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THE GREEN'S FUNCTION AND MOBILITY OF AN ELECTRON IN A RANDOM POTENTIAL

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The Green's function of an electron in a random potential obeying the normal-distribution law is calculated. The form of the wave function suggested by Feynman[2] is employed. It is possible to find the Green's function by taking into account a uniform electric field without assuming this field to be small.

WE calculate here the averaged Green's function of an electron moving in a random potential $U(\mathbf{r})$, which is assumed to have a normal distribution. Such a problem was considered by Andreev[1], who also presented a bibliography of this problem. In the present article we obtain Andreev's result by a different method, which makes it also possible to obtain the next term of the expansion in the exponential. In addition, it is possible to find with the aid of the proposed method (in the same approximation) the Green's function in a random field $U(\mathbf{r})$ and in an alternating homogeneous electric field without assuming the latter to be small. This problem is important for calculations of the conductivity of a disordered system (in particular, in strong fields).

The idea of the method consists in using the space-time approach to quantum mechanics developed by Feynman[2]. The Green's function (just like the wave function) is written in the form of a continuous integral over the virtual paths of a classical particle. It turns out that at this stage it is easy to average the G-function over the ensemble corresponding to the random field $U(\mathbf{r})$. The latter circumstance is, of course, the consequence of the assumed Gaussian character of the distribution. It is then possible to calculate approximately the continual integral and to obtain a closed expression for the G-function. The suggestion that the Feynman formulation of quantum mechanics be used belongs to V. L. Pokrovskiĭ.

In a time-independent random field $U(\mathbf{r})$, the Green's function depends on $\mathbf{r}$, $\mathbf{r}'$, and the time difference $t - t'$, which we shall denote by t. We break up the interval $[0, t]$ into N elementary intervals of the duration $\epsilon = t/N$. For $G(\mathbf{r}, \mathbf{r}'; t)$ we can write (atomic units)

$$G(\mathbf{r},\mathbf{r}';t)= i\sigma(t) \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \frac{1}{A}\int \exp\left[\frac{i}{2\varepsilon}\sum_{k=1}^{N}(\mathbf{r}_k-\mathbf{r}_{k-1})^2 - i\varepsilon\sum_{k=1}^{N}U(\mathbf{r}_k)\right] d\mathbf{r}_1\ldots d\mathbf{r}_{N-1}. \tag{1}$$

We have chosen here a retarded Green's function; $\sigma(t)$ is the step function, equal to zero at $t < 0$ and to unity at $t > 0$; $A = (2\pi i\epsilon)^{3N/2}$; $\mathbf{r}_0 = \mathbf{r}'$, $\mathbf{r}_N = \mathbf{r}$; the integration is carried out only over the internal points $\mathbf{r}_k$, where $0 < k < N$.

The averaging of $G(\mathbf{r}, \mathbf{r}'; t)$ reduces to determining the mean value of $\exp\left[-i\epsilon\sum_{k=1}^{N}U(\mathbf{r}_k)\right]$ over the ensemble. We choose the zero energy such that $\langle U\rangle = 0$; the angle brackets denote averaging. By virtue of the normal distribution of the random function $U(\mathbf{r})$, all the mean values of the products are expressed in terms of the binary correlation function $W(\mathbf{r})$:

$$\langle U(\mathbf{r}_1)U(\mathbf{r}_2)\rangle = W(\mathbf{r}_1-\mathbf{r}_2). \tag{2}$$

We then obtain for the sought mean value

$$\left\langle \exp\left[-i\varepsilon\sum_{k=1}^{N}U(\mathbf{r}_k)\right]\right\rangle = \exp\left[-\frac{\varepsilon^2}{2}\sum_{k,k'=1}^{N}W(\mathbf{r}_k-\mathbf{r}_{k'})\right]. \tag{3}$$

The correctness of (3) can be readily verified by expanding the exponential in a series and averaging term by term.

We introduce new variables $\mathbf{x}_k = \mathbf{r}_k - \mathbf{r}_{k-1}$ and complement (1) by integration with respect to $\mathbf{r}_0$, multiplying the integrand by $\delta(\mathbf{r}_0 - \mathbf{r}')$. The double sum in (3) can be transformed by using the evenness of $W(\mathbf{r})$ (see (2)):

$$\frac{1}{2}\varepsilon^2\sum_{k,k'}^{N}W(\mathbf{r}_k-\mathbf{r}_{k'})=\varepsilon^2\sum_{k>k'}^{N}W(\mathbf{r}_k-\mathbf{r}_{k'})+\frac{1}{2}\varepsilon^2 NW(0). \tag{4}$$

The second term in (4) vanishes in the limit as $\epsilon \to 0$, $N\epsilon \to t$. As a result we obtain

$$\langle G(\mathbf{r},\mathbf{r}';t)\rangle = \langle G(\mathbf{r}-\mathbf{r}';t)\rangle = i\sigma(t)\lim_{\varepsilon\to 0}\frac{1}{A}\int\exp\left[\frac{i}{2\varepsilon}\sum_{k=1}^{N}\mathbf{x}_k^2-\varepsilon^2\sum_{k>k'}^{N}W(\mathbf{x}_k+\ldots+\mathbf{x}_{k'+1})\right] \times\delta\left(\mathbf{r}-\mathbf{r}'-\sum_{k=1}^{N}\mathbf{x}_k\right)d\mathbf{x}_1\ldots d\mathbf{x}_N. \tag{5}$$

It is convenient to go over to a Fourier representation in $\mathbf{r} - \mathbf{r}'$:

$$\langle G(\mathbf{p},t)\rangle = G_0(\mathbf{p},t)\lim_{\varepsilon\to 0}\frac{1}{A}\int\exp\left[\frac{i}{2\varepsilon}\sum_{k=1}^{N}(\mathbf{x}_k-\varepsilon\mathbf{p})^2 - \varepsilon^2\sum_{k>k'}^{N}W(\mathbf{x}_k+\ldots+\mathbf{x}_{k'+1})\right]d\mathbf{x}_1\ldots d\mathbf{x}_N, \tag{6}$$

where $G_0(\mathbf{p}, t)$ is the Green's function of the free particle: $G_0 = i\sigma(t)\exp(-ip^2t/2)$. Finally, we reduce the quadratic form in (6) to diagonal form by shifting the origin, $\mathbf{x}_k = \mathbf{z}_k + \epsilon\mathbf{p}$:

$$\langle G(\mathbf{p},t)\rangle = G_0(\mathbf{p},t)\lim_{\varepsilon\to 0}\frac{1}{A}\int\exp\left[\frac{i}{2\varepsilon}\sum_{k=1}^{N}\mathbf{z}_k^2 - \varepsilon^2\sum_{k>k'}^{N}W(\varepsilon\mathbf{p}(k-k')+\mathbf{z}_k+\ldots+\mathbf{z}_{k'+1})\right]d\mathbf{z}_1\ldots d\mathbf{z}_N. \tag{7}$$

So far we have made no approximations. We now expand $W(\epsilon\mathbf{p}(k-k') + \mathbf{z}_k + \ldots + \mathbf{z}_{k'+1})$ in a Taylor series about the point $\epsilon\mathbf{p}(k-k')$. To this end it is necessary in each case that the sum $\mathbf{z}_k + \ldots + \mathbf{z}_{k'+1}$ be much smaller than the correlation length L, that is, than the characteristic dimension of the function $W(\mathbf{r})$. Inasmuch as $|\mathbf{z}_k| \sim \sqrt{\epsilon}$ in the important integration region, and the number of terms in this sum is of the order of N, we obtain the condition $\sqrt{t} \ll L$ (in order of magnitude $|\mathbf{z}_k + \ldots + \mathbf{z}_{k'+1}| \sim (\mathbf{z}_k^2 + \ldots + \mathbf{z}_{k'+1}^2)^{1/2} \sim \sqrt{N\epsilon}$). Thus,

$$W(\varepsilon\mathbf{p}(k-k') + \mathbf{z}_k + \ldots + \mathbf{z}_{k'+1}) = W(\varepsilon\mathbf{p}(k-k')) + \nabla W(\varepsilon\mathbf{p}(k-k'))(\mathbf{z}_k + \ldots + \mathbf{z}_{k'+1}) + \ldots . \quad (8)$$

Retaining the first term in (8), we obtain Andreev's result

$$\langle G_1(\mathbf{p}, t)\rangle = G_0(\mathbf{p}, t) \lim_{\varepsilon\to 0} \exp\Big[-\varepsilon^2 \sum_{k>k'}^{N} W(\varepsilon\mathbf{p}(k-k'))\Big]$$

$$= G_0(\mathbf{p}, t) \exp\Big[-\frac{1}{2}\int_0^t W(\mathbf{p}t' - \mathbf{p}t'')\,dt'\,dt''\Big]. \quad (9)$$

To clarify the character of the corrections to formula (9), we take into account the next term of the expansion (8). We emphasize that this results in a correction in the exponential.

The problem consists of calculating the coefficient $\mathbf{c}_k$ preceding $\mathbf{z}_k$ in the triple sum

$$\sum_{k=1}^{N}\sum_{k'=1}^{k-1} \nabla W(\varepsilon\mathbf{p}(k-k')) \sum_{i=k'+1}^{k} \mathbf{z}_i.$$

Representing this sum in the form $\sum_{k=1}^{N} \mathbf{c}_k\mathbf{z}_k$, we can again shift along $\mathbf{z}_k$ so as to make the quadratic form in (7) canonical. After straightforward but cumbersome manipulations, we get

$$\mathbf{c}_k = \sum_{j=1}^{k} j\nabla W(\varepsilon\mathbf{p}j) + k\sum_{j=k+1}^{N} \nabla W(\varepsilon\mathbf{p}j), \quad k \leqslant \frac{N}{2};$$

$$\mathbf{c}_k = \sum_{j=1}^{N-k} j\nabla W(\varepsilon\mathbf{p}j) + (N-k+1)\sum_{j=N-k+1}^{k} \nabla W(\varepsilon\mathbf{p}j) + k\sum_{j=k+1}^{N} \nabla W(\varepsilon\mathbf{p}j), \quad k > \frac{N}{2}. \quad (10)$$

Going over from sums to integrals in accordance with the rule $\epsilon\sum \to \int dt$, we obtain the following expression for $\langle G_2(\mathbf{p}, t)\rangle$:

$$\langle G_2(\mathbf{p}, t)\rangle = \langle G_1(\mathbf{p}, t)\rangle + \exp\frac{i}{4}\Big\{\int_0^{t/2}\Big[\int_0^{t'} \tau\nabla W\,d\tau + t'\int_0^{t} \nabla W\,d\tau\Big]^2 dt'$$

$$+ \int_{t/2}^{t}\Big[\int_0^{t-t'} \tau\nabla W\,d\tau + (t-t')\int_{t-t'}^{t'} \nabla W\,d\tau + t'\int_{t'}^{t} \nabla W\,d\tau\Big]^2 dt'\Big\}, \quad (11)$$

where $\nabla W \equiv \nabla W(\mathbf{p}\tau)$.

Thus, the average Green's function is

$$\langle G(\mathbf{p}, t)\rangle = G_0(\mathbf{p}, t)\exp(I_1 + I_2 + \ldots), \quad (12)$$

where I_1 and I_2 are given by formulas (9) and (11). At large values of the time $t \gg L/p$, both terms in the exponent of (12) increase linearly in t, so that the Green's function decreases like $\langle G(\mathbf{p}, t)\rangle \sim G_0(\mathbf{p}, t)\exp(-t/T)$. This region exists if the condition $t \gg L/p$ is compatible with the condition $\sqrt{t} \ll L$ presented above, that is, we must have $pL \gg 1$. When $t \ll L/p$ we get

$$\langle G(\mathbf{p}, t)\rangle \approx G_0(\mathbf{p}, t)\exp\Big[-\frac{W(0)t^2}{2} + i\frac{13}{240}(\nabla W(0))^2 t^5 + \ldots\Big]. \quad (13)$$

The second term in the exponential of (13) can be neglected only if $W^2(0)t^5 \ll L^2$. This limits the applicability of formula (9) at small values of the time. For large values of t ($t \gg L/p$) the correction to (9) is given by the factor $\exp[i\gamma W^2(0)L^2p^{-4}t]$, where γ is of the order of unity.

The calculation of the next terms of the expansion in (12) is much more complicated and leads apparently to even more cumbersome (and therefore useless) formulas than (11). We can, however, draw a general conclusion from the foregoing calculations: when t is large the law governing the decrease of $G(\mathbf{p}, t)$ is exponential, and when t is small it is Gaussian. The last statement is exact, since its validity follows from the exact formula (7):

$$\varepsilon^2\sum_{k>k'}^{N} W(\mathbf{z}_k + \ldots + \mathbf{z}_{k'+1}) \to \frac{1}{2}W(0)t^2 \quad \text{as} \quad t = N\varepsilon \to 0.$$

This means that the farthest "tail" of the density of states at negative energies is always described by the Gaussian exponential $\exp[-E^2/2W(0)]$. The asymptotic form of the density of states obtained in the paper of Zittartz and Langer[3] has a different form and depends on the dimensionality of space. However, the applicability of their formula is limited by the condition $|E| \ll L^{-2}$, whereas in this paper we assume in fact that $L \to 0$ (the correlator is equal to a δ-function).

We shall now show how to take into account, within the framework of the proposed method, a homogeneous electric field $\mathbf{F}(t)$. We represent the Green's function $G(\mathbf{r}, \mathbf{r}'; t, t_0)$, in a form similar to (1):

$$G = i\sigma(t-t_0)\lim_{\varepsilon\to 0}\frac{1}{A}\int \exp\Big[\frac{i}{2\varepsilon}\sum_{k=1}^{N}(\mathbf{r}_k - \mathbf{r}_{k-1})^2 - i\varepsilon\sum_{k=1}^{N}(U(\mathbf{r}_k) - \mathbf{F}(t_k)\mathbf{r}_k)\Big]d\mathbf{r}_1\ldots d\mathbf{r}_{N-1}. \quad (14)$$

Here, as before, $\mathbf{r}_0 = \mathbf{r}'$, $\mathbf{r} = \mathbf{r}_N$; $t_k = t_0 + k\epsilon$, $t - t_0 = N\epsilon$. Introducing the variables $\mathbf{x}_k = \mathbf{r}_k - \mathbf{r}_{k-1}$, we obtain an expression for the G-function, where the exponential under the integral sign will contain terms that are quadratic and linear in $\mathbf{x}_k$. The coefficients of the linear terms depend on $\mathbf{F}(t)$. We then shift the origin of $\mathbf{x}_k$ and obtain, in an approximation corresponding to (9), the Fourier component G with respect to the difference $\mathbf{r} - \mathbf{r}'$:

$$\langle G(\mathbf{p}; \mathbf{r}; t, t_0)\rangle = i\sigma(t-t_0)\exp\Big\{i\mathbf{r}\int_{t_0}^{t}\mathbf{F}(\tau)\,d\tau - \frac{i}{2}\int_{t_0}^{t}\Big[\mathbf{p} + \int_{t_0}^{\tau}\mathbf{F}(\tau')\,d\tau'\Big]^2 d\tau\Big\}$$

$$\times\exp\Big\{-\frac{1}{2}\int_0^{t} W\Big[\int_{t''}^{t'}\Big(\mathbf{p} + \int_{t_0}^{\tau}\mathbf{F}(\tau')\,d\tau'\Big)d\tau\Big]dt'\,dt''\Big\}. \quad (15)$$

An investigation of (15) shows that the character of the attenuation of the Green's function with time, and consequently also the mobility of the particle[1)], depends strongly on the field. In the case of a constant field there appears in the problem a new characteristic time $t_F \equiv \sqrt{L/F}$. It turns out that when $|t - t_0| \gg t_F$ the func-

[1)]If the G-function attenuates like $\exp(-t/\tau)$, in the $\mathbf{p}$,t representation, then τ is the relaxation time which determines the mobility of the electron.

tion $G(\mathbf{p}; \mathbf{r}; t - t_0)$ attenuates not exponentially but in accordance with a power law.

Let us assume that the correlation function is isotropic, that is, it depends only on the modulus $|\mathbf{r} - \mathbf{r}'| \equiv \rho$. We determine the antiderivative of $W(\rho)$ in accordance with the formula

$$V(\rho) = \int_0^\rho W(\rho')\,d\rho'.$$

We then obtain from (15) for $t - t_0 \gg t_F$ (see the Appendix)

$$\langle G(\mathbf{p}; \mathbf{r}; t - t_0)\rangle = G_0 \exp\left(\int_0^\infty \frac{V_\infty - V(2t|\mathbf{p} + \mathbf{F}t|)}{|\mathbf{p} + \mathbf{F}t|}\,dt\right)$$
$$\times \left|\frac{\mathbf{pn} + F(t - t_0) + |\mathbf{p} + \mathbf{F}(t - t_0)|}{\mathbf{pn} + p}\right|^{-V_\infty/F}, \tag{16}$$

where $\mathbf{n}$ is a unit vector along the field and G_0 is the Green's function in the field without account of relaxation.

Besides the already mentioned inequalities $t_F \ll |t - t_0| \ll L^2$, the applicability of formula (16) calls for satisfaction of one more condition. Namely, the values of the momentum p should lie outside the region in which $|t - t_0||\mathbf{p} + \mathbf{F}(t - t_0)| \lesssim L$ (the dimension of the region is of the order $\Delta p \sim L/(t - t_0)$). In particular, it is sufficient to have $p \lesssim Ft_F = \sqrt{FL}$, since $t_F \ll (t - t_0)$.

As $F \to 0$ there follows from (16) an exponential attenuation law

$$G \sim G_0 \exp(-V_\infty t/p).$$

The next term of the expansion in powers of F in the exponential is $V_\infty F t^2 \mathbf{p}\cdot\mathbf{n}/2p^2[p + \mathbf{p}\cdot\mathbf{n}]$. If this correction is not small compared with unity at characteristic values $t - t_0 \sim p/V_\infty$, then the mobility of the particle depends on the field (nonlinear regime). From this we determine the limit of the region of strong fields, namely $F \gtrsim V_\infty \sim W(0)L$. In the limit of strong nonlinearity, $F \gg V_\infty$, the Green's function is proportional to

$$\exp\left[-\frac{V_\infty}{p}\ln\frac{F(t - t_0)}{p}\right]$$

and practically does not decrease in absolute value up a time on the order of $t - t_0 \sim (p/F)\exp(F/V_\infty)$. This circumstance denotes apparently that the relaxation time of the particle decreases exponentially with increasing field.

Let us consider now a periodic field that is turned on adiabatically at $t \to -\infty$, that is,

$$\mathbf{F}(t) = \mathbf{F}_0 e^{\gamma t}\cos\omega t, \quad \gamma \to +0.$$

Putting $t_0 \to -\infty$ we get from (15)

$$\langle G(\mathbf{p}; \mathbf{r}; t)\rangle = G_0 \exp\left\{-\frac{1}{2}\int_0^{t-t_0} W\left[\mathbf{p}(t' - t'')\right.\right.$$
$$\left.\left.-\frac{\mathbf{F}_0}{\omega^2}(\cos\omega t' - \cos\omega t'')\right]dt'\,dt''\right\}. \tag{17}$$

Confining ourselves to the case of a high frequency field $F_0 \ll L\omega^2$, neglecting the last two terms in the argument of W in (17), and then estimating the correction to the exponential necessitated by these terms, we obtain the condition for the applicability of such an approximation, $F_0 W(0) L/\omega^2 p^2 \ll 1$. If this inequality is satisfied, then, as seen from (17), the damped factor of the Green's function has the same form as in the absence of a field:

$$\exp\left[-\frac{(t - t_0)}{p}\int_0^\infty W(z)\,dz\right].$$

Consequently, the relaxation time of a particle with momentum p is

$$\tau = p\left[\int_0^\infty W(z)\,dz\right]^{-1} \sim \frac{p}{W(0)L}.$$

In conclusion we note that within the framework of the proposed method it would be easy to take into account also a homogeneous magnetic field with arbitrary time dependence. Indeed, the vector potential of such a field is linear in r. This allows us to perform the same manipulations that led to formula (15).

APPENDIX

We shall obtain here the asymptotic form of the Green's function at large time $(t - t_0 \gg t_F)$ in a strong constant electric field. The problem consists of calculating the double integral contained in (15):

$$I = \frac{1}{2}\int_{t_0}^{t} W\left\{\mathbf{p}(t' - t'') + \frac{1}{2}\mathbf{F}[(t' - t_0)^2 - (t'' - t_0)^2]\right\}dt'dt''$$
$$= \frac{1}{2}\int_0^{t-t_0} W[\mathbf{p}(t' - t'') + \frac{1}{2}\mathbf{F}(t'^2 - t''^2)]\,dt'\,dt''. \tag{A.1}$$

The correlation function W(r) will be assumed to depend only on the modulus of the argument. We replace the variables in (A.1) in accordance with the formulas $t' - t'' = x$ and $t' + t'' = 2y$ (the Jacobian of the transformation is equal to unity). The integration in the (x, y) plane is within a rhombus with sides satisfying the equations $y \pm x/2 = 0$ and $y \pm x/2 = t - t_0$. We first integrate with respect to $x = t' - t''$, and the fact that W is even allows us to confine ourselves only to that half of the rhombus in which $x > 0$:

$$I = \int_0^{\frac{1}{2}(t-t_0)} dy \int_0^{2y} W(|\mathbf{p} + \mathbf{F}y|x)\,dx + \int_{\frac{1}{2}(t-t_0)}^{t-t_0} dy \int_0^{2(t-t_0-y)} W(|\mathbf{p} + \mathbf{F}y|x)\,dx$$
$$= \int_0^{\frac{1}{2}(t-t_0)} \frac{V(2y|\mathbf{p} + \mathbf{F}y|)}{|\mathbf{p} + \mathbf{F}y|}\,dy + \int_{\frac{1}{2}(t-t_0)}^{t-t_0} \frac{V[2(t - t_0 - y)|\mathbf{p} + \mathbf{F}y|]}{|\mathbf{p} + \mathbf{F}y|}\,dy. \tag{A.2}$$

We transform (A.2), replacing the variable in the last integral $t - t_0 - y = z$:

$$I = \int_0^{\frac{1}{2}(t-t_0)} \frac{V(2y|\mathbf{p} + \mathbf{F}y|) - V(\infty)}{|\mathbf{p} + \mathbf{F}y|}\,dy + V(\infty)\int_0^{t-t_0}\frac{dy}{|\mathbf{p} + \mathbf{F}y|}$$
$$+ \int_0^{\frac{1}{2}(t-t_0)} \frac{V(2z|\mathbf{p} + \mathbf{F}(t - t_0) - \mathbf{F}z|) - V(\infty)}{|\mathbf{p} + \mathbf{F}(t - t_0) - \mathbf{F}z|}\,dz. \tag{A.3}$$

The first integral in (A.3) converges when $t - t_0 \to \infty$, and it can be extended to infinity. The last term tends to zero, since the numerator of the integrand is bounded, and the denominator is not smaller than $[F(t - t_0)/2 - p] \to \infty$.

Finally, the second integral in (A.3) is equal to

$$V(\infty)\int_0^{t-t_0}\frac{dy}{|\mathbf{p} + \mathbf{F}y|} = V(\infty)\int_0^{t-t_0}[p^2 + 2\mathbf{pn}Fy + F^2y^2]^{-1/2}\,dy$$
$$= \frac{V(\infty)}{F}\ln\left|\frac{\mathbf{pn} + F(t - t_0) + |\mathbf{p} + \mathbf{F}(t - t_0)|}{\mathbf{pn} + p}\right|$$

From this we get the asymptotic formula (16) for the Green's function in the case of a constant field.

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