SIMULTANEOUS PHYSICAL SINGULARITY IN SPACE FILLED BY A DUSTLIKE MEDIUM

L. P. GRISHCHUK

Astronomy Institute of the Moscow State University

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A solution of the gravitational equations which possesses a significantly nonsimultaneous true singularity and which depends on eight physically arbitrary functions of three variables was previously considered[5] for space filled by a dustlike medium. A solution with a simultaneous physical (true) singularity is investigated in the present paper. The simultaneous nature of singularity can be ensured only by decreasing the number of physically arbitrary functions by one, the respective solution now depending on seven functions of three variables.

As is well known[1-3], in a synchronous coordinate system, by virtue of one of the equations of gravitation, the metric determinant g must vanish during a finite time. (This is called the Landau theorem.) The vanishing of the metric determinant denotes that a singularity exists in the metric in the given coordinate system. E. Lifshitz, Sudakov, and Khalatnikov[2] determined the geometrical causes of such a singularity, namely that in a synchronous system of coordinates the time lines which form a family of geodetics cross on an envelope (caustic) hypersurface. In the general case, the caustic hypersurface is oriented in time, since it contains length elements of the time-like lines tangent to it. Consequently, the fictitious (coordinate) singularity which results from the crossing of the time lines is in general not simultaneous. After using the permissible spatial transformations, the corresponding solution of the Einstein equations in free space contains five arbitrary functions of three coordinates[2]. One of these functions is connected with the "mathematical" arbitrariness in the choice of the initial hypersurface from which the time coordinate is reckoned. This arbitrariness denotes the possibility of changing the caustic hypersurface, while retaining at the same time the form of the solution in its vicinity.

It is indicated in[2,3] that in a synchronous coordinate system it is also possible to construct a solution in which the fictitious singularity is attained by all points of space simultaneously. In a certain sense, this case corresponds to a certain "limiting" choice of a mathematically arbitrary function, which is contained in the solution with the non-simultaneous singularity. As a result of this choice, the geodetics constructed along normals to the initial hypersurface are focused simultaneously on a geometrical image having a smaller number of dimensions, namely on a two-dimensional surface. The solution of this problem in free space was obtained analytically by Belinskiĭ and Khalatnikov[4].

As shown earlier[5], two families of geodetics—a family of world lines of particles of a dustlike medium and a family time lines of a synchronous coordinate system—behave in a certain sense in the same manner. The world lines of particles of a dustlike medium also have a time-like envelope hypersurface. The corresponding solution of the gravitational equations depends on the maximum number (eight) of physically arbitrary functions. The hypersurface, in whose points the world lines of the particles intersect, and consequently the physical singularity arises, can be chosen as the caustic for the time lines of the synchronous coordinate system. This case corresponds to coinciding physical and coordinate singularities (we emphasize that the time lines themselves do not coincide with the world lines of the dust particles; such a coincidence is possible only if the matter does not rotate, something which we do not postulate). Of course, for another choice of the synchronous coordinate system, the physical and coordinate singularities will occur on different hypersurfaces[5].

We have seen that for a specially constructed synchronous coordinate system it is possible to obtain simultaneous intersection of all the time lines. It can also be stated that the simultaneous character of the intersection of the time lines is ensured by a "limiting" choice of the mathematically arbitrary function. This choice does not decrease the number of physically arbitrary three-dimensional functions (i.e., functions of three variables) contained in the solution. As to the simultaneous intersection of the world lines of the particles with the medium, it can be attained only at the expense of specifying one function out of the physically arbitrary ones. This means (and apparently not only for dustlike matter) that the broadest class of solutions with simultaneous physical singularity can contain not more than seven physical arbitrary three-dimensional functions.

We shall construct the solutions with the simultaneous singularity, confining ourselves to the case when the physical and the coordinate singularities coincide. This solution depends on seven physically arbitrary functions of three variables.

ANALYTIC CONSTRUCTION OF THE SOLUTION

Following[2-4], we write down the metric near the singularity T = 0 in the form 1):

$$
\begin{aligned}
g_{ab} &= \sum_{n=0}^{\infty} g_{ab}^{(n)} t^n = a_{ab} + b_{ab}t + c_{ab}t^2 + \ldots, \\
g_{a3} &= \sum_{n=0}^{\infty} g_{a3}^{(n+2)} t^{n+2} = a_{a3}t^2 + b_{a3}t^3 + \ldots, \\
g_{33} &= \sum_{n=0}^{\infty} g_{33}^{(n+2)} t^{n+2} = a_{33}t^2 + b_{33}t^3 + \ldots .
\end{aligned}
\tag{1}
$$

1) The greek indices $\alpha, \beta, \mu, \nu, \ldots$ run through the values 0, 1, 2, and 3; the latin indices i, k, l, ... run through the values 1, 2, 3. The indices a, b, c, and d can assume values 1 and 2. The speed of light and the Einstein gravitational constant are assumed equal to unity.

We assume here, for the time being, that all the coefficients are arbitrary functions of three spatial coordinates. We also find the contravariant components of the metric tensor (see[2] concerning this procedure):

$$g^{ab} = a^{ab} - b^{ab}t + (-c^{ab} + b_c{}^a b^{bc} + a_3{}^a a^{b3})t^2 + \dots,$$
$$g^{a3} = -a^{a3} + (-b^{a3} + b_c{}^a a^{c3} + b^{33}a_3{}^a)t + \dots,$$
$$g^{33} = a^{33}t^{-2} - b^{33}t^{-1} + (-c^{33} + b^{33}b_3{}^3 + a^{a3}a_a{}^3) + \dots$$

The raising and lowering of the indices is effected with the aid of the metric a_{ab}, a_{33}.

Assume that a physical singularity takes place at $t = 0$. The expansions for the energy density and for the velocity components (in analogy with the case of the nonsimultaneous singularity[5]) will be sought in the form

$$\varepsilon = \varepsilon^{(-1)}t^{-1} + \varepsilon^{(0)} + \dots, \qquad u^\alpha = u^{\alpha(0)} + u^{\alpha(1)}t + \dots \tag{2}$$

The coefficients of these series are arbitrary three-dimensional functions.

We write down Einstein's equations in the form[1]

$$R_{00} \equiv \frac{1}{2}\frac{\partial}{\partial x^0}\varkappa_i^i + \frac{1}{4}\varkappa_k^i\varkappa_i^k = -T_{00} + \frac{1}{2}T, \tag{3}$$

$$R_{0i} \equiv -\frac{1}{2}(\varkappa_{i;k}^k - \varkappa_{k;i}^k) = -T_{0i}, \tag{4}$$

$$R_{ik} \equiv \frac{1}{2}\frac{\partial}{\partial x^0}\varkappa_{ik} + \frac{1}{4}(\varkappa_{ik}\varkappa_l^l - 2\varkappa_i^l\varkappa_{kl}) + P_{ik} = -T_{ik} + \frac{1}{2}g_{ik}T, \tag{5}$$

where $\kappa_{ik} = \partial g_{ik}/\partial x^0$, and P_{ik} is the Ricci spatial tensor.

We first transform the coefficient a_{33} into -1 by means of the coordinate transformation $x^3 = x^3(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$, $x^a = x^a(\tilde{x}^1, \tilde{x}^2)$ which does not change the form of the expansions (1) and (2) (after which we omit the symbol ~). Calculations of the components of the Ricci tensor show that the first terms of the expansions which do not vanish automatically are $R_0^{(-1)}$, $R_{ab}^{(-2)}$, $R_{a3}^{(0)}$, and $R_{33}^{(0)}$. The components of the energy-momentum tensor $T_{\mu\nu} = \epsilon u_\mu u_\nu$ have the following order:

$$T_{00} \sim t^{-1}, \quad T_{0a} \sim t^{-1}, \quad T_{03} \sim t, \quad T_{ab} \sim t^{-1}, \quad T_{a3} \sim t, \quad T_{33} \sim t^3.$$

We shall show that the sum of the gravitation equations, in their first approximation, impose such limitations on the functions a_{ab} that by using the permissible transformation $x^a = x^a(\tilde{x}^1, \tilde{x}^2)$ they can be transformed into

$$a_{ab} = q\delta_{ab}. \tag{6}$$

Here q is an arbitrary function of the coordinates x^1 and x^2; δ_{ab} is the two-dimensional Kronecker symbol (in this section and in a few others our derivations duplicate completely the calculations of Belinskiĭ and Khalatnikov[4]).

Let us consider the following equations:

$$R_{03}^{(-1)} \equiv -{}^1/{}_2 a^{cd}a_{cd}' = 0, \tag{7}$$

$$R_{ab}^{(-2)} \equiv {}^1/{}_2(a_{ab}'' - a^{cd}a_{ac}'a_{bd}') = 0, \tag{8}$$

$$R_{33}^{(0)} \equiv {}^1/{}_2(a_{cd}'a^{cd})' + {}^1/{}_4 a^{ab}a^{cd}a_{ad}'a_{bc}' = 0 \tag{9}$$

(the prime denotes differentiation with respect to x^3). Introducing the notation $a_{ab}' = \alpha_{ab}$, we can transform the system (7)—(9) into

$$\alpha_a{}^{b\prime} = 0, \quad \alpha_a{}^b\alpha_b{}^a = 0, \quad \alpha_a{}^a = 0. \tag{10}$$

Solving (10), we obtain ultimately

$$a_{ab} = \varphi_{ab} + x^3\varphi_{ac}\chi_b{}^c.$$

The arbitrary two-dimensional functions φ_{ab} and χ_a^b (i.e., functions of the two variables x^1 and x^2), are connected by the relations

$$\varphi_{1a}\chi_2{}^a = \varphi_{2a}\chi_1{}^a, \quad \chi_a{}^a = 0, \quad \chi_a{}^b\chi_b{}^a = 0. \tag{11}$$

The quadratic form $dr^2 = \varphi_{ab}dx^a dx^b$ can be reduced, by means of the coordinate transformations $x^a = x^a(\tilde{x}^1, \tilde{x}^2)$, to $dr^2 = q(dx^{1^2} + dx^{2^2})$. As follows from relations (11), all the functions χ_a^b vanish here, i.e., we arrive at the equality (6). Now the first terms of the metric (1) are as follows:

$$g_{ab} = q\delta_{ab} + b_{ab}t + \dots, \qquad g_{a3} = a_{a3}t^2 + b_{a3}t^3 + \dots,$$
$$g_{33} = -t^2 + b_{33}t^3 + \dots. \tag{1'}$$

Not one of the coordinate transformations containing the three-dimensional arbitrary function and not violating the form of the metric (1′) in expansion (2) is now left. The only admissible transformation with two-dimensional arbitrariness is the transformation of the type

$$x^3 = \tilde{x}^3 + f(\tilde{x}^1, \tilde{x}^2). \tag{12}$$

Calculating the components of the Ricci tensor by means of the metric (1′), we arrive at the conclusion that $R_{00} \sim t^{-1}$, $R_{0a} \sim t^{-1}$, $R_{03} \sim t$, $R_{ab} \sim t^{-1}$, $R_{a3} \sim t$, and $R_{33} \sim t$. Let us see what all the Einstein equations yield in the principal order in t:[2)]

$$R_{00}^{(-1)} \equiv b_3{}^3 = -\varepsilon^{(-1)}(u_0^{(0)^2} - {}^1/{}_2), \tag{13}$$

$$R_{0a}^{(-1)} \equiv a_{a3}' = -\varepsilon^{(-1)}u_0^{(0)}u_a^{(0)}, \tag{14}$$

$$R_{03}^{(1)} \equiv -{}^1/{}_2(-c_a{}^{a\prime} + {}^1/{}_2 b_3{}^3 b_a{}^{a\prime} + {}^1/{}_2 b_{ab}'b^{ab} + 2a_{a3}'a^{a3}) = -\varepsilon^{(-1)}u_0^{(0)}u_3 \tag{15}$$

$$R_{ab}^{(-1)} \equiv -{}^1/{}_2(b_{ab}'' - b_{ab}) = -\varepsilon^{(-1)}(u_a^{(0)}u_b^{(0)} - {}^1/{}_2 a_{ab}), \tag{16}$$

$$R_{a3}^{(1)} \equiv {}^1/{}_2(3b_{a3} + b_c{}^c a_{a3} - b_3{}^3 a_{a3} + a^{c3}b_{ac}'' - b_{a;d}^{d\prime} + b_{c;a}^{c\prime})$$
$$= -\varepsilon^{(-1)}u_a^{(0)}u_3^{(2)} + {}^1/{}_2 a_{a3}\varepsilon^{(-1)}. \tag{17}$$

$$R_{33}^{(1)} \equiv b_{33} + {}^1/{}_2 b_a{}^{a\prime\prime} - {}^1/{}_2 b_a{}^a = -{}^1/{}_2\varepsilon^{(-1)}. \tag{18}$$

The purpose of the next step of the investigation is as follows. We shall show that we can determine all the remaining coefficients in the expansions (1′) and (2) with the aid of the six series of equations (5), four series of equations $T^\nu_{\mu;\nu} = 0$ and the equations (14) with respect to the three arbitrary three-dimensional functions $\epsilon^{(-1)}$, u and one two-dimensional function q. The determination of the quantities b_{ab}, c_{ab}, d_{ab}, etc. entails here the appearance of six series of two-dimensional arbitrary functions. Two more functions of two variables appear in the solution of (14). From Eqs. (3) and (4) we require so far only the satisfaction of (14). It will be shown later that if (5) is satisfied and $T^\nu_{\mu;\nu} = 0$, the remaining relations that follow from (3) and (4) lead to certain connections between the two-dimensional functions which arise during the determination of $g_{ab}^{(n)}$ (n = 1, 2, 3, ...).

The method of expressing the next coefficients in the expansions (1′) and (2) in terms of the first ones (i.e., in terms of $\epsilon^{(-1)}$, $u_a^{(0)}$, q) will be illustrated by using the determination of the quantities b_{ik}, $u_a^{(1)}$, $u_3^{(2)}$, and $\epsilon^{(0)}$ as an example. As will be shown later, the next terms of the expansions: c_{ik}, $u_a^{(2)}$, $U_3^{(3)}$, and $\epsilon^{(1)}$ etc., can be obtained in similar fashion.

2) In some places we continue to use the quantities a_{ab}, recalling that $a_{ab} = q(x^1, x^2)\delta_{ab}$. The covariant differentiation in (17) is carried out in a two-dimensional space with metric a_{ab}.

The energy and momentum conservation laws $T^{\nu}_{\mu;\nu} = 0$ are written in the following form:

$$u^{\nu}u_{a;\nu} = 0, \quad (19)$$

$$u^{\nu}u_{3;\nu} = 0, \quad (20)$$

$$u^{\nu}\partial\varepsilon / \partial x^{\nu} + \varepsilon u^{\nu}{}_{;\nu} = 0. \quad (21)$$

In the principal order in t, we obtain from these equations

$$u_a^{(1)} = \frac{1}{u_0^{(0)}}\left(\frac{1}{2}u^{b(0)}u^{d(0)}\frac{\partial a_{bd}}{\partial x^a} - u^{i(0)}\frac{\partial u_a^{(0)}}{\partial x^i}\right), \quad (19')$$

$$u_3^{(2)} = \frac{1}{4u_0^{(0)}} b_{ab}' u^{a(0)}u^{b(0)}, \quad (20')$$

$$\varepsilon^{(0)} = -\frac{1}{u_0^{(0)}}\left\{u^{i(0)}\frac{\partial\varepsilon^{(-1)}}{\partial x^i} + \varepsilon^{(-1)}\left[u_0^{(1)} + \frac{1}{2}b_i{}^i u_0^{(0)} - u_3^{(2)\prime} + (a_3{}^a u_a^{(0)})' + a^{ab}\frac{\partial u_a^{(0)}}{\partial x^b} - \frac{1}{2}a^{ad}u^{d(0)}\left(2\frac{\partial a_{ad}}{\partial x^b} - \frac{\partial a_{ab}}{\partial x^d}\right)\right]\right\} \quad (21')$$

The velocity component u_0 will be expressed in terms of the remaining quantities with the aid of the identity $u_\mu u^\mu = 1$. For $u_0^{(0)}$ and $u_0^{(1)}$ we have

$$(u_0^{(0)})^2 + u_a^{(0)}u_b^{(0)}a^{ab} = 1, \qquad u_0^{(1)} = \frac{1}{2u_0^{(0)}}(u^{a(0)}u^{b(0)}b_{ab} - 2u^{a(0)}u_a^{(1)}).$$

From (14) we determine the functions a_{a3} from the specified values of $\epsilon^{(-1)}$, $u_a^{(0)}$, and q. They are determined with two arbitrary functions A_{a3} of two variables

$$a_{a3} = A_{a3}(x^1, x^2) + j_{a3}.$$

The functions j_{a3} depend entirely on $\epsilon^{(-1)}$, $u_a^{(0)}$, and q. We can then consider all the a_{ik} known.

The scheme for obtaining the coefficients b_{ik}, $u_a^{(1)}$, $u_3^{(2)}$, and $\epsilon^{(0)}$ (which then repeats exactly for the following coefficients) is as follows: From (16) we determine b_{ab}:

$$b_{ab} = \Phi_{ab}^{(1)}(x^1, x^2)e^{x^3} + F_{ab}^{(1)}(x^1, x^2)e^{-x^3} + j_{ab}^{(1)}. \quad (16')$$

The first two terms are the general solution of the corresponding homogeneous equation, and $j_{ab}^{(1)}$ is the particular solution of the inhomogeneous equation. $\Phi_{ab}^{(1)}$ and $F_{ab}^{(1)}$ are six arbitrary two-dimensional functions. Using (16′), we determine b_{33} from (18). Substituting (16′) in (20′), we obtain the function $u_3^{(2)}$. (We note that $u_3^{(2)} \equiv u^{a(0)}a_{a3} - u^{3(0)}$.) From (19′) we get $u_a^{(1)}$. We then substitute the functions b_{ab}, b_{33}, $u_a^{(1)}$, and $u_3^{(2)}$ in (21′) and (17) (the functions $u_a^{(1)}$ do not enter in (17)), and thus express $\epsilon^{(0)}$ and b_{a3}. An important fact here is that the entire resultant arbitrariness lies in the functions $\Phi_{ab}^{(1)}$ and $F_{ab}^{(1)}$. The determination of the values of b_{33}, $u_3^{(2)}$, $u_a^{(1)}$, b_{a3}, and $\epsilon^{(0)}$ introduces no new arbitrariness.

To determine the succeeding coefficients in the expansions (1′) and (2) we shall proceed in similar fashion. Equations (5) with indices i = a and k = b serve to determine $g_{ab}^{(n)}$ (n = 1, 2, ...). From (5) with indices i = k = 3 we get $g_{33}^{(n+2)}$. We then determine $u_3^{(n+1)}$ from (20) and $u_a^{(n)}$ from (19). Substituting the corresponding coefficients in (5) with indices i = a and k = 3 and in (21), we get $g_{a3}^{(n+2)}$, and $\epsilon^{(n-1)}$.

The quantities $g_{ab}^{(n)}$, $g_{33}^{(n+2)}$, $u_3^{(n+1)}$, $u_a^{(n)}$, $g_{a3}^{(n+2)}$, and $\epsilon^{(n-1)}$ are unknown functions of the corresponding equations. To verify the applicability of the indicated scheme of obtaining the coefficients, it is necessary to know how the unknown functions are "generated" in each approximation of the equations (5) and (19)–(21). It is easy to verify that the unknown functions in Eq. (5) with indices i i = a and k = b (in (n – 2)nd order in t), the unknown functions appear in the form of the combinations

$$-{}^1/_2(g_{ab}^{(n)\prime\prime} - n^2 g_{ab}^{(n)}). \quad (22)$$

All the remaining functions that enter in this case are known. We include among the unknown also the two-dimensional functions which have appeared in the preceding orders. Consequently, the solution of this equation (in analogy with the solution of (16′) which is a particular case of this solution) is

$$g_{ab}^{(n)} = \Phi_{ab}^{(n)}(x^1, x^2)e^{nx^3} + F_{ab}^{(n)}(x^1, x^2)e^{-nx^3} + j_{ab}^{(n)}. \quad (23)$$

The unknown functions $g_{33}^{(n+2)}$ enter in Eq. (5) with indices i = k = 3 in the form of the term $(1/2)n(n+1)g_{33}^{(n+2)}$. After substituting (23), all the remaining terms are combinations of known functions. Consequently, the $g_{33}^{(n+2)}$ are determined from (5) (i = k = 3). The unknown functions $u_3^{(n+1)}$ are contained in (20) in the form of the term $(n+1)u_0^{(0)}u_3^{(n+1)}$. Substituting here (23), the remaining functions are expressed in terms of known ones, i.e., $u_3^{(n+1)}$ is determined from (20). The functions $u_a^{(n)}$ enter in (19) in the form $u_0^{(0)}u_a^{(n)}$. After substitution of (23) and $u_3^{(n+1)}$, the remaining terms are known, and we can find $u_a^{(n)}$. Finally, $g_{a3}^{(n+2)}$ and $\epsilon^{(n-1)}$ enter in (5) (i = a, k = 3) and in (21) in the form $(1/2)n(n+2)g_{a3}^{(n+2)}$ and $u_0^{(0)}\epsilon^{(n-1)}$, respectively. The remaining terms will be expressed in terms of known functions after the substitution of $g_{ab}^{(n)}$, $g_{33}^{(n+2)}$, $u_a^{(n)}$, and $u_3^{(n+1)}$. Thus, $g_{a3}^{(n+2)}$ and $\epsilon^{(n-1)}$ are also expressed in terms of known functions with the aid of (5) (i = a, k = 3) and (21).

We have thus shown that all the coefficients of the series (1′) and (2) are expressed in final analysis in terms of the functions $\epsilon^{(-1)}$, $u_a^{(0)}$, q, and a certain aggregate of the appearing two-dimensional functions. We used for this purpose Eqs. (5), (19)–(21), and (14). Satisfaction of these equations does not denote, of course, that (3) and (4) will now be satisfied identically. Equations (3) and (4) impose certain constraints on the arbitrary functions. We now proceed to determine these constraints, assuming that Eqs. (5), (19)–(21), and (14) are already satisfied.

Without calculating directly the left-hand and right-hand sides of (3) and (4), let us ascertain the limitations to which they lead if we substitute in them the result of the solution of Eqs. (5), (19)–(21), and (14). To this end, we make use of the Bianchi identities

$$(R^{\mu\nu} - {}^1/_2 g^{\mu\nu}R)_{;\nu} \equiv 0. \quad (24)$$

We introduce the notation $\sqrt{-g}(R_\alpha^0 + T_\alpha^0 - (1/2)\delta_\alpha^0 T) = S_\alpha$. Recognizing that (5) and $T^{\mu\nu}{}_{;\nu} = 0$ are already satisfied, we can reduce (24) to the form

$$\dot{S}_i - \frac{1}{2}S_{0,i} + \frac{1}{4}\frac{g_{,i}}{g}S_0 \equiv 0, \quad (25)$$

$$\frac{1}{2}\dot{S}_0 + \frac{1}{4}\frac{\dot{g}}{g}S_0 + S_{k,i}g^{ik} + S_k\frac{\partial g^{ik}}{\partial x^i} \equiv 0 \quad (26)$$

(the dot denotes differentiation with respect to t). We note that Eq. (13) is satisfied identically by the virtue of (16) and (18). Taking this into consideration, we obtain $S_0 \sim t$. It is easy to verify that the remaining quantities entering into (25) and (26) have the following order in t: $S_a \sim t$, $S_3 \sim t^2$, $g_{,a} \sim t^2$, $g' \sim t^3$, $g \sim t$, $\sqrt{-g} \sim t$.

Let us assume that Eqs. (3) are satisfied up to order t^{n-1} inclusive, i.e., $S_0^{(1)} = 0$, $S_0^{(2)} = 0$, ..., $S = 0$. We then get from (25)

$$S_a^{(1)} \equiv 0, \ldots, S_a^{(n+1)} \equiv 0;\ S_3^{(2)} \equiv 0, \ldots, S_3^{(n+1)} \equiv 0,$$

i.e., Eqs. (4) are satisfied identically up to n-th order inclusive. As to the quantities $S_0^{(n+1)}$, $S_a^{(n+2)}$, and $S_3^{(n+2)}$, we get for them from (25) and (26) the relations

$$(n+2)S_a^{(n+2)} - \frac{1}{2}S_0^{(n+1)}{}_{,a} + \frac{1}{2}\frac{\partial \ln q}{\partial x^a}S_0^{(n+1)} \equiv 0, \qquad (27)$$

$$(n+2)S_3^{(n+2)} - {}^1/_2 S_0^{(n+1)}{}_{,3} \equiv 0, \qquad (28)$$

$${}^1/_2(n+1)S_0^{(n+1)} + {}^1/_2 S_0^{(n+1)} - S_3^{(n+2)}{}_{,3} \equiv 0. \qquad (29)$$

We eliminate $S_3^{(n+2)}$ from (28) and (29). We then obtain

$$(n+2)^2 S_0^{(n+1)} - S_0^{(n+1)}{}_{,33} \equiv 0$$

or

$$S_0^{(n+1)} \equiv P^{(n+2)}(x^1, x^2)\, e^{(n+2)x^3} + Q^{(n+2)}(x^1, x^2)\, e^{-(n+2)x^3}. \qquad (30)$$

The two-dimensional functions $P^{(n+2)}$ and $Q^{(n+2)}$ represent certain algebraic combinations of the functions q, A_{a3}, $\Phi_{ab}^{(k)}$, and $F_{ab}^{(k)}$ up to (n + 2)nd order, inclusive. The concrete form of these expressions can be established after substituting in (3) the result of the solution of Eqs. (5), (19)–(21), and (14). In order for $S_0^{(n+2)}$ to vanish, it is necessary to satisfy the equations

$$P^{(n+2)}(x^1, x^2) = 0, \qquad Q^{(n+2)}(x^1, x^2) = 0.$$

It then follows from (27)–(29) that $S_a^{(n+2)} \equiv 0$, $S_3^{(n+2)} \equiv 0$, and everything is repeated with allowance for the fact that $S_0^{(n+1)} = 0$.

Let us start the analysis presented above with n = 0; then

$$S_0^{(1)} \equiv P^{(2)}(x^1, x^2)\, e^{2x^3} + Q^{(2)}(x^1, x^2)\, e^{-2x^3}.$$

The quantities $P^{(2)}$ and $Q^{(2)}$ are made up of the functions q, A_{a3}, $\Phi_{ab}^{(1)}$, $F_{ab}^{(1)}$, $\Phi_{ab}^{(2)}$, and $F_{ab}^{(2)}$. In order for $P^{(2)}$ and $Q^{(2)}$ to vanish it is necessary to impose on these functions two constraints. Then, for example, not all the six functions $\Phi_{ab}^{(2)}$ and $F_{ab}^{(2)}$ to remain arbitrary, but only four of them. The same situation is repeated for each group of six functions $\Phi_{ab}^{(3)}$ and $F_{ab}^{(3)}$, $\Phi_{ab}^{(4)}$ and $F_{ab}^{(4)}$, etc. Only $\Phi_{ab}^{(1)}$ and $F_{ab}^{(1)}$ are not subject to additional limitations. These functions (all six) remain arbitrary. Thus, Eqs. (3) impose conditions on the functions $\Phi_{ab}^{(n)}$ and $F_{ab}^{(n)}$, two conditions in each order, starting with n = 2. After satisfaction of these conditions, Eqs. (4) will be satisfied identically.

DETERMINATION OF THE NUMBER OF ARBITRARY FUNCTIONS

We shall now calculate the number of arbitrary functions contained in the solution. Let us consider the components of the metric g_{ab}. As shown, g_{ab} can be represented in the form

$$g_{11} = q + \sum_{n=1}^{\infty} (\Phi_{11}^{(n)} e^{nx^3} + F_{11}^{(n)} e^{-nx^3}) t^n + \sum_{n=1}^{\infty} j_{11}^{(n)} t^n,$$

$$g_{12} = \sum_{n=1}^{\infty} (\Phi_{12}^{(n)} e^{nx^3} + F_{12}^{(n)} e^{-nx^3}) t^n + \sum_{n=1}^{\infty} j_{12}^{(n)} t^n, \qquad (31)$$

$$g_{22} = q + t(\Phi_{22}^{(1)} e^{x^3} + F_{22}^{(1)} e^{-x^3}) + \sum_{n=2}^{\infty} (\Phi_{22}^{(n)} e^{nx^3} + F_{22}^{(n)} e^{-nx^3}) t^n + \sum_{n=1}^{\infty} j_{22}^{(n)} t^n.$$

Using (3), all the functions $\Phi_{22}^{(n)}$ and $F_{22}^{(n)}$ (n = 2, 3, ...) can be expressed in terms of q, A_{a3}, $\Phi_{22}^{(1)}$, $F_{22}^{(1)}$, $\Phi_{11}^{(m)}$, $\Phi_{12}^{(m)}$, $F_{11}^{(m)}$, and $F_{12}^{(m)}$ (m = 1, 2, ..., n), and consequently

$$g_{22} = q + t(\Phi_{22}^{(1)} e^{x^3} + F_{22}^{(1)} e^{-x^3}) + tG_{22}, \qquad (32)$$

where the function G_{22} which depends on all four coordinates is determined completely from

$$\varepsilon^{(-1)},\ u_a^{(0)},\ q,\ A_{a3},\ \Phi_{22}^{(1)},\ F_{22}^{(1)},\ \Phi_{11}^{(n)},$$
$$\Phi_{12}^{(n)},\ F_{11}^{(n)},\ F_{12}^{(n)} \qquad (n = 1, 2, \ldots). \qquad (33)$$

As regards the components g_{11} and g_{12}, they contain four series of two-dimensional arbitrary functions, which is equivalent to specifying four three-dimensional functions. In order to separate the three-dimensional arbitrariness in g_{11} and g_{12} in explicit form, we can proceed, for example, as follows. We represent $\exp(nx^3)$ and $\exp(-nx^3)$ in the form of series in x^3; then g_{11} and g_{12} can be written in the form

$$g_{11} = q + \sum_{n=1}^{\infty} t^n \overline{\Phi_{11}^{(n)}} + x^3 \sum_{n=1}^{\infty} t^n \overline{F_{11}^{(n)}} + tG_{11},$$

$$g_{12} = \sum_{n=1}^{\infty} t^n \overline{\Phi_{12}^{(n)}} + x^3 \sum_{n=1}^{\infty} t^n \overline{F_{12}^{(n)}} + tG_{12}. \qquad (34)$$

We have introduced here the notation $\overline{\Phi_{1a}^{(n)}} = \Phi_{1a}^{(n)} + F_{1a}^{(n)}$ and $\overline{F_{1a}^{(n)}} = n(\Phi_{1a}^{(n)} - F_{1a}^{(n)})$. The functions G_{11} and G_{12} are completely determined from (33). The two-dimensional functions $\overline{\Phi_{1a}^{(n)}}$ and $\overline{F_{1a}^{(n)}}$ are perfectly arbitrary, and consequently

$$g_{11} = \beta_{11} + x^3\gamma_{11} + tG_{11}, \qquad g_{12} = \beta_{12} + x^3\gamma_{12} + tG_{12},$$

where β_{11}, β_{12}, γ_{11}, and γ_{12} are arbitrary functions of the three coordinates x^1, x^2, and t. Strictly speaking, these functions are not perfectly arbitrary, since it follows from (34) that they should satisfy the conditions

$$\beta_{11}(x^1, x^2, 0) = q, \quad \gamma_{11}(x^1, x^2, 0) = 0,$$
$$\beta_{12}(x^1, x^2, 0) = 0, \quad \gamma_{12}(x^1, x^2, 0) = 0.$$

Out of the remaining components of the metric g_{a3} and g_{33}, two-dimensional arbitrariness in two functions A_{a3} is contained in the component g_{a3}. However, one of these functions, say A_{13}, can be set equal to zero by the remaining admissible transformation (12).

Separating the arbitrary functions, the solution of the gravitation equation is now written in the final form:

$$g_{11} = \beta_{11} + x^3\gamma_{11} + \psi_{11}, \qquad g_{12} = \beta_{12} + x^3\gamma_{12} + \psi_{12},$$
$$g_{22} = q + t(\Phi_{22}^{(1)} e^{x^3} + F_{22}^{(1)} e^{-x^3}) + \psi_{22},$$
$$g_{13} = \psi_{13}, \qquad g_{23} = t^2 A_{23} + \psi_{23},$$
$$\varepsilon = \varepsilon^{(-1)} t^{-1} + E, \quad u^a = u^{a(0)} + U^a, \quad u^3 = U^3.$$

Here β_{11}, γ_{11}, β_{12}, γ_{12}, $\epsilon^{(-1)}$, $u^{1(0)}$, and $u^{2(0)}$ are seven arbitrary functions of three variables; q, $\Phi_{22}^{(1)}$, $F_{22}^{(1)}$, and A_{23} are four arbitrary functions of two variables. The quantities ψ_{ab}, ψ_{a3}, E, and U^i are determined completely from the specified arbitrary functions. Since there are no more coordinate transformations left, the entire indicated arbitrariness is physical. Thus, the solution with simultaneous physical singularity depends on seven arbitrary functions of three variables, one less than the maximum possible number. It follows from this fact, in particular, that the obtained solution is unstable. There exists a type of perturbation which upsets the regime described by this solution. This does not mean, however, that the physical singularity will disappear as a result of the perturbations. The perturbations lead only to a defocusing of the world lines of the particles of a dust-like medium and to a change in the character of the singularity from simultaneous to non-simultaneous.

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