PROCESSES INVOLVED IN THE MOTION OF HIGH ENERGY PARTICLES IN A MAGNETIC FIELD

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An operator method is proposed for studying quantum effects involved in the motion of charged particles in a magnetic field which enables us to consider processes in an arbitrary field for particles of arbitrary spin. With the aid of this method studies have been made of quantum phenomena in magnetic bremsstrahlung synchrotron radiation, and also of the production of pairs of particles by photons and of one-photon annihilation of pairs of particles.

1. INTRODUCTION

In classical electrodynamics the emission of radiation accompanying the motion of a charged particle in a magnetic field has been investigated in detail. Of great interest is the problem as to what changes will be introduced into the picture of the emission of radiation by a particle in an external magnetic field by taking into account the quantum nature of the motion and of the process of radiation. The investigation of such quantum effects is important for applications (for example, in the case of the motion of particles in accelerators), and is also of general theoretical interest.

With this aim in mind the problem should be solved within the framework of quantum electrodynamics taking the motion of the particle in the magnetic field into account rigorously (without utilizing perturbation theory), while the process of emission of radiation can be treated within the framework of perturbation theory. The calculation of quantum effects has usually been carried out in the so-called Furry representation utilizing the exact solutions of the corresponding wave equations (Dirac, Klein-Gordon) in a constant and homogeneous magnetic field. With the aid of this method a number of important results has been obtained, but the approach itself is quite complicated and technically awkward and enables one to obtain results only in a homogeneous and constant magnetic field. When for the study of certain phenomena it became necessary to consider quantum effects in an inhomogeneous field this led to a sharp increase in the complexity of the calculations even in a weakly inhomogeneous field. (For a detailed review of the papers along such lines including effects in an inhomogeneous field cf. in[1].) At the same time even when exact solutions of the wave equations have been utilized in order to obtain the final result one finally considers the quasiclassical asymptotic behaviour of the functions obtained, so that in this sense all the results obtained are approximate. As will be seen below this circumstance is not accidental.

In this paper an operator method is proposed for studying quantum effects involved in the motion of charged particles in a magnetic field. This method is suitable for studying any quantum phenomena in magnetic bremsstrahlung and also for the investigation of arbitrary other processes involving electrons and photons in a magnetic field. (As examples of such processes in this paper we consider the production of pairs by photons and the one-photon annihilation of a pair.) The method is technically sufficiently simple and enables us to obtain in a unified manner results for particles of arbitrary spin moving in an arbitrary electromagnetic field.

At the basis of the method lies the fact that quantum effects involved in the motion of ultrarelativistic particles in a magnetic field are of two types. The first of these is related to the quantum nature of the motion itself of the particles in a magnetic field. The noncommutativity of the dynamic variables of the particle[1)] arising in this case is of order $\hbar\omega_0/E$ (where $\omega_0 = v_t/R$, R is the instantaneous radius of curvature, E is the particle energy, v_t is the component of the velocity perpendicular to the magnetic field). The quantity

$$\hbar\omega_0 / E = H / H_0\gamma^2, \tag{1}$$

where $\gamma = E/mc^2$, H is the magnetic field, $H_0 = m^2c^3/e\hbar = 4.41 \times 10^{13}$ Oe (for an electron) is the critical field, is quite small and falls off with increasing energy. Thus, the motion of an electron in a magnetic field becomes more and more "classical" as the energy is increased.

The second type of quantum effects is associated with the recoil of the particle accompanying the emission of radiation and is of order $\hbar\omega/E$, where ω is the frequency of the emitted photon.

We shall characterize the quantum effects in magnetic bremsstrahlung by the invariant parameter[2)]

$$\chi = \frac{H}{H_0}\frac{p_t}{mc} = \frac{|\mathbf{v}|}{c}\frac{\hbar}{mc^2}\gamma^2 = \frac{\hbar\omega_0}{E}\frac{v_t}{c}\gamma^3 = \frac{\hbar e(|(F_{\mu\nu}p^\nu)^2|)^{1/2}}{m^3c^4}. \tag{2}$$

For $\chi \ll 1$ the recoil (and, therefore, the magnitude of the quantum effects) is small, and in this case

1) This problem is discussed in Appendix A.

2) Two other invariant parameters

$$f = \frac{e\hbar}{m^2c^3}(|F_{\mu\nu}F^{\mu\nu}|)^{1/2}, \quad g = \frac{e\hbar}{m^2c^3}(|\varepsilon_{\alpha\beta\gamma\delta}F^{\alpha\beta}F^{\gamma\delta}|)^{1/2}$$

depend only on the field. Since in this whole approach we assume $\gamma \gg 1$, then we always have $\chi \gg f,g$. Moreover, we assume $g,f \ll 1$, and this means that the field $H \ll H_0$. The condition indicated above is satisfied by a large margin for all known fields.

$\omega \sim \omega_0\gamma^3$. In the essentially quantum region $\chi \gtrsim 1$ the energy of the emitted photon is $\hbar\omega \sim E$. Thus, for arbitrary χ quantum effects of the first type are negligibly small compared to the effects of the emission of radiation. Therefore, one can neglect the noncommutativity with each other of the operators for the dynamical variables of the particle (of magnitude $\sim\hbar\omega_0/E$) and take into account only their commutators with the field of the emitted photon (of magnitude $\sim\hbar\omega/E$). This circumstance is utilized systematically below.

We note that the proposed approach is applicable, generally speaking, for studying quantum effects involved in the interaction between particles and photons in an arbitrary external field.

2. MAGNETIC BREMSSTRAHLUNG

We consider the radiation emitted by a charged particle in the course of its motion in a magnetic field. The matrix element for the transition from the initial state of the particle $|i\rangle$ into the corresponding final state $|f\rangle$ with the emission of a photon can be written in the lowest order of perturbation theory in the form[3)]

$$U_{fi} = \left\langle f \left| \frac{e}{(2\pi)^{3/2}\sqrt{2\hbar\omega}} \int e^{i\omega t} M(t)\, dt \right| i \right\rangle, \tag{3}$$

where

$$eM(t) = \Psi_{s'}(\mathbf{P}) \{(j\varepsilon), e^{-i\mathbf{kr}}\} \Psi_s(\mathbf{P}). \tag{4}$$

Here $j_\mu(t)$, $\mathbf{r}(t)$ are respectively the operators for the current and for the coordinate of the particle, ϵ_μ is the polarization vector for the photon, the brackets $\{\,,\}$ denote the symmetrized product of the operators, $\Psi(\mathbf{P})$ is the wave function for a particle with a given spin in an external field in operator form[4)] the indices s and s′ refer to the spin characteristics of the particle.

In accordance with what has been stated in the Introduction in the functions $\Psi(\mathbf{P})$ we can adopt any arbitrary order of writing down the operators occurring in the expressions. For example, for a particle of zero spin we have

$$M_s = \frac{1}{\sqrt{\mathcal{H}}} \left\{ \frac{(\varepsilon P)}{m}, e^{-i\mathbf{kr}} \right\} \frac{1}{\sqrt{\mathcal{H}}}, \tag{5}$$

for a particle of spin ½ we have

$$M_e = u_{s'}^+(\mathbf{P})\, (\alpha\varepsilon)\, e^{-i\mathbf{kr}} u_s(\mathbf{P}), \tag{6}$$

where

$$u = \sqrt{\frac{\mathcal{H}+m}{2\mathcal{H}}} \begin{pmatrix} \varphi(\zeta(t)) \\ \dfrac{\sigma\mathbf{P}}{\mathcal{H}+m}\varphi(\zeta(t)) \end{pmatrix} \tag{7}$$

Here $\varphi(\zeta(t))$ is a two-component spinor which describes the spin states of the electron at time t. In a similar manner one can also write down the expressions for particles of higher spin.

We shall be interested in the probability of a transition accompanied by the emission of a photon summed over all the final states of the particle. Carrying out such a summation we obtain the following expression for the probability of a radiative transition:

$$dw = \frac{e^2}{4\pi\hbar} \frac{d^3k}{(2\pi)^2\omega} \left\langle i \left| \int dt_1 \int dt_2 e^{i\omega(t_1-t_2)} M^*(t_2) M(t_1) \right| i \right\rangle, \tag{8}$$

where $e^2/4\pi\hbar = \alpha = 1/137$.

Multiplying by the energy of the emitted photon $\hbar\omega$ we will obviously obtain an expression for the intensity of the radiation:

$$dI = \frac{e^2}{4\pi} \frac{d^3k}{(2\pi)^2} \left\langle i \left| \int dt_1 \int dt_2 e^{i\omega(t_1-t_2)} M^*(t_2) M(t_1) \right| i \right\rangle. \tag{9}$$

The expressions (8) and (9) given above can be utilized for the study of any arbitrary phenomena involved in the emission of a photon by a particle in an external field.

In accordance with what has been stated above in the expression for M (4)–(6) one should take into account only the commutators of the photon field ($\exp(-i\mathbf{k}\cdot\mathbf{r})$) with the momentum $\mathbf{P}$. In our subsequent discussion we shall utilize the relations

$$\mathbf{P}e^{-i\mathbf{kr}} = e^{-i\mathbf{kr}}(\mathbf{P} - \hbar\mathbf{k}), \quad \mathcal{H}e^{-i\mathbf{kr}} = e^{-i\mathbf{kr}}(\mathcal{H} - \hbar\omega), \tag{10}$$

the first of which is a consequence of the fact that the operator $\exp(-i\mathbf{k}\cdot\mathbf{r})$ is a displacement operator in momentum space, and for the derivation of the second of which one should take into account the fact that

$$[\mathcal{H}, e^{-i\mathbf{kr}}] = -i\hbar\frac{d}{dt}e^{-i\mathbf{kr}}, \tag{11}$$

and carry out an integration by parts in expression (3). Utilizing (10) one can in $M(t_1)$ bring the operator $\exp(-i\mathbf{k}\cdot\mathbf{r}(t_1))$ to the left side of the expression, and bring in $M(t_2)$ the operator $\exp(i\mathbf{k}\cdot\mathbf{r}(t_2))$ to the right side of the expression. After this it is necessary to investigate the combination $\exp(i\mathbf{k}\cdot\mathbf{r}(t_2))\exp(-i\mathbf{k}\cdot\mathbf{r}(t_1))$ appearing in (8) and (9). The noncommutativity of the operators appearing here is essential, so that, generally speaking, it is not possible to restrict ourselves to the expansion of this combination in terms of the lowest commutators. The central point of the present approach is the unfolding of this combination.

For the following discussion it is convenient to carry out in the integrals (8) and (9) the change of variables

$$t_1 = t, \quad t_2 = t + \tau. \tag{12}$$

An essential contribution to the integral over τ comes from the region $|\dot{\mathbf{v}}|\tau \sim 1/\gamma$, so that in carrying out further calculations we shall systematically expand all quantities in powers of $|\dot{\mathbf{v}}|\tau$ and keep only the leading terms of the expansion.

Moreover, for the sake of simplicity, we shall consider fields satisfying the condition

$$|\dot{\mathbf{H}}|\tau / |\mathbf{H}| \ll 1, \tag{13}$$

where $|\dot{\mathbf{H}}|$ characterizes the change of the magnetic field along the trajectory. Physically this criterion means that the field along the trajectory does not change much during a characteristic time of emission. If we introduce the index of inhomogeneity

$$n = |\partial \ln H / \partial \ln r|, \tag{14}$$

then the condition (13) can be written in the form

$$n/\gamma \ll 1. \tag{15}$$

In all interesting cases the fields satisfy this criterion.

As a result of unfolding (cf., Appendix B) we obtain

[3)]In what follows c = 1.

[4)]In order to obtain $\Psi(\mathbf{P})$ it is sufficient to replace in the free wave functions the momentum $\mathbf{p} \to \mathbf{P}(t)$, $E \to \mathcal{H} = \sqrt{P^2 + m^2}$.

$$e^{i\mathbf{kr}(t_2)}e^{-i\mathbf{kr}(t_1)} = \exp\left\{i\left[\omega\tau + \frac{\mathscr{H}}{\mathscr{H}-\hbar\omega}(\mathbf{k}\rho - \omega\tau)\right]\right\}, \quad (16)$$

where $\rho = \mathbf{r}(t_2) - \mathbf{r}(t_1)$.

The combination $\exp(i\mathbf{k}\cdot\mathbf{r}(t_2))\exp(-i\mathbf{k}\cdot\mathbf{r}(t_1))$ obtained above evidently commutes with $\mathscr{H}$ (cf., (10)). In considering its commutation with the operator $\mathbf{P}$ we must take into account the fact that in order to utilize relation (10) it is necessary that all the operators should depend on a single time. Carrying out the corresponding expansions and omitting terms $\sim 1/\gamma^2$ we find that the quantity $\exp(i\mathbf{k}\cdot\mathbf{r}(t_2))\exp(-i\mathbf{k}\cdot\mathbf{r}(t_1))$ commutes with $\mathbf{P}$. Thus, all the operators in expressions (8) and (9) turn out to commute to our degree of accuracy, and therefore all these operators appearing as expectation values in the initial state can be replaced by their classical values.

We can now write the square of the matrix element in the form

$$\langle i|M^*(t_2)M(t_1)|i\rangle = \exp\{i[\omega\tau + E(\mathbf{k}\rho - \omega\tau)/E']\}R^*(t_2)R(t_1), \quad (17)$$

where

$$eR(t) = {}^1/_2\Psi_{s'}^+(\mathbf{P}')([j(\mathbf{P}) + j(\mathbf{P}')]\varepsilon)\Psi_s(\mathbf{P}). \quad (18)$$

Here $E' = E - \hbar\omega$, $\mathbf{P}' = \mathbf{P} - \hbar\mathbf{k}$; E, $\mathbf{P}$ are no longer operators, but c-numbers. We note that all the information on the spin and polarization states is contained in R(t).

Thus, in the unfolding operation the spin and the polarization characteristics of the particles are not at all affected, and this is connected with the fact that in our approximation we neglect the effect of the spin on the motion (terms $\sim\hbar\omega_0/E$). But the function R(t) describing them has the form of a matrix element for the transition for free particles taking conservation laws into account. This enables us to consider in a unified manner problems for arbitrary spin (cf., also Appendix C).

For the sake of definiteness we now consider particles of spin $1/2$. Then we have

$$R(t) = \varphi_f^+[A + i\sigma\mathbf{B}]\varphi_i, \quad (19)$$

where

$$A = \frac{1}{2}(\varepsilon\mathbf{P})\left[\frac{1}{E+m} + \frac{1}{E'+m}\right], \quad B = \frac{1}{2}\left(\frac{[\varepsilon\mathbf{P}]}{E+m} - \frac{[\varepsilon\mathbf{P}]}{E'+m}\right). \quad (20)^*$$

Here we have neglected terms $\sim 1/\gamma$, and, moreover, in this whole approach it is assumed that the final electrons remain ultrarelativistic.

If we utilize the equations of motion for a spin in an external field[2], then it can be easily shown that with an accuracy up to terms $\sim 1/\gamma$ we have

$$\varphi(\zeta(t_1)) = \varphi(\zeta(t_2)) = \varphi(\zeta(t)); \quad (21)$$

and taking this into account we have

$$R^*(t_2)R(t_1) = {}^1/_4\,\mathrm{Sp}\,[(1+\sigma\zeta_i)(A(t_2) - i\sigma\mathbf{B}(t_2))(1+\sigma\zeta_f)(A(t_1) + i\sigma\mathbf{B}(t_1))]. \quad (22)$$

Expression (22) can be utilized for the investigation of any arbitrary polarization and spin phenomena involved in the emission of radiation by electrons in a magnetic field.

We now consider the intensity of the radiation in the case of a motion of an electron in an external field summed over the polarizations of the photons and over the spins of the final electrons and averaged over the spins of the initial electrons. Then we have

$$S_iS_f[R^*(t_2)R(t_1)] = A^*(t_2)A(t_1) + \mathbf{B}^*(t_2)\mathbf{B}(t_1). \quad (23)$$

The rest of the calculation is carried out as in the classical problem of magnetic bremsstrahlung (cf., for example,[3]). Summing over the polarizations of the photon we have

$$\sum_\lambda A^*(t_2)A(t_1) = \frac{1}{4}\left(1+\frac{E}{E'}\right)^2(\mathbf{v}(t_2)\mathbf{v}(t_1) - 1),$$

$$\sum_\lambda \mathbf{B}^*(t_2)\mathbf{B}(t_1) = \frac{1}{2}\left(\frac{\hbar\omega}{E'}\right)^2\left(\mathbf{v}(t_2)\mathbf{v}(t_1) - 1 + \frac{2}{\gamma^2}\right), \quad (24)$$

where we have discarded terms of highest order in $1/\gamma$. Up to our degree of accuracy we have

$$\mathbf{v}(t_2)\mathbf{v}(t_1) = 1 - 1/\gamma^2 - {}^1/_2\dot{\mathbf{v}}^2\tau^2. \quad (25)$$

Substituting (23)--(25) into (22), and (22) into (9) we obtain the following expression for the intensity of radiation per unit time:

$$\frac{dI_e}{dt} = -\frac{e^2}{4\pi}\frac{d^3k}{(2\pi)^2}\int_{-\infty}^{\infty} d\tau\left[\frac{1+\alpha}{\gamma^2} + \frac{1}{2}\left(1+\alpha+\frac{\alpha^2}{2}\right)\dot{\mathbf{v}}^2\tau^2\right]$$
$$\times\exp\left[-\frac{i\alpha E\tau}{\hbar}\left(1 - \mathbf{nv} - \frac{\mathbf{n}\dot{\mathbf{v}}}{2}\tau + \frac{1}{6}\dot{\mathbf{v}}^2\tau^2\right)\right], \quad (26)$$

where we have introduced $\alpha = \hbar\omega/E'$, $\mathbf{n} = \mathbf{k}/\omega$. This formula gives the angular and the spectral distribution of the intensity of radiation.

We now introduce ϑ—the angle between the $(\mathbf{v}, \dot{\mathbf{v}})$ plane and the vector $\mathbf{n}$ and ψ--the angle between the projection of the vector $\mathbf{n}$ on the $(\mathbf{v}, \dot{\mathbf{v}})$ plane and the vector $\mathbf{v}$. Of interest is the intensity of radiation integrated over the azimuthal angle of photon emission ψ. It turns out to be convenient to carry out the integration over τ and ψ simultaneously. Expressing the scalar combinations appearing in the above expressions

$$\mathbf{nv} = |\mathbf{v}|\cos\psi\cos\vartheta, \quad \mathbf{n}\dot{\mathbf{v}} = |\dot{\mathbf{v}}|\sin\psi\cos\vartheta \quad (27)$$

and taking into account the fact that up to terms of highest order in $1/\gamma^2$ the principal contribution comes from small ψ and ϑ we obtain

$$(1-\mathbf{nv})\tau - \frac{\mathbf{n}\dot{\mathbf{v}}}{2}\tau^2 + \frac{\dot{\mathbf{v}}^2}{6}\tau^3 = \frac{\mu^{3/2}}{2|\dot{\mathbf{v}}|}\left(x + \frac{1}{3}x^3 + y + \frac{1}{3}y^3\right), \quad (28)$$

where in the integral (26) we have made the substitution $\tau \to \tau + \psi/|\dot{\mathbf{v}}|$ and we have introduced the notation

$$\mu = 1 - \mathbf{v}^2\cos^2\vartheta = 1/\gamma^2 + \vartheta^2,$$
$$y = \frac{1}{\sqrt{\mu}}|\dot{\mathbf{v}}|\tau, \quad x = \frac{1}{\sqrt{\mu}}\psi. \quad (29)$$

Utilizing the well known integrals

$$\int_0^\infty \cos b\left(x+\frac{1}{3}x^3\right)dx = -\int_0^\infty x^2\cos b\left(x+\frac{1}{3}x^3\right)dx = \frac{1}{\sqrt{3}}K_{1/3}\left(\frac{2}{3}b\right),$$
$$\int_0^\infty x\sin b\left(x+\frac{1}{3}x^3\right)dx = \frac{1}{\sqrt{3}}K_{2/3}\left(\frac{2}{3}b\right), \quad (30)$$

we obtain the following expression for the angular and the spectral distribution of the intensity of the radiation per unit time:

*$[\epsilon\mathbf{P}] \equiv \epsilon \times \mathbf{P}_0$.

$$\frac{dI_e}{dt}=\frac{e^2}{4\pi}\frac{E^3}{3\pi^2|\dot{\mathbf{v}}|\hbar^3}\frac{\alpha^2}{(1+\alpha)^4}\mu\Big\{\mu\Big(1+\alpha+\frac{\alpha^2}{2}\Big)[K_{1/3}^2(\eta)+K_{2/3}^2(\eta)]-\frac{(1+\alpha)}{\gamma^2}K_{1/3}^2(\eta)\Big\}d\alpha\, d\sin\vartheta, \qquad (31)$$

where

$$\eta=\frac{1}{3}\frac{\alpha E\mu^{3/2}}{\hbar|\dot{\mathbf{v}}|}=\frac{\alpha}{3\chi}(\gamma^2\mu)^{3/2}. \qquad (32)$$

The calculation of the intensity of radiation from particles of zero spin which is carried out in the same manner (starting with formulas (17) and (18)) turns out to be even simpler. The expression for the intensity is obtained from (31) if in the first term in the figure brackets we omit the term $\alpha^2/2$.

As another illustration we consider magnetic bremsstrahlung from particles of spin unity.

In this case the quantity M(t) can be written in the form

$$M=\frac{1}{\sqrt{\mathcal{H}}}(t_f)_\mu\{[g^{\mu\nu}(\varepsilon P)-\varepsilon^\mu P^\nu-\varepsilon^\nu P^\mu],\ e^{-ikr}\}(t_i)_\nu\frac{1}{\sqrt{\mathcal{H}}}, \qquad (33)$$

where $t_i(\mathbf{P})$ and $t_f(\mathbf{P})$ are the polarizations of the initial and the final vector particles.

The further discussion is analogous to the case of particles of spin ½--one must go over to the description of the polarizations in terms of quantities in the rest system. In doing this it is easy to show that up to terms $\sim 1/\gamma$ one can assume that the polarizations in the rest system depend on a single time (cf., (21)).

Carrying out an expansion in powers of $|\dot{\mathbf{v}}|\tau$ and integrating over the variables τ and ψ (cf., (26)—(30)) we obtain for the angular and the spectral distribution of the intensity of radiation from a vector particle per unit time

$$\frac{dI_v}{dt}=\frac{dI_e}{dt}+\frac{dI_1}{dt}, \qquad (34)$$

the quantity dI_e/dt is given by formula (31), while

$$\frac{dI_1}{dt}=\frac{e^2}{4\pi}\frac{E^3}{9\pi^2|\dot{\mathbf{v}}|\hbar^3}\frac{\alpha^4\mu^2}{(1+\alpha)^4}\Big[\frac{1}{4}\frac{\alpha^2}{1+\alpha}+\gamma^2\mu\Big(1+\frac{1}{2}\frac{\alpha^2}{1+\alpha}\Big)\Big]\times(K_{1/3}^2(\eta)+K_{2/3}^2(\eta))\,d\alpha\, d\sin\vartheta. \qquad (35)$$

In order to obtain the total intensity of radiation it is necessary to integrate (31), (35) over the angle of emission and over the frequency of the photon. For the evaluation of the integral over α it is convenient to introduce the representation[4]

$$\frac{1}{(1+\alpha)^m}=\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\frac{\Gamma(-s)\Gamma(m+s)}{\Gamma(m)}\alpha^s ds, \qquad (36)$$

where $1-m<\lambda<0$. After this the integrals over α can be easily evaluated. Carrying out in the same manner the elementary integration over ϑ (from which it can be seen that the basic method yields the region $\vartheta\sim 1/\gamma$), we obtain for the electron

$$\frac{dI_e}{dt}=\frac{\sqrt{3}}{32\pi^2\hbar^2}e^2m^2\chi^2\frac{1}{2\pi i}\cdot\int_{\lambda-i\infty}^{\lambda+i\infty}(3\chi)^s(8+2s+s^2)\Gamma(-s)\Gamma(s+2)\Gamma\Big(\frac{s}{2}+\frac{2}{3}\Big)\Gamma\Big(\frac{s}{2}+\frac{4}{3}\Big)ds. \qquad (37)$$

An analogous calculation for particles of zero spin yields

$$\frac{dI_s}{dt}=\frac{3\sqrt{3}}{16\pi^2\hbar^2}e^2m^2\chi^2\frac{1}{2\pi i}\cdot\int_{\lambda-i\infty}^{\lambda+i\infty}(3\chi)^s\Gamma(-s)\Gamma(s+2)\Gamma\Big(\frac{s}{2}+\frac{2}{3}\Big)\Gamma\Big(\frac{s}{2}+\frac{7}{3}\Big)ds, \qquad (38)$$

while for particles of spin unity the additional term dI_1/dt (cf., (34)) will be given by

$$\frac{dI_1}{dt}=\frac{3\sqrt{3}}{32\pi^2\hbar^2}e^2m^2\chi^4\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}(3\chi)^s\Gamma(-s)\Big[(s+3)\Gamma(s+4)\Gamma\Big(\frac{s}{2}+\frac{4}{3}\Big)\Gamma\Big(\frac{s}{2}+\frac{5}{3}\Big)+\frac{(3\chi)^2}{8}\Big(\frac{3}{2}s+\frac{22}{3}\Big)\Gamma(s+5)\Gamma\Big(\frac{s}{2}+\frac{7}{3}\Big)\Gamma\Big(\frac{s}{2}+\frac{8}{3}\Big)\Big]ds. \qquad (39)$$

The integrals (37)—(39) can be evaluated by closing the contour of integration to the right for $\chi \ll 1$ (in this case a series in χ is obtained) and to the left for $\chi \gg 1$ (in this case a series in inverse powers of χ is obtained). In view of the awkwardness of these series we write out here only the first terms of the corresponding expansions.

For $\chi \ll 1$ we have

$$\frac{dI_{s,e,v}}{dt}=\frac{e^2m^2\chi^2}{6\pi\hbar^2}\Big(1-\frac{55\sqrt{3}}{16}\chi+\delta_{s,e,v}\chi^2+\ldots\Big), \qquad (40)$$

where $\delta_s=42$, $\delta_e=48$, $\delta_v=105/2$. The first term in these expansions is the classical expression for the intensity, the second term is the first quantum correction, both terms do not depend on the spin of the radiating particle, such a dependence appears only starting with the third term.

For $\chi \gg 1$ we have

$$\frac{dI_s}{dt}=\frac{e^2m^2(3\chi)^{2/3}\Gamma(2/3)}{2\pi 3^3\hbar^2}+\ldots,$$
$$\frac{dI_e}{dt}=\frac{e^2m^2(3\chi)^{2/3}8\Gamma(2/3)}{\pi 3^5\hbar^2}+\ldots,$$
$$\frac{dI_v}{dt}=\frac{e^2m^2(3\chi)^{4/3}35\Gamma(1/3)}{2\pi 3^8\hbar^2}+\ldots. \qquad (41)$$

For $\chi \gg 1$ the photons carry away an energy of the order of the energy of the radiating particle, but at the same time it is possible to show that the principal contribution to the integrals (31), (35) is given by the region $\alpha \sim 1$, and this means that $E' \sim E$ (thereby justifying the assumption concerning the ultrarelativistic nature of the final electrons). In this case the mean angle for the emission of a photon is $\sim\chi^{1/3}/\gamma$. Thus, in the essentially quantum region the nature of the radiation changes appreciably compared to the classical region.

It is of interest to note that for $\chi \gg 1$ the intensity of emission from a vector particle grows with the energy faster than in the case of particles of spin zero and ½; such a situation, generally speaking, is characteristic for the quantum electrodynamics of a vector particle.

We note further that one can also obtain closed formulas for dI/dt which are particularly convenient for the case $\chi \sim 1$. Utilizing the formula[4]

$$\int_0^\infty x^{\mu-1}K_\nu(x)\,dx=2^{\mu-2}\Gamma\Big(\frac{\mu-\nu}{2}\Big)\Gamma\Big(\frac{\mu+\nu}{2}\Big), \qquad (42)$$

we obtain

$$\frac{dI_s}{dt}=\frac{e^2m^2}{6\sqrt{3}\pi\hbar^2}\left[\frac{1}{2\chi}-\frac{1}{4\sqrt{3}}+\frac{2\pi}{9}\left(1-\frac{1}{\chi^2}\right)\Phi_{2/3}(\xi)-\frac{\pi}{9\chi}\Phi_{1/3}(\xi)\right],$$

$$\frac{dI_e}{dt}=\frac{e^2m^2}{2\pi3^4\hbar^2}\left\{\frac{2\pi}{3\sqrt{3}}\left[\left(16+\frac{13}{\chi^2}\right)\Phi_{2/3}(\xi)+\frac{1}{\chi}\left(47+\frac{2}{\chi^2}\right)\Phi_{1/3}(\xi)\right]-19-\frac{6\sqrt{3}}{\chi}-\frac{1}{\chi^2}\right\}, \quad (43)$$

where

$$\Phi_\nu=e^{-i\pi\nu/2}(J_\nu-\mathbf{J}_\nu)+e^{i\pi\nu/2}(J_{-\nu}-\mathbf{J}_{-\nu}),$$

$\mathbf{J}_\nu$ is the Anger function, $\xi = 2i/3\chi$.

All the expressions obtained for the intensity of emission depend on the kinematic characteristics of the particle $\mathbf{v}(t)$, $\dot{\mathbf{v}}(t)$ in the given field. In a homogeneous field for the cases of particles of spin $1/2$ and zero they go over into the well known expressions obtained by Klepikov[5] and Matveev[6]. All the expressions for the vector particles are obtained here for the first time.

Expression (37) was also obtained in the paper by Nikishov and Ritus[7] who have studied the intensity of emission from an electron in the field of a plane electromagnetic wave and a constant crossed field. They have noted that for $f, g \ll \chi$ the same expression also describes the radiation from an electron in an arbitrary homogeneous field. The investigation carried out here shows that such a generality of the result is physically associated with the fact that in essence in order to obtain it, it is sufficient to take into account recoil during emission. In this sense the approach is applicable to quite a wide class of external fields. Both Klepikov[5] and also Nikishov and Ritus[7] have utilized the solution of the equations in a definite external field; the analysis which is carried out here shows that it is not necessary to do so, and that in order to obtain the given set of results it is sufficient to know the Heisenberg equations of motion in the given external field.

Evidently in all the expressions the characteristics of the inhomogeneity of the magnetic field are contained only in χ. This question has recently given rise to a discussion (for the first term of the expansion for $\chi \ll 1$, cf., reference[1]). We also note that a similar method for discussing the radiation in a magnetic field, but using an expansion of the commutators up to the first term in $\hbar\omega/E$ has been utilized in a number of specific problems[8-10].

3. PRODUCTION OF A PAIR OF PARTICLES BY A PHOTON

The method developed in section 2 can be utilized for the study of a number of other processes. Here we shall consider the production of a pair of charged particles by a photon in an external field. In the lowest approximation of perturbation theory the matrix element for the process has the form (3) (only one has to make the substitution $k_\mu \to -k_\mu$):

$$U_{fi}=\left\langle q\left|\frac{e}{(2\pi)^{3/2}\sqrt{2\hbar\omega}}\int e^{-i\omega t}M(t)\,dt\right|\bar{q}\right\rangle, \quad (44)$$

$$eM(t) = \Psi_s^+(\mathbf{P})\{(j\varepsilon),\ e^{i\mathbf{kr}}\}\Psi_{\bar{s}}(\mathbf{P}), \quad (45)$$

where $|q\rangle$ and $|\bar{q}\rangle$ are respectively the state of the particle and of the antiparticle, s and $\bar{s}$ are the indices of the spin states.

We are interested in the transition probability summed over the final states of the produced pair. We carry out the procedure in two stages. First we sum over the final states of the antiparticle, and obtain

$$dw=\frac{\alpha}{(2\pi)^2}\frac{1}{\omega}\sum_q\left\langle q\left|\int dt_1\int dt_2 e^{i\omega(t_1-t_2)}M(t_2)M^*(t_1)\right|q\right\rangle. \quad (46)$$

This expression is analogous to (8) and further discussion proceeds as in Sec. 2. Utilizing formulas (10) we bring the quantity $\exp(i\mathbf{k}\cdot\mathbf{r}(t_2))$ to the right in $M(t_2)$, and we bring the quantity $\exp(-i\mathbf{k}\cdot\mathbf{r}(t_1))$ to the left in $M^*(t_1)$.

Further we must take into account the fact that we are considering the case when the produced electron and positron are both ultrarelativistic. The principal contribution to the probability is given by the range of velocities of the final particles for which $1-\mathbf{n}\cdot\mathbf{v}\sim 1/\gamma^2$, $\mathbf{n}$ is the direction of motion of the photon, and for the same reason $|\dot{\mathbf{v}}|\tau\sim 1/\gamma$.

Physically this means that the created particle is moving at the instant of creation in the direction of motion of the photon, and the photon-particle interaction remains significant until the particle has turned through an angle $\sim 1/\gamma$, so that the situation is very similar to magnetic bremsstrahlung. Therefore the unfolding operation and the transition to the classical values of the quantities evaluated as expectation values in the state $|q\rangle$ are carried out under the same assumptions as in section 2. We have

$$\langle q|M(t_2)M^*(t_1)|q\rangle=\exp\left\{i\left[\omega\tau-\frac{E}{E'}(\mathbf{k}\rho-\omega\tau)\right]\right\}R(t_2)R^*(t_1), \quad (47)$$

where

$$eR(t)={}^1/_2\overset{+}{\Psi}_s(\mathbf{P})([j(P)+j(-P')]\varepsilon)\Psi_{\bar{s}}(-\mathbf{P}'),$$
$$\hbar\omega-E=E',\quad \hbar\mathbf{k}-\mathbf{P}=\mathbf{P}'.$$

Taking into account the change in the sign of the momenta compared with formula (18) associated with the fact that now the photon is in the initial state and both particles are in the final state, the calculation is carried out as in Sec. 2 if we take into account the fact that $\sum_q \to \int d^3P/\hbar^3$. After carrying out the integration over the relative time τ and over the azimuthal angle of emission of the particle ψ, after summing over the spin of the final particle and after averaging over the photon polarizations we obtain for the probability of creation of an electron-positron pair per unit time

$$\frac{dw_e}{dt}=\frac{8\alpha}{3\pi^2}\frac{m^2}{\hbar^2\omega\varkappa}\int_0^\infty dx\int_0^\infty dy\,\mathrm{ch}^3x$$
$$\times\{K_{1/3}^2(\eta)+\mathrm{ch}^2x(2\mathrm{ch}^2y-1)[K_{1/3}^2(\eta)+K_{2/3}^2(\eta)]\}, \quad (48)$$

where

$$\varkappa=\frac{H}{H_0}\frac{\hbar k_t}{m}=\frac{e\hbar^2(|(F_{\mu\nu}k^\nu)^2|)^{1/2}}{m^3},\quad \eta=\frac{4}{3\varkappa}\mathrm{ch}^2y\,\mathrm{ch}^3x,$$
$$\mathrm{ch}^2y=\frac{1}{4}\frac{\hbar^2\omega^2}{EE'},\quad \mathrm{ch}^2x=\gamma^2\mu, \quad (49)$$

k_t is the photon momentum perpendicular to the direction of the magnetic field.

In discussing pair creation the assertion made in Sec. 2 regarding the separation of the spin characteristics from the unfolding operation remains valid. Thus, one can consider the creation of a pair of particles of arbitrary spin. For example, for scalar particles we obtain

$$\frac{dw_s}{dt}=\frac{4\alpha m^2}{3\pi^2\hbar^2\omega\varkappa}\int_0^\infty dx\int_0^\infty dy\,\mathrm{ch}^3 x\{\mathrm{ch}^2 xK_{2/3}{}^2(\eta)+\mathrm{sh}^2 xK_{1/3}{}^2(\eta)\}. \quad (50)$$

Evaluating these integrals we obtain for $\kappa \ll 1$

$$\frac{dw_e}{dt}=\frac{3\sqrt{3}}{16\sqrt{2}}\frac{\alpha m^2\varkappa}{\hbar^2\omega}e^{-8/3\varkappa},\quad \frac{dw_s}{dt}=\frac{1}{6}\frac{dw_e}{dt}. \quad (51)$$

For $\kappa \gg 1$ we have

$$\frac{dw_e}{dt}=\frac{5\left(\frac{2}{3}\right)^{1/3}\Gamma\left(\frac{5}{6}\right)}{14\Gamma\left(\frac{7}{6}\right)}\frac{\alpha m^2\varkappa^{2/3}}{\hbar^2\omega};\quad \frac{dw_s}{dt}=\frac{1}{5}\frac{dw_e}{dt}. \quad (52)$$

In a homogeneous field the probability for electrons goes over into that obtained by Klepikov[5].

4. SINGLE PHOTON PAIR ANNIHILATION

The matrix element for this process is the Hermitian conjugate of U_{fi} (44). The expression for $M^*(t_2)M(t_1)$ now contains $|\bar{q}\rangle\langle\bar{q}|$. We make use of the artificial device:

$$|\bar{q}_{\mathbf{p}'}\rangle\langle\bar{q}_{\mathbf{p}'}|\rightarrow\sum_{\bar{q}}|\bar{q}\rangle\langle\bar{q}|\delta(\mathbf{p}'+\mathbf{P})\hbar^3\rightarrow\delta(\mathbf{p}'+\mathbf{P})\hbar^3 \quad (53)$$

(where $\mathbf{P}$ is the momentum operator), with the aid of which the problem reduces to single particle expectation values as in Secs. 2 and 3.

Since we are considering the single photon annihilation of a pair of particles moving along curvilinear trajectories in a magnetic field it is difficult to describe this process (in contrast to the process of single photon annihilation in a Coulomb field) in terms of the language of cross sections. Apparently, the most convenient description in this case is the introduction of the lifetime of particles moving in a medium of antiparticles in a magnetic field (or vice versa).

Carrying out all the required commutations and the unfolding we obtain for the probability of single photon annihilation of a particle moving in a medium of antiparticles in a magnetic field per unit time:

$$\frac{dw}{dt}=\alpha n\frac{\pi d^3k}{\omega}\hbar^3\int d\tau\exp\left\{-i\frac{E}{E'}[\mathbf{k}\rho-\omega\tau]\right\}R^*(t_2)R(t_1)\delta(\mathbf{p}+\mathbf{p}'-\hbar\mathbf{k}), \quad (54)$$

where n is the density of antiparticles. This expression, as in Secs. 2 and 3, can be utilized for particles of arbitrary spin.

For the single photon annihilation of an electron-positron pair after averaging over the azimuthal angle of the relative motion of the initial particles we have

$$\frac{dw_e}{dt}=\frac{\alpha}{3}\frac{n}{|\dot{\mathbf{v}}|}\frac{m^4\hbar}{(E+E')E'^2E^4}\left(1+\frac{p_z{}^2}{m^2}\right)$$
$$\times\left\{\left[(E+E')^2+(E^2+E'^2)\frac{p_z{}^2}{m^2}\right]K_{1/3}{}^2(\eta)+(E^2+E'^2)\left(1+\frac{p_z{}^2}{m^2}\right)K_{2/3}(\eta)\right\}, \quad (55)$$

where

$$\eta=\frac{1}{3\varkappa}\frac{(E+E')^2}{EE'}\left(1+\frac{p_z{}^2}{m^2}\right)^{3/2};$$

the quantity appearing in κ is $k_t = \sqrt{\omega^2 - k_{\parallel}^2}$; $p_z = p\sin\vartheta \ll E$ is the z-component of the momentum of the electron and the positron in the system in which the photon moves perpendicular to the field.

The principal contribution to this expression is given by the region in which the electron and the positron move in the same direction while the angle between their momenta is $\sim 1/\gamma$. In a homogeneous field this expression coincides with that obtained by Klepikov[5].

APPENDIX A

COMMUTATION PROPERTIES OF THE DYNAMIC VARIABLES OF A PARTICLE IN A MAGNETIC FIELD

In the case of motion of charged particles in an external magnetic field the following operator equation holds

$$\{\mathcal{H},\mathbf{v}\}=c^2(\mathbf{p}-e\mathbf{A})=c^2\mathbf{P}. \quad (A.1)$$

Here $\mathcal{H}$ is the Hamiltonian, $\mathbf{v}$ is the velocity operator, the brackets $\{,\}$ denote the symmetrized product of the operators. From this we obtain the equation for $\mathbf{v}$:

$$\mathbf{v}=c^2\{\mathbf{P},\mathcal{H}^{-1}\}-{}^1/_4[[\mathbf{v},\mathcal{H}],\mathcal{H}^{-1}]. \quad (A.2)$$

Solving this equation by iteration we obtain a series in powers of $\hbar$. In the first approximation with respect to $\hbar$ we have

$$\mathbf{v}=c^2\{\mathbf{P},\mathcal{H}^{-1}\}, \quad (A.3)$$

from where we obtain

$$[r_i,v_j]=i\hbar\frac{\partial v_j}{\partial p_i}=i\hbar\frac{c^2}{\mathcal{H}}\left[\delta_{ij}-\frac{v_iv_j}{c^2}\right]. \quad (A.4)$$

Here we have neglected the noncommutativity of the components of the velocity v_i and v_j. The latter in the same approximation is given by the expression

$$\frac{1}{c^2}[v_m,v_n]=\frac{ie\hbar c}{\mathcal{H}^2}\varepsilon_{mnl}\left[H_l\left(1-\frac{\mathbf{v}^2}{c^2}\right)+\frac{1}{c^2}v_l\mathbf{vH}\right]. \quad (A.5)$$

From here follows the uncertainty relation for the components of the velocity. In the general case for ultrarelativistic electrons we have

$$\frac{\Delta v_i\Delta v_j}{c^2}\gtrsim\frac{e\hbar cH}{2E^2}=\frac{\hbar\omega_0}{2E}, \quad (A.6)$$

But if the motion takes place in the plane perpendicular to the magnetic field then we have

$$\frac{\Delta v_i\Delta v_j}{c^2}\gtrsim\frac{\hbar\omega_0}{E\gamma^2}. \quad (A.7)$$

We have considered the noncommutativity of the components of the velocity in the first order with respect to $\hbar$. From equation (A.2) it follows that terms of highest order with respect to $\hbar$ have the form of a series in $\hbar\omega_0/E$.

APPENDIX B

THE UNFOLDING OF THE COMBINATION $e^{i\mathbf{k}\cdot\mathbf{r}(t_2)}e^{-i\mathbf{k}\cdot\mathbf{r}(t_1)}$

In order to carry out the unfolding operation we write

$$\mathbf{r}(t_2)=\mathbf{r}(t_1)=\rho. \quad (B.1)$$

Then it is convenient to represent $e^{i\mathbf{k}\cdot\mathbf{r}(t_2)}$ in the form

$$e^{i\mathbf{kr}(t_2)}=e^{i\mathbf{k}(\mathbf{r}(t_1)+\rho)}=Le^{i\mathbf{kr}(t_1)}. \quad (B.2)$$

Here L is the operator to the determination of which the problem is reduced. Replacing for the sake of brevity

$$a=i\mathbf{kr}(t),\quad b=i\mathbf{k}\rho-i\omega\tau, \quad (B.3)$$

we have

$$\exp[\xi(a+b)] = e^{-i\omega\tau}L(\xi)e^{\xi a}, \qquad (B.4)$$

where ξ is a parameter. The operator $L(\xi)$ satisfies the equation

$$dL/d\xi = L(\xi)e^{\xi a}be^{-\xi a}. \qquad (B.5)$$

We now evaluate

$$e^{\xi a}be^{-\xi a} = \sum_{n=0}^{\infty}\frac{\xi^n}{n!}[a,[a,\ldots[a,b]\ldots]]. \qquad (B.6)$$

We find the commutator

$$[a,b] = -[\mathbf{kr},\mathbf{k\rho}] = -i\hbar(\mathbf{k}\nabla_{\mathbf{p}})(\mathbf{k\rho}) \qquad (B.7)$$

by expanding ρ in powers of τ:

$$\rho = \mathbf{v}(t)\tau + \frac{1}{2!}\dot{\mathbf{v}}(t)\tau^2 + \frac{1}{3!}\ddot{\mathbf{v}}(t)\tau^3 + \ldots \qquad (B.8)$$

Then, replacing $\mathbf{v}(t) = \mathbf{P}(t)/\mathscr{H}$ and utilizing the Heisenberg equation of motion in a magnetic field we obtain

$$[a,b] = -[\mathbf{kr},\mathbf{k\rho}] = (2\hbar\omega/\mathscr{H})b. \qquad (B.9)$$

In evaluating the commutator along with expanding in powers of $1/\gamma$ we have also taken into account the fact that relation (13) holds.

The operators $\mathscr{H}$ and b appearing in (B.9) commute up to terms involving $\hbar\omega_0/E$, and, therefore, the order in which they appear in (B.9), and also in further expressions is immaterial. The fact that the commutator [a, b] is expressed in terms of b enables us to evaluate all the terms of the series (B.6), if we take into account the fact that

$$\left[a,\frac{1}{\mathscr{H}}\right] = \frac{\hbar\omega}{\mathscr{H}^2}; \qquad (B.10)$$

then we have

$$e^{\xi a}be^{-\xi a} = \sum_{n=0}^{\infty}\frac{\xi^n}{n!}(n+1)!\left(\frac{\hbar\omega}{\mathscr{H}}\right)^n b = \frac{b}{(1-\xi\hbar\omega/\mathscr{H})^2}. \qquad (B.11)$$

Solving the differential equation (B.5) with the boundary condition $L(0) = e^{i\omega\tau}$ taking into account what has been said regarding the operators b and $\mathscr{H}$ we obtain

$$L(\xi) = \exp\left\{b\frac{\xi\mathscr{H}}{\mathscr{H}-\hbar\omega\xi} + i\omega\tau\right\}. \qquad (B.12)$$

Then, taking into account (B.2) and (B.3) we obtain

$$e^{i\mathbf{kr}(t_2)}e^{-i\mathbf{kr}(t_1)} = L(1) = \exp\left\{i\left[\omega\tau + \frac{\mathscr{H}}{\mathscr{H}-\hbar\omega}(\mathbf{k\rho}-\omega\tau)\right]\right\}, \qquad (B.13)$$

and this completes the solution of the unfolding problem.

APPENDIX C

ANOTHER METHOD OF INTEGRATING OVER TIME

In some cases, in particular in the discussion of processes involving particles of higher spin, it turns out to be convenient to carry out the integration with respect to time immediately after carrying out the unfolding operation (over t_1 and t_2 in formula (17)). Since the whole information concerning spins and polarizations is contained in R(t) such an approach enables us to carry out the discussion directly at the level of the matrix elements (while the method of integration over time adopted in this paper is necessarily associated with discussing the combination $R^*(t_2)R(t_1)$ as a whole).

As an illustration we consider motion in a circular orbit in a homogeneous magnetic field with frequency ω_0. Then the integrals appearing with respect to time (for one revolution) have the form

$$T_{mn} = \frac{1}{\omega_0}\int_{-\pi}^{\pi}\exp\{i\nu(|\mathbf{v}|\cos\vartheta\sin\varphi - \varphi)\}\cos^m\varphi\sin^n\varphi\,d\varphi, \qquad (C.1)$$

where $\varphi = \omega_0 t$, $\nu = E\omega/E'\omega_0 = \alpha E/|\dot{\mathbf{v}}|$; for example,

$$T_{00} = \frac{2\pi}{\omega_0}J_\nu(\nu|\mathbf{v}|\cos\vartheta). \qquad (C.2)$$

In the general case T_{mn} is expressed in terms of J_ν and its derivatives. These quantities are very similar to the classical ones and differ from them by the factor E/E' in ν. If we utilize the well known asymptotic behavior

$$J_\nu(\nu|\mathbf{v}|\cos\vartheta) = \frac{1}{\pi}\frac{\mu^{1/2}}{\sqrt{3}}K_{1/3}\left(\frac{\nu}{3}\mu^{3/2}\right) \qquad (C.3)$$

and the corresponding expressions for the derivatives of J_ν, then we can easily obtain all the expressions for the intensity of the radiation given in the text of this article. The domain of applicability of such an approach is the same as for the basic method.

[1] Sinkhrotronnoe izluchenie (Synchrotron Radiation), Editors A. A. Sokolov and I. M. Ternov, Nauka, 1966.

[2] D. Fradkin and R. Good, Revs. Modern Phys. **33**, 343 (1961).

[3] J. Schwinger, Phys. Rev. **75**, 1912 (1949).

[4] I. S. Gradshteĭn and I. M. Ryzhik, Tablitsi integralov, summ, ryadov i proizvedeniĭ (Tables of Integrals, Sums, Series, and Products), Fizmatgiz, 1962.

[5] I. P. Klepikov, Zh. Eksp. Teor. Fiz. **26**, 19 (1954).

[6] A. N. Matveev, Zh. Eksp. Teor. Fiz. **31**, 479 (1956) [Sov. Phys.-JETP **4**, 396 (1957)].

[7] A. I. Nikishov and V. I. Ritus, Zh. Eksp. Teor. Fiz. **46**, 776 (1964) [Sov. Phys.-JETP **19**, 529 (1964)].

[8] J. Schwinger, Proc. Natl. Acad. Sci. U. S. **40**, 132 (1954).

[9] V. N. Baĭer and V. M. Katkov, Phys. Letters **24A**, 327 (1967).

[10] V. N. Baĭer and V. M. Katkov, Zh. Eksp. Teor. Fiz. **52**, 1422 (1967) [Sov. Phys.-JETP **25**, 944 (1967).

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