STABILITY OF A HOMOGENEOUS PLASMA WITH ISOTROPIC DISTRIBUTION FUNCTIONS IN A MAGNETIC FIELD

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It is shown that potential oscillations can build up in a homogeneous plasma with isotropic particle-velocity distribution functions located in a stationary magnetic field if fast ions with $f \sim \delta(v - v_0)$ are also present besides the main plasma. Criteria for the development of such a plasma are obtained. The theory is applied to a thermonuclear deuterium-tritium plasma in which fast He^4 ions with an isotropic distribution function are produced in the fusion reaction. If the thermonuclear plasma temperature is T = 35–11 keV, then instability develops at ion cyclotron harmonics with $n \gtrsim 5$–6 and $n \lesssim 7$–14.

In the course of operation of the future self-maintaining magnetic thermonuclear reactor (MTR), the plasma will contain fast ions, which are the products of the fusion reactions, and which are grouped, with relatively small scatter, about the velocity v_0 acquired during the time of the reaction. Therefore in closed systems the ion distribution function will be an isotropic but not monotonic function of the velocity v. If such a plasma turns out to be unstable, then such an instability will be also intrinsically present in the MTR, just like the "cone" instability[1] due to the very nature of the method of containing the plasma in magnetic-mirror traps. This problem apparently begins to attract the attention of the theoreticians. Oraevskii and Kolesnichenko consider the problem of the stability of an inhomogeneous thermonuclear plasma with frequencies far from the ion cyclotron frequency[2]. In addition, although it has been shown long ago that a homogeneous nonrelativistic plasma with isotropic distribution functions is stable, in the absence of a magnetic field, against buildup of potential oscillations[3], the question of the stability of such a plasma in a magnetic field still remains open.

It will be shown in the article that under definite conditions such a plasma is unstable against potential oscillations propagating almost strictly across the magnetic field. In the first section of the article we derive the dispersion relation, and in the second we consider the idealized case of a plasma with one type of ions, while in the third we present some estimates of the stability of a thermonuclear plasma.

1. DERIVATION OF DISPERSION RELATION

We consider a plasma in which the velocity distribution functions f_α of the particles of type α are isotropic in velocity space, i.e., $f_\alpha \equiv f_\alpha(v)$. A homogeneous magnetic field H directed along the z axis is present in the plasma. For potential oscillations, $E = -\nabla\varphi$, the following dispersion relation is valid:

$$k^2 = 8\pi^2 \sum_\alpha \frac{e_\alpha^2}{m_\alpha} \int v_\perp dv_\perp dv_\parallel \sum_n \frac{J_n^2(k_\perp v_\perp/\Omega_\alpha)}{k_\parallel v_\parallel + n\Omega_\alpha - \omega} \times \left[k_\parallel \frac{\partial f_\alpha}{\partial v_\parallel} + \frac{n\Omega_\alpha}{v_\perp}\frac{\partial f_\alpha}{\partial v_\perp}\right]. \quad (1)$$

The notation in (1) is standard. We change over to spherical coordinates in velocity space:

$$v_\parallel = v\cos\theta, \quad v_\perp = v\sin\theta, \quad \frac{\partial f_\alpha}{\partial v_\parallel} = \frac{v_\parallel}{v}\frac{\partial f_\alpha}{\partial v};$$

$$\frac{\partial f_\alpha}{\partial v_\perp} = \frac{v_\perp}{v}\frac{\partial f_\alpha}{\partial v}; \quad v_\perp dv_\perp dv_\parallel = v^2 dv \sin\theta\, d\theta.$$

Substituting these expressions in (1) and taking into account the identity $\sum J_n^2 = 1$, we get

$$k^2 = 8\pi^2 \sum_\alpha \frac{e_\alpha^2}{m_\alpha} F_\alpha, \quad (2)$$

where

$$F_\alpha = \int_0^\infty v\,dv \int_0^\pi \sin\theta\, d\theta \frac{\partial f_\alpha}{\partial v} \sum J_n^2\left(\frac{k_\perp v \sin\theta}{\Omega_\alpha}\right) \frac{\omega}{k_\parallel v\cos\theta + n\Omega_\alpha - \omega}.$$

We drop the index α and take the substitution

$$\frac{1}{k_\parallel v\cos\theta + n\Omega - \omega} \to i\int_{-\infty}^0 d\tau \exp\{i\tau(k_\parallel v\cos\theta + n\Omega - \omega)\},$$

and to satisfy the rule for going around the poles in the integration with respect to v and θ, we assume, as usual, that $\omega = \omega + i\epsilon$. Then

$$F = i\omega \int_0^\infty v\,dv \int_0^\pi \sin\theta\, d\theta \frac{\partial f}{dv} \int_{-\infty}^0 d\tau \exp\{i\tau(-\omega + k_\parallel v\cos\theta)\} \times \sum_n J_n^2\left(\frac{k_\perp v\sin\theta}{\Omega}\right)\exp(i\tau n\Omega). \quad (3)$$

We use the theorem for the addition of cylindrical functions

$$J_0(z\sin\alpha) = \sum_n J_n^2(z/2)e^{2in\alpha}.$$

We obtain

$$F = 2i\omega \int_0^\infty v\,dv \frac{\partial f}{\partial v} \int_{-\infty}^0 d\tau e^{-i\omega\tau} \times \int_0^{\pi/2} \sin\theta\, d\theta \cos(\tau k_\parallel v\cos\theta) J_0\left(\frac{2k_\perp v\sin\theta}{\Omega}\sin\frac{\Omega\tau}{2}\right). \quad (4)$$

Recognizing that[4]

$$\int_0^{\pi/2} \sin\theta\, d\theta \cos(\beta\cos\theta) J_0(\alpha\sin\theta) = \sqrt{\frac{\pi}{2}}(\alpha^2 + \beta^2)^{-1/4} J_{1/2}[(\alpha^2+\beta^2)^{1/2}]$$

and that $J_{1/2}(z) = \sqrt{2/\pi z}\sin z$, we carry out the integration with respect to θ. As a result we get

$$F = 2i\omega \int_0^\infty v\,dv \int_{-\infty}^0 d\tau e^{-i\omega\tau} \frac{\partial f}{\partial v} \frac{\sin Av}{Av}, \quad (5)$$

where

$$A=\left[(\tau k_{\|})^2+\left(\frac{2k_\perp}{\Omega}\sin\frac{\Omega\tau}{2}\right)^2\right]^{1/2}.$$

If we now integrate by parts with respect to v and take into account the fact that $vf(v) = 0$ when $v = 0$ and $v = \infty$, then we obtain the final result

$$F=-2i\omega\int_0^\infty dv\int_{-\infty}^0 d\tau e^{-i\omega\tau}f(v)\cos Av. \quad (6)$$

Since the integral with respect to $d\tau$ cannot be expressed in terms of known functions, let us consider a few particular cases. If there is no magnetic field, then $\Omega \to 0$ and $A \to k\tau$, and the dispersion relation takes the form

$$1=\sum_\alpha\frac{16\pi^2e_\alpha^2}{m_\alpha}\int_0^\infty\frac{f_\alpha v^2\,dv}{\omega^2-k^2v^2}. \quad (7)$$

Expression (7) can be derived in standard fashion. It is easy to deduce with its aid the stability of the plasma without the magnetic field and with isotropic distribution functions. Indeed, let us assume that the plasma is unstable, i.e., that $\omega = \omega_r = i\gamma, \gamma > 0$. Then the integration with respect to v is carried out in (7) along the real semiaxis of v. Separating the imaginary part in (7), we get

$$\sum_\alpha\frac{16\pi^2e_\alpha^2}{m_\alpha}\int_0^\infty v^2\,dvf_\alpha\frac{\omega_r\gamma}{(\omega_r^2-\gamma^2-k^2v^2)^2+4\omega_r^2\gamma^2}=0.$$

Since $f_\alpha(v) \geq 0$, this equality is not satisfied, and the aforementioned assumption $\gamma > 0$ is incorrect.

When $k_{\|} \gg k_\perp$, we have $A \approx |k_{\|}|\tau$ and the dispersion relation assumes the form (7) with k replaced by $k_{\|}$. This means that a plasma with isotropic distribution functions in a magnetic field is stable against buildup of potential oscillations with $k_{\|} \gg k_\perp$.

If $\omega = \omega_r + i\gamma$, with $a \equiv |k_{\|}|\Omega_\alpha/2k_\perp\gamma \ll 1$, then the integral with respect to τ in (6) can be represented as the sum of two integrals:

$$\int_{-\infty}^0 d\tau\ldots=\int_{-\infty}^{-1/a\gamma}d\tau\ldots+\int_{-1a/\gamma}^0 d\tau\ldots$$

The first integral is proportional to $-1/\alpha$ and is exponentially small. In the second we can put, with the same degree of accuracy,

$$A=\left|\frac{2k_\perp}{\Omega_\alpha}\sin\frac{\Omega_\alpha\tau}{2}\right|,$$

and replace the lower limit by $-\infty$. Using the identity

$$\cos(z|\sin\varphi|)\equiv\cos(z\sin\varphi)=\sum_{n=-\infty}^{n=\infty}J_{2n}(z)e^{i2n\varphi},$$

we obtain

$$F=-2\int_0^\infty f_\alpha(v)\,dv\omega\sum_{n=-\infty}^{\infty}\frac{J_{2n}(2k_\perp v/\Omega_\alpha)}{n\Omega_\alpha-\omega}. \quad (8)$$

Thus, under the condition $|k_{\|}|\Omega_\alpha/2k_\perp\gamma \ll 1$, the dispersion equation takes the form

$$k^2=\sum_\alpha\frac{16\pi^2e_\alpha^2}{m_\alpha}\int_0^\infty f_\alpha\,dv\left(-1+\omega\sum_n\frac{J_{2n}(z_\alpha)}{\omega-n\Omega_\alpha}\right),$$

where

$$z_\alpha=2k_\perp v/\Omega_\alpha. \quad (9)$$

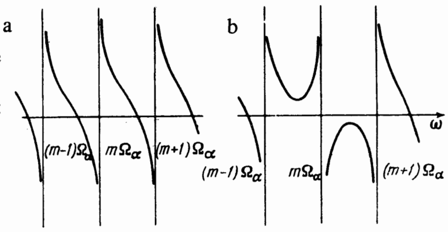

FIG. 1. Dependence of the right-side of Eq. (9) on the frequency ω: a—case of normal dispersion; b—case of anomalous dispersion with $\omega = m\omega_{H\alpha}$.

In particular, for a cold plasma ($f_\alpha = (n_\alpha/4\pi v_0^2)\delta(v - v_0)$, $v_0 \to 0$), the dispersion relation, as usual, will be

$$k^2=\sum_\alpha\frac{\omega_{0\alpha}^2k_\perp^2}{\omega^2-\Omega_\alpha^2},\quad \omega_{0\alpha}^2=\frac{4\pi e_\alpha^2n_\alpha}{m_\alpha}. \quad (10)$$

If

$$\int_0^\infty f_\alpha\,dvJ_{2n}(z_\alpha)\geqslant 0$$

for any n, then the dependence of the right-hand side of the dispersion equation on ω takes the form shown in Fig. 1a (for one of the terms of the sum over the particle types). In this case the plasma is stable and has oscillation modes with frequencies in the intervals $n\Omega_\alpha < \omega < (n + 1)\Omega_\alpha$, $|n| = 1, 2, 3 \ldots$ However, if this integral is smaller than zero for n = m, then the dependence of the right-side of (9) on ω has the form shown in Fig. 1b, and the plasma may turn out to be unstable.

Thus, the sufficient but not necessary condition for the stability of the plasma against the oscillation modes under consideration is

$$\int_0^\infty f_\alpha\,dvJ_{2n}(z_\alpha)\geqslant 0 \text{ for all } n\neq 0. \quad (11)$$

In particular, it is satisfied for an arbitrary distribution function $f_\alpha(v)$ which decreases monotonically with increasing v.

2. PLASMA WITH ONE TYPE OF IONS

Let us consider the case when a plasma with cold ions of mass M and density n_{0i} has a group of similar fast ions with density $n_{1i} \ll n_{0i}$ and a distribution function $f_{1i} = (n_{1i}/4\pi v_0^2)\delta(v - v_0)$. Taking into account the fact that $k^2T_e/m\omega_{He}^2 \approx mT_e/ME_0 \ll 1$ (E_0 is the energy of the fast ions), the dispersion relation is written in the form

$$1=\frac{k_{\|}^2\omega_{0e}^2}{k^2\omega^2}-\frac{\omega_{0e}^2}{\omega_{He}^2}+\frac{\omega_{0i}^2}{\omega^2-\omega_{Hi}^2}+\frac{\omega_{1i}^2}{k^2v_0^2}\left(-1+\omega\sum_{n=-\infty}^{\infty}\frac{J_{2n}(z_0)}{\omega-n\omega_{Hi}}\right), \quad (12)$$

where $z_0 = 2k_\perp v_0/\omega_{Hi}$; ω_{0e}, ω_{0i}, and ω_{1i} are the corresponding plasma frequencies and ω_{He} and ω_{Hi} are the cyclotron frequencies.

Instability can set in when $J_{2n} < 0$ and $n \geq 2$ with frequencies $\omega \approx n\omega_{Hi}$ (see Fig. 2). Putting $\omega = \omega_{Hi}(n + x)$ and recognizing that $x \ll 1$ and $n_{1i} \ll n_{0i}$, we rewrite (12) in the form

$$1-\frac{k_{\|}^2\omega_{0e}^2}{k^2n^2\omega_{Hi}^2}-\frac{\omega_{0i}^2}{(n^2-1)\omega_{Hi}^2}=-x\left(\frac{2k_{\|}^2\omega_{0e}^2}{k^2n^3\omega_{Hi}^2}+\frac{2n}{(n^2-1)^2}\frac{\omega_{0i}^2}{\omega_{Hi}^2}\right)+\frac{\omega_{1i}^2nJ_{2n}(z_0)}{k^2v_0^2x} \quad (13)$$

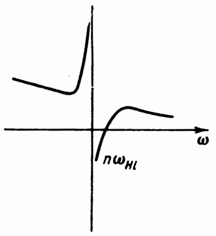

FIG. 2. Dependence of the right-side of (12) on the frequency ω at $\omega \approx n\omega_{Hi}$.

For instability to set in, it is necessary to satisfy the condition

$$\left|1-\frac{k_\parallel^2\omega_{0e}^2}{k^2n^2\omega_{Hi}^2}-\frac{\omega_{0i}^2}{(n^2-1)\,\omega_{Hi}^2}\right| \leqslant 2^{3/2}\left[\frac{\omega_{1i}^2 n\mu}{\omega_{Hi}^2 a_n^2}\left(\frac{k_\parallel^2\omega_{0e}^2}{k^2n^3\omega_{Hi}^2}+\frac{n}{(n^2-1)^2}\frac{\omega_{0i}^2}{\omega_{Hi}^2}\right)\right]^{1/2}, \tag{14}$$

where μ is the maximum value of $-J_{2n}(z_0)$ and a_n is the value of $k_\perp v_0/\omega_{Hi}$ at which this maximum is reached. The condition (14) determines the range of densities of the cold plasma, in which the n-th cyclotron harmonic is unstable. Neglecting the corrections $\sim 1/n^2$, we obtain the magnitude of this range

$$\max,\ \min\left(\frac{\omega_{0i}^2}{\omega_{Hi}}\right)=\frac{n^2-1}{(1+k_\parallel^2M/k^2m)}\left[1\pm\sqrt{2}\frac{n}{a_n}\left(\frac{\mu n_{1i}}{n_{0i}(1+k_\parallel^2M/k^2m)}\right)^{1/2}\right]. \tag{15}$$

Consequently, for the n-th cyclotron ion harmonic the critical value is $\omega_{0i}^2/\omega_{Hi}^2 \leq (n^2-1)$ and decreases with increasing $|k_\parallel|/k$.

To determine the lower limit of the critical value of $(\omega_{0i}^2/\omega_{Hi}^2)_{min}$, we use the condition for the applicability of the dispersion equation $(k_\parallel/k_\perp)^2 < 4\gamma^2/\omega_{Hi}^2$. The maximum value of γ is

$$\gamma_{max} \approx \frac{n}{a_n}\,\omega_{0i}\sqrt{\frac{n_{1i}}{n_{0i}}\frac{\mu}{2}}. \tag{16}$$

When $n_{1i}/n_{0i} < a_n^2 m/8n^4\mu M$ it is necessary to have $(k_\parallel/k_\perp)^2 < M/m$ and $(\omega_{0i}^2/\omega_{Hi}^2)_{min} > (n^2-1)/2$. When $n_{1i}/n_{0i} \gg a_n^2 m/8\mu n^4 M$ the critical value is $(\omega_{0i}^2/\omega_{Hi}^2)_{min} \approx a_n\sqrt{n_{0i}m/n_{1i}M}$. Thus, at $n_{1i}/n_{0i} = 10^{-2}$–10^{-3} the cold-plasma density ranges in which the n-th cyclotron ion harmonics can be excited overlap. We note that the value of the increment is very large. Thus, for $n_{0i} \approx 10^{14}$ cm^{-3}, $n_{1i}/n_{0i} = 10^{-3}$, and n = 10 we have $\gamma = 6 \times 10^7$ sec^{-1}.

The maximum value of the cold-plasma density at which the instability can set in determined by the largest number of the unstable harmonic n and by satisfaction of the potential condition $v_A^2 \gg (\omega/k)^2 = n^2v_0^2/a_n^2$ or else $\omega_{0i}^2/\omega_{Hi}^2 \ll c^2/v_0^2$. For He^4 ions which are the products of the d-t reaction we have $\omega_{0i}^2/\omega_{Hi}^2 \ll 500$. It will be shown below that the thermal scatter of the fast ions leads to a suppression of the instability at cyclotron harmonics with $n^2 \gtrsim 2E_0/T$, where E_0 is the energy of the fast ion and T is the temperature of the cold ions. At T = 17 keV we have $2E_0/T \approx 400$.

From condition (14) it follows that at a fixed value of $\omega_{0i}^2/\omega_{Hi}^2$ the instability develops in a narrow range $|k_\parallel|/k_\perp$ and can therefore be stabilized by using shear of the magnetic force lines. Simple estimates show that to suppress the instability the required shear is

$$\theta > n\frac{a}{\rho_0}\sqrt{\frac{m}{M}}\left(\frac{n_{1i}}{n_{0i}}\right)^{1/4},\quad k_\parallel \ll \sqrt{\frac{m}{M}};$$

$$\theta > n\frac{a}{\rho_0}\sqrt{\frac{m}{M}}\left(\frac{n_{1i}}{n_{0i}}\right)^{1/2},\quad k_\parallel > \sqrt{\frac{m}{M}}, \tag{17}$$

where a is the transverse dimension of the system and ρ_0 the Larmor radius of the fast ion.

The foregoing results describe the case of the strongest instability and therefore the magnitude of the transverse component of the wave vector was fixed. It was determined by the requirement $k_\perp v_0/\omega_{Hi} = a_n$, where a_n is half the argument of the function J_{2n} at which J_{2n} has the first minimum. If we consider the dependence of the character of the instability on $k_\perp$, then a curious detail appears: the instability at the n-th cyclotron harmonic vanishes at $J_{2n}(z_0) \geq 0$, and can exist only when $J_{2n}(z_0) < 0$. Thus, the regions of the phase velocities where the plasma is stable alternate with regions where instability can set in. It is obvious that when $z_0 < \alpha_{2n}$, where α_{2n} is the first zero of the function J_{2n}, $J_{2n}(\alpha_{2n}) = 0$, the instability cannot develop at all.

So far we have disregarded the finite temperature of the "cold" ionic component and the velocity scatter of the fast ions. If the ion temperature is T_i, then the dispersion relation takes the following form (we neglect the electron contribution):

$$1=\frac{\omega_{0i}^2}{k^2}\frac{M}{T_i}\left(-1+\omega\sum_n\frac{\exp\{-k_\perp^2T_i/M\omega_{Hi}^2\}I_n(k_\perp^2T_i/M\omega_{Hi}^2)}{\omega-n\omega_{Hi}}\right)+\frac{\omega_{1i}^2}{k^2v_0^2}\left(-1+\omega\sum_n\frac{J_{2n}(2k_\perp v_0/\omega_{Hi})}{\omega-n\omega_{Hi}}\right). \tag{18}$$

The plasma will be stable at the n-th cyclotron harmonic if the coefficient of $1/(\omega - n\omega_{Hi})$ is positive. It follows therefore that the criterion for the occurrence of the instability is

$$n_{0i}\frac{M}{T_i}\exp\left\{-\frac{k_\perp^2T_i}{M\omega_{Hi}^2}\right\}\ I_n\left(\frac{k_\perp^2T_i}{M\omega_{Hi}^2}\right)<-\frac{n_{1i}}{v_0^2}J_{2n}\left(\frac{2k_\perp v_0}{\omega_{Hi}}\right) \tag{19}$$

Assuming that $T_i/E_0 \ll 1$, we obtain for $k_\perp = \omega_{Hi}a_n/v_0$ the following form of the instability criterion:

$$\frac{n_{1i}}{n_{0i}}>\frac{a_n^2}{2\mu n!}\left(\frac{a_n^2T_i}{4E_0}\right)^{n-1}. \tag{20}$$

It is obvious that the stability of the plasma depends strongly on the temperature of the "cold" ions and, for example, when T_i/E_0 increases to 10^{-1}, the instability at any cyclotron harmonic can develop only if $n_{1i} \gtrsim n_{0i}$, which is of no interest in the case of a thermonuclear plasma.

The velocity scatter of the fast particles about the value v_0 causes the quantity

$$\int_0^\infty f_{1i}\,dv\,J_{2n}\left(\frac{2k_\perp v}{\omega_{Hi}}\right)$$

to decrease when the velocity scatter Δv begins to exceed the distance between the zeroes of the function $J_{2n}(2k_\perp v/\omega_{Hi})$, i.e., when $k_\perp\Delta v/\omega_{Hi} \gtrsim 1$. This results in one more instability criterion:

$$k_\perp \lesssim \frac{\omega_{Hi}}{v_0}\frac{v_0}{\Delta v}. \tag{21}$$

This denotes in fact that we have used above the value $a_n \lesssim v_0/\Delta v$. The latter inequality is the condition that determines the highest unstable cyclotron harmonic.

Since in a thermonuclear plasma $v_0/\Delta v \approx (E_0/T_i)^{1/2}$, then we get $a_n \lesssim 14$ for $E_0/T_i = 200$, and the harmonics up to $n < 12$ will be unstable. However, a more accurate analysis shows that the condition (21) is much less stringent and therefore the main stabilizing factor will be the finite temperature T_i of the "cold" ions (see (20)).

3. INSTABILITY OF A THERMONUCLEAR PLASMA

Let us consider a plasma consisting of an equal-component mixture of deuterium and tritium with a temperature on the order of several times 10 kV. The fusion reaction

$$d + t \rightarrow He^4 + n$$

takes place in the plasma, and the energy of the He^4 ions is approximately 3.5 MeV. The number of nuclear reactions in 1 cm^3 per second is $(1/4)n_{0i}^2\overline{\sigma v}$, where n_{0i} is the total number of ions in 1 cm^3 and $\overline{\sigma v}$ for a Maxwellian distribution of the ions in the plasma is 1.5×10^{-16}, 4×10^{-16}, and 10^{-15} cm^3/sec for T_i equal to 11, 17, and 35 keV respectively. Since the cyclotron frequencies of the ions d and He^4 are equal, the main criterion for the development of instability will be a somewhat modified condition (19), namely

$$\frac{n_{1i}}{n_{0i}} > \frac{E_0}{\mu T_i}\exp\left\{-\frac{a_n^2 T_i}{E_0}\right\} I_n\left(\frac{a_n^2 T_i}{E_0}\right), \quad E_0 = \frac{M_{He}v_0^2}{2}. \tag{22}$$

The table lists the values for temperatures T_i equal to 11, 17, and 35 keV, above which the instability can set in.

We see from the table that the instability develops easiest at high cyclotron harmonics and it can be suppressed by increasing the temperature of the thermonuclear plasma.

As already noted above, the largest number of the unstable cyclotron harmonic is determined by the velocity scatter of the fast He^4 ions, namely, by the condition $k\Delta v/\omega_{Hi} < 1$. Rough estimates show that $\Delta v \approx \sqrt{2T_i/M}$ and that allowance for the velocity scatter leads to the appearance of a factor $\approx \exp\{-(k\Delta v/\omega_{Hi})^2\}$ in that term that takes into account the contribution of the fast ions in the dispersion equation.

One of the stabilizing factors will be the broadening of the distribution function of the fast particles in velocity space on account of the Coulomb collisions. The principle role in this mechanism will be played by the dynamic friction of the ions against the electrons. The behavior of the distribution function f_{1i} of the fast He ions is described by the following equation:

$$\frac{\partial f_{1i}}{\partial t} - \frac{64e^4 n_{0i} L\sqrt{\pi}}{3mM_{He}}\left(\frac{m}{2T_e}\right)^{3/2}\frac{\partial}{\partial v} v f_{1i} = \frac{\partial f_{1i}}{\partial t} - 10^{-12}\frac{n_{0i}}{T_e^{3/2}(\text{keV})}\frac{\partial}{\partial v} v f_{1i} = \varphi(v), \tag{23}$$

where $\varphi(v)$ is proportional to the distribution of the He^4 ions after the fusion reaction. We can assume with sufficient degree of accuracy that $\varphi(v) = b\exp\{-(v_0 - v)^2/(\Delta v)^2\}$, where $\Delta v \approx \sqrt{2T_i/M}$.

The solution of (23) is

$$f_{1i} = \frac{1}{v}\int_v^{ve^\tau}\frac{\varphi(z)}{a}dz, \quad a = \frac{10^{-12}n_{0i}}{T_e^{3/2}(\text{keV})}, \quad \tau = at.$$

For the chosen form of $\varphi(v)$ we get

$$f_{1i} = \frac{\Delta v b\sqrt{\pi}}{av\ 2}\left[\Phi\left(\frac{ve^\tau - v_0}{\Delta v}\right) - \Phi\left(\frac{v - v_0}{\Delta v}\right)\right], \tag{24}$$

Values of n_{1i}/n_{0i} at which instability sets in for T_i equal to 11, 17, and 35 keV and for n = 2 – 7.

T_i	n					
	2	3	4	5	6	7
35	1.52	0.18	$3.8\cdot10^{-2}$	$1.5\cdot10^{-2}$	$3.9\cdot10^{-3}$	$5.5\cdot10^{-4}$
17	0.85	0.05	$5.8\cdot10^{-3}$	$1.3\cdot10^{-3}$	$1.8\cdot10^{-4}$	$1.4\cdot10^{-5}$
11	0.5	$2.2\cdot10^{-2}$	$1.6\cdot10^{-3}$	$2.4\cdot10^{-4}$	$2.1\cdot10^{-5}$	$1.1\cdot10^{-6}$

where Φ is the probability integral. It is seen from this expression that as a result of the Coulomb collisions the scatter of the fast ions with respect to velocity increases with time, and that at the instant t the half-width of the scatter is approximately equal to

$$\Delta v(t) = \Delta v + {}^1/_2 v_0(1 - e^{-t/a}).$$

If the value of $2\Delta v(t)$ does not exceed $v_2 - v_1$, where v_1 and v_2 are determined by the equations $J_{2n}(2k_\perp v_{1,2}/\omega_{Hi}) = \alpha_1, \alpha_2$, while α_1 and α_2 are the first and second zeroes of the function, then the corrections that must be introduced into the dispersion equation as a result of the velocity scatter of the fast ions can be neglected.

From this we can roughly estimate the time t during which the instability can develop. It is equal to $t_0 = 0.3 - 0.15\ a^{-1}$ for n = 2–7. Within that time there will accumulate in 1 cm^3, as the result of the fusion reaction,

$$n_{1i} = {}^1/_4 n_{0i}^2\overline{\sigma v}t_0 = 0{,}7 - 0.3\cdot 10^{11} n_{0i}\overline{\sigma v}T_e^{3/2}(\text{keV})$$

fast He^4 ions. If $T_i = T_e = T$, then

$$\frac{n_{1i}}{n_{0i}} = (1.5 - 0.7)\cdot 10^{-2} \text{ for } T = 35 \text{ keV},$$

$$\frac{n_{1i}}{n_{0i}} = (2 - 1)\cdot 10^{-3} \text{ for } T = 17 \text{ keV},$$

$$\frac{n_{1i}}{n_{0i}} = (3{,}7 - 1{,}8)\cdot 10^{-4} \text{ for } T = 11 \text{ keV}.$$

Comparison with the table shows that in the temperature range T = 11–35 keV instability can develop at the n-th cyclotron harmonics with n > 5–6.

Finally, for the development of the nonlinear mode, it is necessary to satisfy the condition $\gamma t_0 \gg 1$, i.e.,

$$(1 - 0.3)\cdot 10^{19}\frac{(n^2 - 1)}{a_n}\sqrt{\frac{\mu}{2}}\frac{H}{n_{0i}}\left[T_e^{9/4}\overline{\sigma v}\right]^{1/2} \gg 1,$$

and when T_e = 17 keV and n = 6 we should have $H \gg 2.5 \times 10^{-13} n_{0i}$, where n_{0i} is per cm^3 and H is in Oersteds.

We note that the condition for the development of instability at the n-th harmonic, $\omega_{0i}^2/\omega_{Hi}^2 < (n^2 - 1)$, determines the value of the cold-ion density, above which no instability can develop. For $n^2 = 200$ and $H = 10^5$ Oe, the plasma is unstable if $n_{0i} < 6 \times 10^{18}$.

When the instability goes over into the nonlinear mode there can occur either a sharp increase of the escape of the particles from the plasma, or an increase of the rate of exchange of energy between the fast ions and the thermonuclear plasma. Rough estimates show that the plasma diffusion coefficient will be $\approx 10^{-2} - 10^{-3}$ D_B, where $D_B = T/M\omega_{Hi}$ is the Boehm diffusion coefficient.

CONCLUSIONS

We have demonstrated, first of all, that instability can develop on potential oscillations in a homogeneous plasma with isotropic particle distribution functions in velocity space, provided a constant magnetic field is present in the plasma. This fact is of interest in itself, even if for no other reason than that such a plasma is stable without a magnetic field.

Such an instability may be observed experimentally in the future self-maintaining closed thermonuclear systems. Although when the temperature of the thermonuclear plasma is increased this instability becomes rapidly stabilized and can develop only during the initial operating stage of the reactor, it can lead either to an increase of the diffusion coefficient or to a change in the cooling time of the fusion reaction products. Since this instability is inherent in stationary thermonuclear reactors, it is apparently of interest to consider the turbulent mode, to take into account the inhomogeneity of the plasma, etc.

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