnetic moment for an antiferromagnet) where for some quantities the correlations oscillate. We shall assume that in those cases there exist other quantities (e.g., the energy density) for which the behavior of the correlations obeys (2.3).

We assume that parameters with the dimension of length, such as the interaction radius for the system, are finite. In the transition point we may expect at distances $r \gg r_0$ that the correlations are determined by constants connected with the properties of the system at the transition point and the magnitude of the distance r. In that case the power indices α_i can be found from dimensional relations. For such a behavior the theory turns out to be a variant of Kolmogorov's theory of universal similarity.[8]

Different arguments in aid of the behavior (2.3) of the correlations were given in[5-7,9,10]. For the case of a wave field methods of estimating diagrams, which were developed in[9], show that the power-law behavior may be reconciled with the microscopic equations of the system. The concrete value of α in such a matching must be found from the equations not used for matching the power-law estimates.

Let the PT point be $\tau = 0$, $h_k = 0$. We consider a system where all these quantities except one h_i vanish and $r_c = r_c(h_i)$. A characteristic length appears in the region of large distances. When $r_0 \ll r \ll r_c$ the correlations should remain constant, within r/r_c, by virtue of the continuity of the correlations in the PT point. We shall assume that the correlation functions for $r \gg r_0$ and r_c finite have the form

$$\varphi_b = r^{-\alpha} C_b(r/r_c). \quad (2.4)$$

We consider the sphere (or circle for a two-dimensional system) S(R) with radius R and calculate the average fluctuation of the quantity

$$B_i(R) = \int\limits_{\mathbf{r}\in S(r)} (b_i(\mathbf{r})-\bar{b}_i)\, dV, \quad \overline{B_i^2(R)} = \int\limits_{\mathbf{r},\mathbf{r}'\in S(R)} \varphi_{b_i}(\mathbf{r}-\mathbf{r}')\, dV\, dV', \quad (2.5)$$

integrated over the region S(R). In the PT point or when $r_0 \ll R \ll r_c$ we find

$$\overline{B_i^2(R)} = \begin{cases} \text{const}\cdot R^{2a-\alpha_i}, & \text{if } \alpha_i < a, \\ \text{const}\cdot R^a \ln R, & \text{if } \alpha_i = a. \end{cases} \quad (2.6)$$

If $\alpha_i \leq a$, the fluctuations turn out to be strong and the mean square fluctuations increase faster than the volume of the region, which is a consequence of the strong correlation.

In the following we need an estimate of the quantities $\overline{B_i^{2n}(R)}$ and accordingly of the correlation functions. To take into account the requirements of the principle of the weakening of correlations we introduce irreducible correlation functions (semi-invariants, see[11]):

$$Q_{b,n}(\mathbf{r}_1,\ldots,\mathbf{r}_n) \equiv \frac{\delta^n}{\delta\lambda(\mathbf{r}_1)\ldots\delta\lambda(\mathbf{r}_n)} \ln\langle \exp\int \lambda(\mathbf{r})b(\mathbf{r})\, dV\rangle. \quad (2.7)$$

For b = m the correlations (2.7) are the same as the correlations Q of[5].

We write the probability distribution of the quantity B(R) in the form

$$dW(B) = W(B)\, dB. \quad (2.8)$$

It can be found if in the Gibbs distribution we sum over all configurations with a given value of B(R). Comparing (2.6) and (2.8), we find

$$\overline{B^2(R)} = \int B^2 W(B)\, dB = \text{const}\cdot R^{2a-\alpha_i}; \quad (2.9)$$

If the probability distribution (2.8) is a Gaussian distribution, (2.9) completely characterizes this distribution and the quantities

$$\langle\langle B^n(R)\rangle\rangle = \frac{\partial^n}{\partial\lambda^n} \ln\langle e^{\lambda B(R)}\rangle \quad (2.10)$$

vanish for $n > 2$, $\lambda = 0$. This is an expression for a weak correlation and corresponds to fluctuations far from the PT point. We shall assume that for $r_0 \ll R \lesssim r_c$ the distribution (2.8) is not Gaussian. The probability density (2.8) is essentially non-vanishing in the range of values $B \sim \sqrt{[\overline{B^2(R)}]}$ so that $B_{iR} = \sqrt{[\overline{B^2(R)}]}$ is a quantity characterizing the spread of the probability distribution.

The similarity hypothesis can be formulated as follows: The probability distribution for the quantity $B_i(R)/B_{iR}$ is in a PT point independent of R if $R \gg r_0$.

Near the PT point, when one of the quantities τ or h is non-vanishing, the quantity r_c is finite though large: $r_c \gg r_0$. The probability distribution for the quantity B(R) depends not only on R but also on the thermodynamic quantities τ or h.

When R increases from $r_0 \ll R \ll r_c$ to $R \gg r_c$, the evolution of the distribution function for B(R) near the PT point can be described as follows:

When R increases to $R \gg r_c$, the fluctuations in the quantity B(R) become basically Gaussian (up to terms of order $r_c/R \ll 1$). This is connected with the fact that the fluctuations in volumes $V \sim r_c^a$ separated by large distances $r \gg r_c$ are independent of the magnitude of r_c. For a system with an interaction which vanishes rigorously when $r > r_0$ it is natural to expect an exponential decrease of the correlations for distances $r \gg r_c$. For $\alpha < a$, we obtain for $\langle B^2(R)\rangle$ with $R \gg r_c$, instead of (2.6), apart from terms small of order r_c/R

$$\langle B^2(R)\rangle = \text{const}\cdot R^a r_c^{a-\alpha}. \quad (2.11)$$

For $\alpha = a$ we find

$$\langle B^2(R)\rangle = \text{const}\cdot R^a \ln r_c. \quad (2.12)$$

For quantities $\beta(r)$ whose fluctuations in the PT point do not become strong the usual dependence

$$\left\langle\left\{\int [\beta(\mathbf{r})-\bar{\beta}]\, dV\right\}^2\right\rangle = \text{const}\cdot R^a \quad (2.13)$$

is valid instead of (2.11) and (2.12). The contribution of order (2.13) to $\langle B^2(R)\rangle$, which is connected with correlations at small distances $r \sim r_0$, always exists but when $\alpha \leq a$ it turns out to be small compared with the contribution of distant ($r \sim r_c$) correlations. When $\alpha > a$, on the other hand, this contribution turns out to be dominant.

It is well known that the probability distribution can be given by its moments. One checks easily that the properties of the distribution (2.8) formulated in terms of the behavior of the moments (2.13) are the same as the properties assumed in[5,6]. The quantities $\langle\langle B^{2n}(R)\rangle\rangle$ of (2.10) can be found from $\overline{B^{2n}(R)}$ and for them the following relation holds

$$\langle\langle B^{2n}(R)\rangle\rangle = D_{2n}B_R^{2n}. \tag{2.14}$$

If the probability distribution is different from a Gaussian one, then $D_{2n} \neq 0$. The quantities $\langle\langle B^{2n}(R)\rangle\rangle$ can be expressed in terms of the correlations Q of (2.7) as follows:

$$\langle\langle B^n(R)\rangle\rangle = \int\limits_{r_i\in S(R)} Q_{bn}(\mathbf{r}_1,\ldots,\mathbf{r}_n)\,dV_1\ldots dV_n. \tag{2.15}$$

Let us formulate the similarity assumption in the correlation language. In a PT point the correlations $Q_{b,2n}(r_1, \ldots, r_{2n})$ are for $|r_i - r_j| \gg r_0$ homogeneous functions of the distances $|r_i - r_j|$ of degree $-n\alpha$:

$$\frac{Q_{b,2n}(\lambda\mathbf{r}_1,\ldots,\lambda\mathbf{r}_{2n})}{Q_{b,2}{}^n(\lambda\mathbf{r}_i-\lambda\mathbf{r}_j)} = \frac{Q_{b,2n}(\mathbf{r}_1,\ldots,\mathbf{r}_{2n})}{Q_{b,2}{}^n(\mathbf{r}_i-\mathbf{r}_j)}. \tag{2.16}$$

We introduce a distance $\widetilde{R}$:

$$\widetilde{R}^2 = \sum_{i>j}|\mathbf{r}_i-\mathbf{r}_j|^2. \tag{2.17}$$

The correlations $Q_{b,2n}$ can be written in the form

$$Q_{b,2n}(\mathbf{r}_1,\ldots,\mathbf{r}_{2n}) = \frac{1}{\widetilde{R}^{n\alpha}}\varphi_{b,2n}\left(\frac{\mathbf{r}_1}{\widetilde{R}},\ldots,\frac{\mathbf{r}_{2n}}{\widetilde{R}}\right). \tag{2.18}$$

Near the PT point the quantity r_c is finite. For large distances there is a characteristic length and the homogeneity of the correlations Q_{2n} must disappear. When $r_0 \ll |r_i - r_j| \ll r_c$ the correlations do not change, apart from terms of order $|r_i - r_j|/r_c$ by virtue of their continuity in a PT while they must vanish for $|r_i - r_j| \gg r_c$. If the finiteness of r_c is connected with the fact that only one of the thermodynamic quantities is different from its critical value, one may assume that when $r_0 \ll |r_i - r_j| \ll r_c$ the homogeneity property occurs when we make simultaneously the changes $r_i \to \lambda r_i$, and $r_c \to \lambda r_c$. We note that when $|r_i - r_j| > r_c$ the form of the correlations is unimportant, since from general considerations it follows that the contribution from such distances is negligibly small. The form of the correlations $Q_{b,2n}$ is according to our assumptions as follows:

$$Q_{b,2n}(\mathbf{r}_1,\ldots,\mathbf{r}_{2n}) = \frac{1}{\widetilde{R}^{n\alpha}}\varphi_{b,2n}\left(\frac{r_c}{\widetilde{R}},\frac{\mathbf{r}_1}{\widetilde{R}},\ldots,\frac{\mathbf{r}_{2n}}{\widetilde{R}}\right), \tag{2.19}$$

where as $y \to \infty$

$$\varphi_{b,2n}(y;\mathbf{r}_1,\ldots,\mathbf{r}_{2n}) \to \varphi_{b,2n}(\mathbf{r}_1,\ldots,\mathbf{r}_{2n}),$$

while as $y \to 0$ this function tends fairly rapidly to zero. We find from (2.19) for

$$\langle\langle B^{2n}\rangle\rangle = \lim_{R\to\infty}\langle\langle B^{2n}(R)\rangle\rangle$$

for finite r_c

$$\langle\langle B^{2n}\rangle\rangle = \widetilde{D}_{2n}\,V r_c^{(2a-\alpha)n-a}, \tag{2.20}$$

where the $\widetilde{D}_{2n}$ are constants and V the volume of the system,

$$\widetilde{D}_{2n}V = \int\frac{1}{\widetilde{R}^{n\alpha}}\varphi_{b,2n}\left(\frac{1}{\widetilde{R}};\frac{\mathbf{r}_1}{\widetilde{R}},\ldots,\frac{\mathbf{r}_{2n}}{\widetilde{R}}\right)dV_1\,dV_2\ldots dV_{2n}. \tag{2.21}$$

In the case where $\alpha = a$, Eq. (2.6) acquires a logarithmic factor $\ln(R/r_0)$. The formulation of the similarity for the distribution function must in this case be changed. We choose a fixed number $\lambda < 1$ and restrict the integration in (2.5) to the interval $\lambda R \leq |\mathbf{r} - \mathbf{r}'| \leq R$ (the contribution of fluctuations of scale R). Using this procedure for all $\langle\langle B^{2n}\rangle\rangle$ we can retain the old formulation for the quantities obtained in this way. The formulation of the similarity hypothesis in terms of the correlations Q is not changed.

3. BEHAVIOR OF THE CORRELATION RADIUS AND OF THE RATIO OF THE SUSCEPTIBILITY POWER INDICES

Assuming that the correlations of the quantities b_i are characterized by a single correlation radius r_c we can find the dependence of r_c on each of the quantities h, τ when the other quantities are equal to their critical values. We consider first of all the case when all external fields are absent and the finiteness of r_c is connected with $\tau \neq 0$; $r_c = r_c(\tau)$. We define the energy density $\epsilon(\mathbf{r})$ when there are no external fields

$$E_0\{m(\mathbf{r})\} = \int\varepsilon(\mathbf{r})\,dV. \tag{3.1}$$

Generally speaking, $\epsilon(\mathbf{r})$ can not be given unambiguously by the definition (3.1). We have assumed that there is such a way of defining $\epsilon(\mathbf{r})$ that the value of this quantity in the point r depends on the configuration $\{m(\mathbf{r}')\}$ only when $|\mathbf{r} - \mathbf{r}'| \leq r_0$ (finite interaction radius r_0 in the system). For instance, for the Ising model[1]

$$\varepsilon(\mathbf{r}) = \sigma(\mathbf{r})\sum_{\mathbf{r}'}J(\mathbf{r}-\mathbf{r}')\sigma(\mathbf{r}'), \tag{3.2}$$

where one assumes that $J(r) \equiv 0$ when $r > r_0$. For the average energy $\overline{E}$ we find

$$\overline{E} = T\frac{\partial}{\partial\tau}\ln Z,\quad Z = \sum_{\{m\}}\exp\left\{\frac{\int\varepsilon(\mathbf{r})\,dV}{T_c}(\tau-1)\right\}. \tag{3.3}$$

(The summation is over all configurations of the system.) Using (2.10) we find easily that when $\tau \ll 1$

$$T_c{}^2\frac{\partial^2}{\partial\tau^2}\langle\langle E_0{}^{2n}\rangle\rangle = \langle\langle E_0^{2n+2}\rangle\rangle. \tag{3.4}$$

Substituting the behavior (2.20) into (3.4) we find for $r_c(\tau)$, if $\alpha_\tau \neq a$,

$$\frac{\partial^2}{\partial\tau^2}r_c^{n(2a-\alpha_\tau)-a} = \text{const}\cdot r_c^{n(2a-\alpha_\tau)+a-\alpha}. \tag{3.5}$$

The constants are different for different n. We obtain

$$r_c(\tau) = r_{0\tau}\tau^{-2/(2a-\alpha)}. \tag{3.6}$$

If $\alpha = a$ (in the known cases this is, apparently, always the case) we get for n = 1

$$\frac{\partial^2}{\partial\tau^2}\ln r_c = \text{const}\cdot r_c{}^a,\qquad r_c = \text{const}\cdot\tau^{-2/a}. \tag{3.7}$$

The specific heat of the system in the case $\alpha \neq a$ is

$$c \approx T_c^{-2}\langle\langle E_0^2\rangle\rangle = \text{const}\cdot\tau^{-2(a-\alpha)/(2a-\alpha)}. \quad (3.8)$$

If $\alpha = a$,

$$c \approx VA \ln \tau, \quad (3.9)$$

where A is a constant.

The connection between the behavior (3.7) of the correlation radius and the logarithmic behavior (3.9) of the specific heat was discovered in[5]. Using (3.8) or (3.9) and (3.4) we can find the quantities $\langle\langle E_0^{2n}\rangle\rangle$ in terms of $\langle\langle E_0^2\rangle\rangle$.

Repeating the derivation of Eqs. (3.6) and (3.8) for an arbitrary quantity we find under the same assumptions

$$r_c(h_i) = r_{0h_i} h_i^{-2/(2a-\alpha_i)}, \quad (3.10)$$

where h_i is a dimensionless field, acting on b_i, α_i the power index in the correlation function:

$$\varphi_i(\mathbf{r}-\mathbf{r}') = \langle(b_i(\mathbf{r}) - \bar{b}_i)(b_i(\mathbf{r}') - \bar{b}_i)\rangle \sim |\mathbf{r}-\mathbf{r}'|^{-\alpha_i}, \quad r_0 \ll |\mathbf{r}-\mathbf{r}'| \ll r_c. \quad (3.11)$$

The susceptibility $\chi_i = \partial^2 F/\partial h_i^2$, where F is the free energy of the system in an external field h_i, is apart from a regular factor the same as the quantity $\langle\langle B_i^2\rangle\rangle = \lim_{R\to\infty} \langle\langle B^2(R)\rangle\rangle$. From Eqs. (3.10) and (3.11) for the quantities b_i, for which the correlations are described by Eq. (3.11) with a correlation radius common to all, we find

$$\chi_i = \text{const}\cdot V h_k^{-2(a-\alpha_i)/(2a-\alpha_k)}. \quad (3.12)$$

Formula (3.12) is the connection between the singularities of the different thermodynamic derivatives. For the relations of the powers in (3.10) to (3.12) it is necessary that the constants D in (2.20) are non-zero. This means that when $R \sim r_c$ the probability distribution (2.8) must differ essentially from a Gaussian distribution. One obtains easily for the case of a Gaussian distribution the limits on the power indices in the form of inequalities.

The behavior of the correlation radius $r_c(h)$ can be interpreted as follows. The asymptotic behavior of the correlations when $|\mathbf{r}-\mathbf{r}'| \gg r_c$,

$$\lim_{|\mathbf{r}-\mathbf{r}'|\to\infty} \langle b(\mathbf{r})b(\mathbf{r}')\rangle = \langle b\rangle^2, \quad (3.13)$$

determines the average value of the quantity b. Let us consider the quantity

$$x(\mathbf{r}) = b(\mathbf{r}) - \langle b\rangle_c, \quad (3.14)$$

where $\langle b\rangle_c$ is the average in the PT point. The correlation $\langle x(\mathbf{r})x(\mathbf{r}')\rangle$ is for $|\mathbf{r}-\mathbf{r}'| \sim r_c$ of the same order as for $|\mathbf{r}-\mathbf{r}'| \gg r_c$. Therefore we get from (3.11)

$$r_c^{-\alpha} \sim \bar{x}^2. \quad (3.15)$$

The quantity $\bar{x} = \bar{b} - \langle b\rangle_c$ is connected with the susceptibility χ through the relation

$$\bar{x} \sim h\chi \sim h r_c^{a-\alpha}. \quad (3.16)$$

Comparing (3.15) and (3.16) we get the estimate

$$r_c \sim h^{-2/(2a-\alpha)},$$

which is the same as (3.6). In this way we found in[5] the dependence of the correlation radius on the magnetic field for a ferromagnetic.

Formula (3.15) gives an estimate of $\bar{x}$ also for the case when the finiteness of r_c is connected with another field h or with the temperature τ and provided symmetry requirements do not lead to $\bar{x} \equiv 0$, as is the case, e.g., for the spontaneous moment in the Ising model for $\tau > 0$. In the last case any arbitrarily small field (for a system with $V \to \infty$) lifts for $\tau < 0$ the degeneracy so that the spontaneous moment below T_c has a temperature dependence described by Eq. (3.15):

$$m_{b_i s} \sim r_c(\tau)^{-\alpha_i/2}. \quad (3.17)$$

The considerations used in deriving (3.17) are due to Ryazanov.[11] Indeed, the derivation of Eq. (3.17) requires the analysis of the behavior of the singular part of the thermodynamic potential also under the additional assumptions given in[5].

The power indices determine the behavior of the correlations at large distances r, $r_0 \ll r \ll r_c$. By virtue of the continuity of the correlations at such distances these quantities are continuous in the transition point, i.e., they are constants for each PT, which are the same for $\tau > 0$ and for $\tau < 0$. In the case of a logarithmic dependence of the specific heat this logarithmic divergence is determined by the asymptotic behavior of the correlations for $r_0 \ll r \ll r_c$ and the logarithms occur therefore with the same coefficients for $\tau > 0$ and for $\tau < 0$. The behavior of the correlations for $r \sim r_c$ is different for $T > T_c$ and for $T < T_c$ so that if the specific heat is $c \sim \ln \tau$, then a discontinuity may be added to the logarithmic singularity. To estimate this discontinuity quantitatively we need additional assumptions, for instance, such as were used in[10]. As to order of magnitude, the discontinuity in the specific heat and also the coefficient of the logarithmic singularity are the same, from dimensional considerations, as the regular part of the specific heat. For the susceptibility with a power-law singularity in the PT point one can have different coefficients in the regions above and below T_c.

4. DISCUSSION OF THE VALUES OF THE CONSTANTS

We have already noted that the structure of the theory is close to the structure of Kolmogorov's theory of universal similarity.[8] In such a kind of theory important conclusions can be obtained from dimensional relations, provided one knows the parameters which determine equilibrium. The appearance in the PT point of a hierarchy of scales from r_0 to the size of the system gives us a possibility to write the fluctuations of strongly fluctuating quantities as a superposition of fluctuations of different scales. One can give arguments favoring the statistical independence of fluctuations with very different scales (see, e.g.,[9,10]), but the degree of their independence requires the solution of the microscopic problem, which up to the present time has not been possible to do mathematically without errors. Models based upon the hypothesis of statistical independence were studied in[12,13], and a formulation of the hypothesis in terms close to those of the theory of turbulence was given in[10]. Basic for the theory is the statement that the fluctuations of scale R are determined by the magnitude of the scale and by constants which determine the equilibrium and which are common for all scales.

To introduce a scale we construct a sequence of divisions of the space in cells with dimensions $d_n = \lambda^n d_0$, $\lambda > 1$:

$$d_0 \sim r_c, \quad d_1 = \lambda d_0, \quad d_2 = \lambda d_1 = \lambda^2 d_0, \ldots, d_n = \lambda^n d_0, \ldots \quad (4.1)$$

We write the fluctuations of the quantity $B = \int (b(\mathbf{r}) - \bar{b}) dV$ as a superposition of independent fluctuations of scale d_k, k = 0, 1, ..., N; a characteristic dimension for the change due to fluctuations in b for a fluctuation of scale d_k is d_k. The choice of divisions (4.1) is stipulated by the requirement to obtain as $r \to \lambda r$ the same structure of the divisions in scale.

For $\langle B^2 \rangle \equiv \langle\langle B^2 \rangle\rangle$ we get

$$\langle B^2 \rangle = \sum_{k=0}^{N} \langle B_k^2 \rangle, \quad (4.2)$$

where $\langle B_k^2 \rangle$ is the mean square fluctuation of scale d_k, $d_N \sim r_c$. We introduce $p = 1/d_k$, $dp = -p \ln \lambda dk$, and in (4.2) we change to integrating over p: $p_0 = 1/d_0$, $p_c = 1/r_c$,

$$\langle B^2 \rangle = \frac{1}{\ln \lambda} \int_{p_c}^{p_0} \frac{dp}{p} \langle B_p^2 \rangle. \quad (4.3)$$

When $B_k \equiv 1$, we find the number of scales N(V) in the volume $V \ll r_c^a$:

$$N(V) = \frac{\ln (V/d_0^a)}{\ln \lambda}. \quad (4.4)$$

Let us consider the fluctuations in the energy E in the system. For the case of a small correlation radius $r_c \sim d_0$ the whole contribution to $\langle\langle E^2 \rangle\rangle$ is connected with the small scale $\sim r_0$ and $\langle\langle E^2 \rangle\rangle \sim V\rho T^2$, where V is the volume of the system, ρ the number of particles per unit volume. We assume that $\langle E_k^2 \rangle$ has the same form for any k < N and for $r_c \gg r_0$, i.e.,

$$\langle E_k^2 \rangle \sim V r_0^{-a} T_c^2. \quad (4.5)$$

Under those assumptions we have for $V \sim r_c^a$

$$\langle\langle E^2 \rangle\rangle \sim \frac{VT_c^2}{r_0^a} N(V) = \frac{VT_c^2}{r_0^a \ln \lambda} \ln \frac{V}{r_0^a}. \quad (4.6)$$

The behavior (4.6) means that $r_c \sim \tau^{-2/a}$, and corresponds to Ryazanov's hypothesis of uniform temperature spread.[10]

Another interpretation of (4.5) is connected with the probability distribution of the quantity $\xi(\mathbf{r}) = \epsilon(\mathbf{r}) - \bar{\epsilon}$. The probability $P[\xi(0)\xi(\mathbf{r})]$ for a value of the product $\xi(0)\xi(\mathbf{r})$ can be written in the form

$$P[\xi(0)\xi(\mathbf{r})] = P[\xi(0)] P_{\xi(0)}[\xi(\mathbf{r})], \quad (4.7)$$

where $P[\xi(0)]$ is the probability for a value ξ in the point $\mathbf{r} = 0$, $P_{\xi(0)}[\xi(\mathbf{r})]$ the probability for a value ξ in the point r for given $\xi(0)$. For $\xi(0) \sim \xi_0 = \langle \xi^2 \rangle^{1/2}$ we assume that

$$\langle \xi(\mathbf{r}) \rangle_0 = \int \xi P_{\xi_0}(\xi) d\xi(\mathbf{r})$$

is determined by the magnitude of T_c and the distance r:

$$\langle \xi(\mathbf{r}) \rangle_0 \sim T_c / r^a. \quad (4.8)$$

For ξ_0 we find from dimensionality considerations

$$\xi_0 \sim T_c / r_0^a \quad (4.9)$$

and

$$\langle \xi(0) \xi(r) \rangle \sim \frac{T_c^2}{r_0^a r^a}, \quad (4.10)$$

which is equivalent to (4.6).

In the assumptions (4.5) or (4.8) the exponent α in (2.3) for the correlations in the energy density is

$$\alpha = -2/a. \quad (4.11)$$

To estimate the correlations in the moment m it is necessary to know the connection between the energy and the moment. Such a connection is determined by the effective interaction in the system and from dimensional considerations it must have the form

$$E \sim Vgm^\nu. \quad (4.12)$$

The "coupling constant" g is connected with the properties of the system at small distances $\sim r_0$. For a system with binary interactions the interaction energy is given in the form

$$E \sim g_2 \int m^4(\mathbf{r}) dV. \quad (4.13)$$

When averaging over small distances $\sim r_0$ it is possible that an interaction of the form

$$E \sim g_k \int m^{2k}(\mathbf{r}) dV, \qquad g_k \sim g_2^n \overline{m^{2(2n-k)}} r_0^{a(n-1)} T_c^{-n+1}, \quad (4.14)$$

may appear where the bar across indicates averaging over a scale of order r_0. For a wave field[9] the constants g_k correspond, e.g., to the contribution of diagrams with $\omega_n \neq 0$ or $p \gtrsim 1/r_0$ in diagrams with 2k boson exits.

We assume that the interaction (4.12) is determined by one of the quantities g_k:

$$E \sim V g_k m^{2k}. \quad (4.15)$$

We evaluate the change in the energy $\Delta\Phi(R, M)$ when the moment M(R) fluctuates in a region with linear dimensions R in the PT point. The estimate

$$\Delta\Phi \sim M(R) h,$$

is valid for $\Delta\Phi$, where h is the field producing the moment M(R). For a not too weak field we can use Eq. (3.12) whence follows that

$$M(R) \sim h\chi \sim R^a h^{\alpha/(2a-\alpha)} \quad (4.16)$$

and

$$\Delta\Phi \approx g R^a (M/R^a)^{2a/k} \quad (4.17)$$

The structure of (4.17) is obvious and is determined by the requirement of additivity of $\Delta\Phi$ for a given moment $m = M(R)/R^a$. Expression (4.17) determines the effective interaction of the fluctuations of scale R and fixes the magnitude of the "coupling constant" g. Comparing (4.17) and (4.15) and assuming that this effective interaction is determined by the quantity g_k we get

$$\alpha = a/k, \quad k = 2, 3, \ldots \quad (4.18)$$

Fixman[14] proposed the spectrum (4.18) on the basis of considerations which were similar in concept. One can give arguments[10,14] in aid of the statement that for a = 3 the case k > 3 is impossible.

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