NONLINEAR INTERACTION OF WAVES IN THE PRESENCE OF TWO-STREAM INSTABILITY

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The dynamics of the development of kinetic two-stream instability in a plasma may be determined by the nonlinear interaction of the excited Langmuir waves, and not by the quasilinear effects. For this, it is sufficient that the waves be confined in a finite volume, and their decay and transformation into other modes be weak. A nonlinear equation is found, which describes the evolution of the energy spectrum of the oscillations in this case, and an asymptotic (large time) stationary solution is obtained for the equation. The solutions describe steady-state energy spectra of oscillations, which differ significantly from those given by quasilinear theory.

1. INTRODUCTION

A beam of charged particles passing through a plasma may excite Langmuir oscillations in it.[1-4] For this, it is necessary that the directed velocity of the beam, u, be large compared to the thermal velocities of the plasma particles $v_{T(e,i)0}$. In addition, the beam must be sufficiently rarefied and have a significant thermal spread:

$$v_{Te1} > u(N_1/N_0)^{1/3}, \tag{1.1}$$

where v_{Te1} is the thermal velocity of the electrons in the beam, and N_1 and N_0 are the beam and plasma densities respectively.

The dynamics of the development of the two-stream kinetic instability that arise under these conditions is customarily described within the framework of quasilinear theory.[5,6] In our paper, we show that the quasilinear effects do not always play a determining role in the development of the instability. On the contrary, if we take into account the finite dimensions of the volume occupied by the plasma oscillations then, as we shall see below, the determining factor may be the effect of nonlinear interaction of the oscillations with each other. We derive an equation that describes the evolution of the energy spectra of the oscillations in such situations, and we find under some simple assumptions a stationary solution of this equation, which is significantly different from the stationary spectra of quasilinear theory.

First of all, let us consider how the laws of conservation of energy and momentum appear in the model of two uniform semi-bounded mutually-penetrating plasmas, used in [5,6] to describe two-stream instability. Since the "wave-beam particle" system is regarded as an isolated subsystem in this case, the total fluxes of energy and momentum in it must be conserved. Thus the fluxes of energy and momentum of the wave at any point are determined by the change of the corresponding fluxes of the beam. For example, at steady state, the flux of energy of the wave at any point sufficiently far from the plasma boundary is[5]

$$\sum_k 2\varepsilon_k v_g \approx \int_{-\infty}^{\infty} v \frac{mv^2}{2}(f_0 - f_\infty)\,dv, \tag{1.2}$$

where v_g is the group velocity of the wave, ε_k is the energy of the electric field in the wave with wave vector k (for simplicity, the one-dimensional case is considered here). Let f_0 and f_∞ denote the distribution functions of the plasma and beam: $f_0 \equiv f(0, v)$ is the undisturbed distribution function at the plasma boundary (x = 0), $f_\infty \equiv f(\infty, v)$ is the steady state distribution for large x (x → 0). In the model considered, the energy and momentum obtained by the plasma waves from the beam particles are carried by them with the group velocity v_g from the boundary into the plasma to infinity. On the other hand, since in actuality the volume occupied by the plasma is finite and plasma oscillations do not get out of it, it is clear that for large x we must have $\sum_k 2\varepsilon_k v_g = 0$. Thus, in the model considered, it is necessary to introduce some third subsystem, which interacts with the waves and absorbs energy and momentum from them. Obviously, the conclusions of Vedenov et al.[5] concerning the steady-state forms of the spectrum and distribution function remain valid, if for example, on the second boundary of the plasma at x = L, the plasma waves are fully absorbed or transformed into radiation.

However, in the case where reflection is important, we must expect the physical results to be completely different. In this case, a stable stationary state in the subsystem of waves and resonant particles can no longer be realized, since, on the one hand, the state in which a beam passes unaltered through a plasma is unstable, and on the other hand, the fluxes of energy and momentum for steady oscillations which occupy a finite volume and form a system of standing waves are equal to zero. Nevertheless, a stationary state becomes possible if we consider the interaction of the oscillations with the nonresonant particles, i.e., the induced scattering of the waves by the particles, which appears as a nonlinear effect (proportional to ε_k^2). The nonresonant particles of the plasma may serve as the third subsystem, which absorbs energy and momentum from the oscillations. Obviously, this is accompanied by an acceleration and heating of the entire plasma mass.

Thus, in the indicated situation, which is apparently realized most often, when the oscillations form standing waves inside some volume, the nonlinear interaction of the waves between themselves may play a determining role.

2. THE BASIC EQUATIONS

In this paper, we examine the influence of induced scattering of waves by particles on the development of two-stream instability under the following assumptions.

First, we assume that the beam passing through a region of oscillations experiences only weak perturbation, so that the linear growth increment of the oscillations $\gamma(k)$ may be assumed to be independent of the energy ε_k and to be determined by the formula:[5]

$$\gamma(k) = \frac{\pi}{2}\frac{\omega_{Le0}^3}{k^2}\frac{\partial f_0}{\partial v}\Big|_{v=\omega/k}. \tag{2.1}$$

Second, the linear dimension L of the region with oscillations will be assumed to be large compared to the wavelength $\lambda = 2\pi k^{-1}$ ($L \gg k^{-1}$), and the plasma in this region assumed to be sufficiently uniform, such that the kinetic equation for waves derived for unbounded plasmas is applicable.

The energy transferred by the beam particles to the plasma oscillations per unit time equals $W_t \approx 2\gamma(k)W(k)\Delta kSL$, where W(k) is the spectral energy density, Δk the width of the wave packet, and S the cross-section area of the beam. The total energy flux of the beam equals $P \approx N_1(mu^2/2)Su$. Obviously, the condition of weak perturbation of the beam may be written as

$$\frac{W_t}{P} \approx \frac{4\gamma(k)W(k)\Delta k}{N_1 mu^2}\frac{L}{u} \ll 1. \tag{2.2}$$

For Maxwellian distributions of plasma and beam particles, the increment (2.1) equals[3, 4]

$$\gamma = -\sqrt{\frac{\pi}{8}}\left[\frac{\omega_{Le0}^4}{k^3v_{Te0}^3}\exp\left\{-\frac{\omega_{Le0}^2}{2k^2v_{Te0}^2}\right\} + \frac{\omega_{Le1}^2\omega_{Le0}(\omega-uk)}{k^3v_{Te1}^3}\exp\left\{-\frac{(\omega-uk)^2}{2k^2v_{Te1}^2}\right\}\right]. \tag{2.3}$$

Using the estimate of the maximum increment

$$\gamma_m \approx \sqrt{\frac{\pi}{8}}\frac{N_1}{N_0}\omega_{Le0}\frac{u^2}{v_{Te1}^2},$$

we write inequality (2.2) as

$$\frac{W_t}{P} \sim \frac{W(k)\Delta k}{N_0\varkappa T_{e1}}\frac{\omega_{Le0}L}{u} \ll 1, \tag{2.4}$$

where T_{e1} is the temperature of the beam electrons and κ is the Boltzmann constant. Since $\omega_{Le0} \approx uk$, we have $\omega_{Le0}L/u \approx kL \gg 1$. Thus the condition we obtained

$$\frac{W(k)\Delta k}{N_0\varkappa T_{e1}} \ll (kL)^{-1} \tag{2.5}$$

is stronger than the weak-turbulence condition $W(k)\Delta k/N_0\kappa T_{e0} \ll 1$, if $T_{e1} \sim T_{e0}$, and is equivalent to the latter if $T_{e1} \gg T_{e0}$ (hot beam in cold plasma).

Considering $kL \gg 1$, we shall assume, however, that the time kL/ω is small in comparison with the characteristic time of increase of the oscillations. For this condition to be satisfied, assuming that the waves are reflected without damping or transformation at the point of reflection, we may regard the energy of the reflected waves to be always equal to the energy of the incident waves: $W(-\mathbf{k}, t) = W(\mathbf{k}, t)$. This permits us not to consider the boundary problem here. It is quite simple to write the kinetic equation for the waves in an unbounded plasma, which determines the evolution of the spectrum $W(\mathbf{k})$ for those waves generated by the beam, and to include in the nonlinear term of this equation the interactions of these waves with the opposite waves having a spectrum $W(\mathbf{k}) = W(-\mathbf{k})$.

Within the approximation considered, the nonlinear kinetic equation for the waves may be written as[7, 8]

$$\frac{dn(\mathbf{k})}{dt} = 2\gamma(\mathbf{k})n(\mathbf{k}) + n(\mathbf{k})\int Q(\mathbf{k},\mathbf{k}')n(\mathbf{k}')d\mathbf{k}', \tag{2.6}$$

where $n(\mathbf{k})$ is the number of quanta of plasma oscillations in a unit interval of the wave vectors $\mathbf{k}$, $n(\mathbf{k}) = W(\mathbf{k})/\omega(\mathbf{k})$. The kernel $Q(\mathbf{k}, \mathbf{k}')$ for a plasma without magnetic field may be represented in the form[7]

$$Q(\mathbf{k},\mathbf{k}') = (2\pi)^3\sum_\alpha\int d\mathbf{p}\left(\mathbf{k}-\mathbf{k}', \frac{\partial f_\alpha}{\partial \mathbf{p}}\right)W_\alpha(\mathbf{p},\mathbf{k},\mathbf{k}'), \tag{2.7}$$

where $W_\alpha(\mathbf{p}, \mathbf{k}, \mathbf{k}')$ is the probability of scattering of a quantum $\mathbf{k}'$ by a particle of type α with momentum $\mathbf{p}$ to form a quantum $\mathbf{k}$, and $f_\alpha(\mathbf{p})$ is the distribution function of the particles of type α. The probability W_α possesses the following symmetry property:

$$W_\alpha(\mathbf{p},\mathbf{k},\mathbf{k}') = W_\alpha(\mathbf{p},\mathbf{k}',\mathbf{k}) \tag{2.8}$$

and is proportional to $\delta(\omega(\mathbf{k}) - \omega(\mathbf{k}') - (\mathbf{k} - \mathbf{k}', \mathbf{v}))$, which reflects the laws of conservation of energy and momentum in the scattering process ($\omega(\mathbf{k})$ is the spectrum of the oscillations).

Assuming small deviations of the distribution function from isotropic distributions ($N_1 \ll N_0$), we can write $\partial f_\alpha/\partial \mathbf{p}$ as $\mathbf{v}\partial f_\alpha/\partial \varepsilon$, where $\varepsilon = p^2/2m_\alpha$ is the particle energy, and then

$$Q(\mathbf{k},\mathbf{k}') = (2\pi)^3[\omega(\mathbf{k}) - \omega(\mathbf{k}')]\sum_\alpha\int d\mathbf{p}\frac{\partial f_\alpha}{\partial \varepsilon}W_\alpha(\mathbf{p},\mathbf{k},\mathbf{k}') \equiv Q_c(\mathbf{k},\mathbf{k}')\,\mathrm{sign}\,[\omega(\mathbf{k}) - \omega(\mathbf{k}')], \tag{2.9}$$

in which $Q_c(\mathbf{k}, \mathbf{k}') = Q_c(\mathbf{k}', \mathbf{k})$.

We shall assume the oscillation spectrum to be one-dimensional, and shall subsequently ascertain when this is valid. Equation (2.7) assumes the form

$$\frac{dn(k)}{dt} = 2\gamma(k)n(k) + n(k)\int_{-\infty}^{\infty}Q_c(k,k')\,\mathrm{sign}\,(|k|-|k'|)\,n(k')dk'. \tag{2.10}$$

Here we assume $\partial\omega/\partial k > 0$. From (2.10) it is immediately clear that the nonlinear effect described by the second term on the right side is confined to shifting the oscillations in the spectrum while the total number of waves $\int n(k)dk$ is conserved. If the distribution function is close to equilibrium, then this shifting occurs toward the side of lower frequencies and moduli of wave numbers ($Q_c(k, k') < 0$), so that the energy of the quanta decreases in the process. The increment $\gamma(k)$ is positive in the finite region of the wave numbers defined by the condition $\partial f_e/\partial v > 0$ (we consider electron oscillations), so that there exists a lower boundary of instability $|k|_m > 0$, and for $0 < |k| < |k|_m$ we have $\gamma(k) < 0$. Under these conditions, on account of the nonlinear wave interaction, the energy of the oscillations is carried out into the region of small $|k|$, where the waves are damped, so that the existence of a stationary state $dn(k)/dt = 0$ is possible. We shall seek just such a steady-state solution of (2.10) for large $t \to \infty$.

First, let us note that because of the condition $W(k) = W(-k)$ we have $n(k) = n(-k)$, so that $\omega(k) = \omega(-k)$. Thus (2.10) easily reduces to an equation for some purely "normal" waves excited by the beam, for which we may assume $k > 0$:

$$\frac{dn(k)}{dt} = 2\gamma(k)n(k) + n(k)\int_0^\infty Q_0(k, k')\,\mathrm{sign}(k-k')n(k')\,dk', \tag{2.10a}$$

where $Q_0(k, k') = Q_c(k, k') + Q_c(k, -k')$. We shall study just this equation.

3. STATIONARY SOLUTIONS

Let us first consider the simplest case, for which we may assume $Q_0(k, k') = Q_0 = \text{const}$ for all k and k' for which $n(k)$ and $n(k')$ differ appreciably from zero. As will be clear from what follows, for this it is necessary that the region of instability be sufficiently narrow (weakly over-critical). Equation (2.10) takes the form

$$\frac{dn(k)}{dt} = \left[2\gamma(k) + Q_0\int_0^\infty \mathrm{sign}(k-k')n(k')\,dk'\right]n(k)$$
$$= \left\{2\gamma(k) + Q_0\left[-\int_0^\infty n(k')\,dk' + 2\int_0^k n(k')\,dk'\right]\right\}n(k). \tag{3.1}$$

It is seen that the stationary solution $dn(k)/dt = 0$ must satisfy the following conditions: In the region where $n(k) \neq 0$, the equation

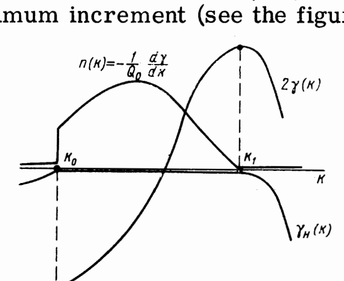
$$\gamma_H(k) = 2\gamma(k) + Q_0\left[-\int_0^\infty n(k')\,dk' + 2\int_0^k n(k')\,dk'\right] = 0 \tag{3.2}$$

must hold, and in all the remaining parts of the axis $0 \le k < \infty$, $n(k) \equiv 0$ must hold. Thus, Eq. (3.1) with $dn/dt = 0$ is formally satisfied by any function, which satisfies (3.2) in some region and vanishes in all the remaining parts of the axis $0 \le k < \infty$. However, in order for some stationary state to be attained at $t \to \infty$ regardless of the form of the initial conditions $n(k, 0)$, it is necessary that $n(k)$ vanish in the stationary state only where $\gamma_H(k) < 0$, since only under this condition will $n(k) \to 0$ as $t \to \infty$. This additional condition makes the solution of the problem single-valued.

From (3.2) it follows that where $n(k) \neq 0$ we must have

$$n(k) = -\frac{1}{Q_0}\frac{d\gamma(k)}{dk}, \tag{3.3}$$

from which, considering the fact that $n(k) \ge 0$ and $Q_0 < 0$, we have that $n(k) \neq 0$ only when $d\gamma(k)/dk > 0$. This implies that $n(k) \equiv 0$ when $k > k_1$, where k_1 is the point of maximum increment (see the figure).



In addition, from the condition $\gamma_H(k) < 0$ when $n(k) = 0$ it follows that we must have

$$Q_0\int_0^{k_1} n(k')\,dk' + 2\gamma(k_1) = 0, \tag{3.4}$$

so that the region where $n(k) > 0$ starts from k_1. Since the function

$$Q_0\left[-\int_0^\infty n\,dk' + 2\int_0^k n\,dk'\right]$$

can either equal $-2\gamma(k)$ or be constant (where $n(k) = 0$) and must be continuous, this function must change continuously from the point k_1 toward the side of smaller k and remain equal to $-2\gamma(k)$ until the point k_0, where $\gamma(k_0) = -\gamma(k_1)$, at which, according to (3.2) and (3.4),

$$Q_0\left[-\int_0^\infty n\,dk' + 2\int_0^{k_0} n\,dk'\right] = -Q_0\int_0^\infty n(k')\,dk'.$$

From this, it follows that $n(k) \equiv 0$ when $k < k_0$. We have thus found the stationary solution $n(k)$. It has the following form:

$$n(k) = -Q_0^{-1}\,d\gamma/dk \quad \text{if } k_0 \le k \le k_1$$
$$n(k) = 0 \quad \text{if } k < k_0 \text{ and } k > k_1.$$

Here

$$\gamma_H(k) = 0 \quad \text{for } k_0 \le k \le k_1;$$
$$\gamma_H(k) = 2[\gamma(k) - \gamma(k_1)] \quad \text{for } k > k_1;$$
$$\gamma_H(k) = 2[\gamma(k) + \gamma(k_1)] \quad \text{for } k < k_0,$$

so that the condition $\gamma_H(k) < 0$ for $n(k) = 0$ is indeed satisfied.

Passing to the more general one-dimensional equation (2.10), we shall seek its asymptotic solution for $t \to \infty$ also in stationary form. To this end, we also assume that $n(k) \neq 0$ on some interval $k_0 \le k \le k_1$, on which the equation

$$\gamma_H(k) \equiv 2\gamma(k) + \int_{k_0}^{k_1} Q_0(k, k')\,\mathrm{sign}(k-k')n(k')\,dk' = 0 \tag{3.5}$$

holds, and $n(k) \equiv 0$ if $k < k_0$ and $k > k_1$, and $\gamma_H(k) < 0$ if $k < k_0$ and $k > k_1$. Equation (3.5) can be reduced to an ordinary Fredholm equation of the second kind

$$\frac{d\gamma_H}{dk} = 2\gamma'(k) + \int_{k_0}^{k}\frac{\partial Q_0(k, k')}{\partial k}n(k')\,dk'$$
$$-\int_k^{k_1}\frac{\partial Q_0(k, k')}{\partial k}n(k')\,dk' + 2n(k)Q_0(k, k') = 0 \tag{3.6}$$

or

$$n(k) - \int_{k_0}^{k_1} Q^*(k, k')n(k')\,dk' = f(k), \tag{3.7}$$

where

$$Q^*(k, k') = -\frac{\mathrm{sign}(k-k')}{2Q_0(k, k)}\frac{\partial Q_0(k, k')}{\partial k}, \quad f(k) = -\frac{\gamma_k'(k)}{Q_0(k, k)} \tag{3.8}$$

However, the limits k_0 and k_1 in (3.7) are not yet determined. In order to determine them, we must consider the following. First of all, in order for $\gamma_H(k)$ to vanish when $k_0 \le k \le k_1$, it is necessary to pose the extra condition $\gamma_H(k_0) = 0$ or $\gamma_H(k_1) = 0$ in addition to (3.7). We choose the second condition

$$\gamma_H(k_1) = 2\gamma(k_1) + \int_{k_0}^{k_1} Q_0(k_1, k')n(k')dk' = 0. \tag{3.9}$$

In the second place, the requirement $\gamma_H < 0$ if $k < k_0$ and $k > k_1$ leads to the conditions

$$\frac{d\gamma_H(k_0 - 0)}{dk} \geqslant 0, \quad \frac{d\gamma_H(k_1 + 0)}{dk} \leqslant 0. \tag{3.10}$$

But according to (3.6)

$$\frac{d\gamma_H(k_0 + 0)}{dk} = 2\gamma'(k_0) - \int_{k_0}^{k_1} \frac{\partial Q_0(k_0, k')}{\partial k} n(k')dk' \tag{3.11}$$
$$+ 2n(k_0 + 0)Q_0(k_0, k_0) = 0.$$

Considering that $Q_0(k, k) < 0$, we see that $d\gamma_H(k_0 - 0)/dk > 0$ if $n(k_0 + 0) > 0$, so that no additional condition arises here. On the other hand, since

$$\frac{d\gamma_H(k_1 - 0)}{dk} = 2\gamma'(k_1) + \int_{k_0}^{k_1} \frac{\partial Q_0(k_1, k')}{\partial k} n(k')dk' \tag{3.12}$$
$$+ 2n(k_1 - 0)Q_0(k_1, k_1) = 0,$$

the second inequality (3.10) can be satisfied only with $n(k_1 - 0) = 0$; with this, $d\gamma_H(k_1 + 0)/dk$ also vanishes.

Thus, the spectrum $n(k)$ in the short-wave edge goes to zero in a continuous manner, as in the idealized case examined earlier, and appears in general as shown in the figure. Moreover, we have obtained a second condition for the location of the limits k_0 and k_1, which may may be written as

$$\int_{k_0}^{k_1} Q^*(k_1, k')n(k')dk' + f(k_1) = 0. \tag{3.13}$$

Since $d\gamma_H(k_1 + 0)/dk = 0$, the requirement $\gamma_H < 0$ for $k > k_1$ leads to the condition

$$d^2\gamma_H(k_1 + 0) / dk^2 \leqslant 0.$$

As is easily shown, this is indeed satisfied.

Equations (3.7), (3.9) and (3.13) form a complete system for the determination of $n(k)$, k_0, and k_1. For their solution, it is in principle sufficient to obtain solutions of the integral equation (3.7) for all possible values of k_0 and k_1, substitute these into (3.9) and (3.13), and solve the resulting algebraic equations for k_0 and k_1. For a practical solution of the problem, however, we must use the method of successive approximations. We choose the zeroth approximation to be $n^{(0)}(k) = f(k)$, $k_0^{(0)}$ and $k_1^{(0)}$ from the conditions $\gamma(k_1^{(0)}) = \max \gamma(k)$, $\gamma(k_0^{(0)}) = -\gamma(k_1^{(0)})$. This zeroth approximation is the exact solution of the problem when $Q_0(k), k') = \text{const}$, as seen above. The first approximation is given by the formulas

$$n^{(1)}(k) = \int_{k_0^{(0)}}^{k_1^{(0)}} Q^*(k, k')n^{(0)}(k')dk' + n^{(0)}(k),$$
$$\int_{k_0^{(1)}}^{k_1^{(1)}} Q_0(k_1^{(1)}, k')n^{(0)}(k')dk' + 2\gamma(k_1^{(1)}) = 0, \tag{3.14}$$
$$\int_{k_0^{(1)}}^{k_1^{(1)}} Q^*(k_1^{(1)}, k')n^{(0)}(k')dk' + f(k_1^{(1)}) = 0.$$

Let us denote the integral operator with limits k_0 and k_1 by $\hat{P}(k_0, k_1)$, so that $n^{(1)}(k) = \hat{P}(k_0^{(0)}, k_1^{(0)})n^{(0)} + n^{(0)}$. The second approximation is given by the formulas

$$n^{(2)}(k) = n^{(0)}(k) + \hat{P}(k_0^{(1)}, k_1^{(1)})n^{(0)} + \hat{P}^2(k_0^{(1)}, k_1^{(1)})n^{(0)},$$
$$\int_{k_0^{(2)}}^{k_1^{(2)}} Q_0(k_1^{(2)}, k')n^{(1)}(k')dk' + 2\gamma(k_1^{(2)}) = 0, \tag{3.15}$$
$$\int_{k_0^{(2)}}^{k_1^{(2)}} Q^*(k_1^{(2)}, k')n^{(1)}(k')dk' + f(k_1^{(2)}) = 0.$$

The higher approximations are constructed in a similar manner, so that the m-th approximation $n^{(m)}(k)$ has the form of the m-th partial sum of the Neumann series for the integral equation with limits $k_0^{(m-1)}$ and $k_1^{(m-1)}$. A rigorous investigation of the convergence of this iteration scheme appears to be a particularly complicated problem; it is possible to show, however, that for sufficiently small $Q^*(k, k')$, the convergence occurs rapidly.

4. STATIONARY SPECTRA IN THE CASE OF TWO-STREAM INSTABILITY

We now apply the formulas obtained above to the actual calculation of the stationary spectra of the oscillations for two-stream instability. To this end, we use the formula for the increment (2.7) and the formulas for the nonlinear coefficient $Q_C(k, k')$ in the one-dimensional case.[7,9] In this connection, we must distinguish some variants of induced scattering by particles.

1. The scattering is from the electrons. This implies that $|\omega(\mathbf{k}) - \omega(\mathbf{k}')| \gg |\mathbf{k} - \mathbf{k}'| v_{Ti_0}$ for almost all $\mathbf{k}$ and $\mathbf{k}'$, or in the one-dimensional case

$$|k + k'| r_{De0} \gg \sqrt{T_i m / T_e M}. \tag{4.1}$$

If $\mathbf{k}$ and $\mathbf{k}'$ are directed to the same side, this means that

$$|k| r_{De0} \gg \sqrt{T_i m / T_e M}, \tag{4.2}$$

while for scattering from opposite waves ($\mathbf{k}$ and $\mathbf{k}'$ in opposite directions)

$$\Delta k r_{De0} \gg \sqrt{T_i m / T_e M}, \tag{4.3}$$

where Δk is the width of the stationary spectrum. For the interaction of waves of the one-dimensional spectrum it is particularly important to take into account the role of the ions in creating a shielding charge of the electrons, from which nonlinear scattering takes place.[7,9] The point is that without allowance for this fact, the nonlinear interaction of waves with parallel $\mathbf{k}$ is very small (it is zero in the first approximation $(kr_{De0})^2$). Therefore, in practice, for all wavelengths except the shortest ($kr_{De} > (m/M)^{1/6}$), the kernel of the equation has the following form:

$$Q_0(k, k') = -\frac{3}{8(2\pi)^{5/2}} \frac{\omega_{Le0}^2}{N_0 \varkappa T_{e0}} |k + k'| r_{De0} \tag{4.4}$$

for

$$|k + k'| r_{De0} < \sqrt{m / M} \tag{4.5}$$

(considering (4.1), we note that this is possible only for $T_i/T_e \ll 1$), and

$$Q_0(k, k') = -\frac{1}{6(2\pi)^{5/2}} \frac{\omega_{Le0}^2}{N_0 \varkappa T_{e0}} \frac{m}{M} \frac{1}{|k + k'| r_{De0}} \tag{4.6}$$

for

$$|k + k'| r_{De0} > \sqrt{m / M}. \tag{4.7}$$

For instability at the long waves $|k|r_{De0} < \sqrt{m/M}$ we have for $\Delta k \ll |k|$

$$Q_0(k, k') \approx -\frac{3}{4(2\pi)^{5/2}}\frac{\omega_{Le0}^2}{N_0\varkappa T_{e0}}|k|r_{De0}, \tag{4.8}$$

and using (3.3) we can immediately write the stationary spectrum $W(k) = \omega_{Le0} n(k)$:

$$W(k) = \frac{2}{3}(2\pi)^3 N_1 \frac{mu^2}{2}\frac{u^2}{\omega_{Le0}v_{Te1}}\frac{T_{e0}}{T_{e1}} \times \frac{1-(\omega - ku)^2/(kv_{Te1})^2}{|k|r_{De0}}\exp\left\{-\frac{(\omega-ku)^2}{2k^2v^2_{Te1}}\right\}. \tag{4.9}$$

In the case (4.7), a large effect is given by scattering from opposite waves ($kk' < 0$), so that Q_0, which is proportional to $1/(|k| - |k'|)$, cannot be approximated by a constant. Thus, we can only estimate $W(k)$ here:

$$W(k) \sim 3(2\pi)^3 N_1 \frac{Mu^2}{2}\frac{u}{\omega_{Le0}}\frac{T_{e0}}{T_{e1}}\left[1-\left(\frac{\omega-ku}{kv_{Te1}}\right)^2\right]|k|r_{De0} \times \exp\left\{-\frac{(\omega-ku)^2}{2k^2v^2_{Te1}}\right\}. \tag{4.10}$$

2. The scattering is from ions ($|k|r_{De0} < \sqrt{T_i m/T_e M}$). Here

$$Q_0 = -\frac{3}{4(2\pi)^{5/2}}\frac{\omega_{Le0}^2}{N_0\varkappa T_{e0}}|k|r_{De0}\sqrt{\frac{T_{e0}M}{T_{i0}m}} \tag{4.11}$$

and

$$W(k) = \frac{2}{3}(2\pi)^3 N_1 \frac{mu^2}{2}\frac{u^2}{\omega_{Le0}v_{Te1}}\sqrt{\frac{m}{M}}\frac{\sqrt{T_{i0}T_{e0}}}{T_{e1}} \times \frac{1-(\omega-ku)^2/(kv_{Te1})^2}{|k|r_{De0}}\exp\left\{-\frac{(\omega-ku)^2}{2k^2v^2_{Te1}}\right\}. \tag{4.12}$$

In the cases examined, the points k_0 and k_1, which define the boundaries of the stationary spectrum, are located symmetrically with respect to the boundaries of the instability region and are defined by the equations

$$\frac{\omega - k_{0,1}u}{{}^1/_2(k_0+k_1)v_{Te1}} = \pm 1. \tag{4.13}$$

In this connection, $W(k_{0,1}) = 0$. Moreover, $W(k) \equiv 0$ if $k < k_0$ and $k > k_1$, so that formulas (4.9), (4.10), and (4.12) are valid only if $k_0 \le k \le k_1$.

The resulting formulas for $W(k)$ are easily integrated with respect to k and readily yield the total energy of the oscillations W_0. For example, for scattering of the waves by ions,

$$W_0 = \int W(k)\,dk \approx \frac{2}{3}(2\pi)^{7/2}N_1\frac{mu^2}{2}\sqrt{\frac{m}{M}}\frac{\sqrt{T_{i0}T_{e0}}}{T_{e1}}(|k|r_{De0})^{-1}. \tag{4.14}$$

5. CONDITIONS OF APPLICABILITY OF THE ONE-DIMENSIONAL MODEL

We now turn to the question of under what conditions the stationary spectrum may be regarded as one-dimensional. In the first place, such a situation exists when oblique waves, in which the components of **k** perpendicular to the beam are nonzero, are strongly damped. This damping occurs, for example, in an axially-symmetric system whose transverse dimensions are small compared to the longitudinal ones (the beam is along the axis). Let the wavelengths of the oscillations be small with respect to all dimensions, and let the plane waves be a sufficiently good approximation of the natural oscillations. The finite dimensions can nevertheless be accounted for by introducing into the kinetic equation for the waves a term proportional to $\mathbf{v}_g \partial n(\mathbf{k})/\partial \mathbf{r}$ which would describe the transport of energy of the waves out of the system in the transverse direction.[10] This term has the order of magnitude of $\omega(\mathbf{k})k_\perp L_\perp n(\mathbf{k})$ ($k_\perp$ is the transverse wave number, $L_\perp$ is the transverse dimension), and in the case $\omega(\mathbf{k}) \gg \gamma(\mathbf{k})$ under consideration this term leads to strong damping of the instability when $k_\perp \neq 0$.

Another cause of the one-dimensional form of the spectrum is the nature of the nonlinear wave interaction. Assume that we have a one-dimensional spectrum $n(\mathbf{k}) = n(k_\parallel)\delta(\mathbf{k}_\perp)$, where $n(k_\parallel)$ satisfies Eq. (2.10) with $dn/dt = 0$. This $n(\mathbf{k})$, obviously, formally satisfies (2.6) with $dn(\mathbf{k})/dt = 0$. However, as before, it is necessary to show that

$$\frac{1}{n(\mathbf{k})}\frac{dn(\mathbf{k})}{dt} < 0$$

for $n(\mathbf{k}) = 0$, in order for this solution to be the true stationary limit for all initial conditions.

According to (2.9), $Q(\mathbf{k}, \mathbf{k}') = \tilde{Q}(\mathbf{k}, \mathbf{k}')[\omega(\mathbf{k} - \omega(\mathbf{k}')]$. Thus for $k_\perp \ll k_\parallel$

$$\begin{aligned}\int Q(\mathbf{k}, \mathbf{k}')n(\mathbf{k}')d\mathbf{k}' &= \int dk_\parallel' Q(\mathbf{k}, \mathbf{k}_\parallel')n(\mathbf{k}_\parallel') \\ &\approx \int dk_\parallel' \tilde{Q}(\mathbf{k}_\parallel, \mathbf{k}_\parallel')n(k_\parallel')[\omega(\mathbf{k}_\parallel) - \omega(\mathbf{k}_\parallel')] \\ &+ \int dk_\parallel' \tilde{Q}(\mathbf{k}_\parallel, \mathbf{k}_\parallel')n(k_\parallel')[\omega(\mathbf{k}) - \omega(\mathbf{k}_\parallel)].\end{aligned} \tag{5.1}$$

Since

$$2\gamma(\mathbf{k}_\parallel) + \int dk_\parallel' \tilde{Q}(\mathbf{k}_\parallel, \mathbf{k}_\parallel')n(k_\parallel')[\omega(\mathbf{k}_\parallel) - \omega(\mathbf{k}_\parallel')] \leqslant 0,$$

and $\tilde{Q} < 0$, $n(\mathbf{k}) > 0$ and $\omega(\mathbf{k}) - \omega(\mathbf{k}_\parallel) > 0$, we get

$$\frac{1}{n(\mathbf{k})}\frac{dn(\mathbf{k})}{dt} = 2\gamma(\mathbf{k}) + \int Q(\mathbf{k}, \mathbf{k}')n(\mathbf{k}')d\mathbf{k} < 0. \tag{5.2}$$

The condition of validity of this investigation is the inequality $\Delta k \ll |k|$.

In conclusion we note that the general nature of the method used here to determine the stationary spectrum in the case of kinetic instability permits us in principle to use this method also for other instabilities of a type wherein, for one reason or another, the dependence of the increment on the oscillation energy can be neglected.

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204