MAGNETIC PROPERTIES OF A SPIN ARRAY WITH TWO SUBLATTICES

V. M. KONTOROVICH and V. M. TSUKERNIK

Institute of Radiophysics and Electronics, Academy of Sciences, Ukrainian S.S.R.; Physico-technical Institute, Academy of Sciences, Ukrainian S.S.R.

Submitted April 27, 1967

Zh. Eksp. Teor. Fiz. 53, 1167–1176 (September, 1967)

We consider an exactly soluble model of a spin array with two sublattices, in which the nearest-neighbor interaction is connected solely with the transverse spin components and is different for left-hand and right-hand neighbors. The system reduces to a perfect quasi-particle Fermi gas. In the excitation spectrum which consists of two branches there is a gap which depends on the magnetic field. At zero temperature the magnetic susceptibility has two square root singularities with respect to the field in the points where the gap vanishes. The magnetic field and temperature dependence of the susceptibility of the array is similar to the corresponding dependence for an antiferromagnetic. We have calculated the spin pair-correlators. There is no long-range order in the system. An important difference in the classical solution of the same problem lies in the fact that in fields below the saturation field there is in the classical case a gapless "acoustic" branch of the spectrum which does not exist in actual fact.

1. A number of authors[1,2] have shown that one can find an exact solution of a one-dimensional model of a chain of spins with anisotropic nearest-neighbor interactions which does not contain the z-components of the spin operators (the x-y model) and that one can thus obtain complete information about the properties of the model.

In the present paper, in contradistinction to the previous ones, we consider a chain with two sublattices where the interaction constants are different for left-hand and right-hand neighbors. A magnetic field along the z axis is applied to the chain and the effective magnetic moments of the spins may also be different. The inequality of the interaction constants leads to a field-dependent gap in the spectrum and a peculiar behavior of the magnetic and mechanical moments which is connected with this. There are at zero temperature two phase transition points, as far as the field is concerned, in the system, where the magnetic susceptibility has a square-root singularity. At finite temperatures the singularity is smeared out.

The system reminds one in many respects of an antiferromagnetic, although there is no long-range order in our case. A comparison between the exact quantum consideration and the classical one shows that even if we forget about the difference in the statistics of the excitations there is an important difference both in the excitation spectrum and in the properties of the ground state.

2. The Hamiltonian of the system considered has the form

$$\mathcal{H} = -J_1 \sum_n (s_{n1}^x s_{n2}^x + s_{n1}^y s_{n2}^y) - J_2 \sum_n (s_{n2}^x s_{n+1,1}^x + s_{n2}^y s_{n+1,1}^y) - H \sum_n (\mu_1 s_{n1}^z + \mu_2 s_{n2}^z); \quad (1)$$

s_{nj} is the spin operator of the j-th sublattice (j = 1, 2) in the n-th elementary cell; μ_1 and μ_2 are, respectively, the effective magnetic moments of the atoms of the first and the second sublattice; H is the external magnetic field; J_1 the interaction constant inside the cell, and J_2 the interaction constant between the cells (we shall assume that $J_1 > 0$, $J_2 > 0$).

We shall change from the spin operators to Fermi operators through the transformations

$$s_{n1}^+ = \prod_{m<n} \sigma_{m1}\sigma_{m2} a_{n1}, \quad s_{n2}^+ = \prod_{m<n} \sigma_{m1}\sigma_{m2}\sigma_{n1} a_{n2},$$

$$s_{n1}^- = \prod_{m<n} \sigma_{m1}\sigma_{m2} a_{n1}^+, \quad s_{n2}^- = \prod_{m<n} \sigma_{m1}\sigma_{m2}\sigma_{n1} a_{n2}^+, \quad (2)$$

where $s_{nj}^{\pm} = s_{nj}^x \pm i s_{nj}^y$, $\sigma_{mj} = 2s_{mj}^z = 1 - 2a_{mj}^+ a_{mj}$, and the operators a_{mj}, a_{mj}^+ satisfy the fermion commutation relations. These transformations are completely analogous to those applied earlier.[3-5] Substituting (2) into the Hamiltonian (1) and transforming to the Fourier components of the Fermi operators

$$a_{\lambda j} = \frac{1}{\sqrt{N}} \sum_n a_{nj} e^{-i\lambda n} \qquad (-\pi \leqslant \lambda < \pi),$$

where N is the number of cells, we get

$$\mathcal{H} = -\frac{1}{2} \sum_\lambda \{(J_1 + J_2 e^{-i\lambda}) a_{\lambda 1}^+ a_{\lambda 2} + (J_1 + J_2 e^{i\lambda}) a_{\lambda 2}^+ a_{\lambda 1}\} + \mu_1 H \sum_\lambda a_{\lambda 1}^+ a_{\lambda 1} + \mu_2 H \sum_\lambda a_{\lambda 2}^+ a_{\lambda 2} - N \frac{\mu_1 + \mu_2}{2} H. \quad (3)$$

To diagonalize the Hamiltonian (3) using a canonical unitary transformation we change to new Fermi operators $b_{\lambda 1}$, $b_{\lambda 2}$:

$$a_{\lambda 1} = u_{11}^\lambda b_{\lambda 1} + u_{12}^\lambda b_{\lambda 2}, \quad a_{\lambda 2} = u_{21}^\lambda b_{\lambda 1} + u_{22}^\lambda b_{\lambda 2}. \quad (4)$$

It is convenient to write Eqs. (3) and (4) in a compact form by introducing matrix notation

$$a_\lambda = \begin{pmatrix} a_{\lambda 1} \\ a_{\lambda 2} \end{pmatrix}, \quad b_\lambda = \begin{pmatrix} b_{\lambda 1} \\ b_{\lambda 2} \end{pmatrix}, \quad U_\lambda = \begin{pmatrix} u_{11}^\lambda & u_{12}^\lambda \\ u_{21}^\lambda & u_{22}^\lambda \end{pmatrix},$$

$$A_\lambda = \begin{pmatrix} \mu_1 H & -\frac{1}{2}(J_1 + J_2 e^{-i\lambda}) \\ -\frac{1}{2}(J_1 + J_2 e^{i\lambda}) & \mu_2 H \end{pmatrix}.$$

In this notation the Hamiltonian (3) and the transformation (4) can be written as follows:

$$\mathcal{H} = \sum_\lambda a_\lambda^+ A_\lambda a_\lambda - N \frac{\mu_1 + \mu_2}{2} H, \quad (3')$$

$$a_\lambda = U_\lambda b_\lambda \quad (U_\lambda^{-1} = U_\lambda^+). \quad (4')$$

Substituting (4′) into (3′) and using the unitarity of the matrix U_λ, we get

$$\mathcal{H} = \sum_\lambda b_\lambda^+ (U_\lambda^{-1} A_\lambda U_\lambda) b_\lambda - N\frac{\mu_1+\mu_2}{2}H.$$

The diagonalization of the Hamiltonian thus reduces to the requirement that the matrix $U_\lambda^{-1}A_\lambda U_\lambda$ be diagonal, i.e.,

$$U_\lambda^{-1}A_\lambda U_\lambda = \varepsilon_\lambda,\ \varepsilon_\lambda = \begin{pmatrix} \varepsilon_{\lambda 1} & 0 \\ 0 & \varepsilon_{\lambda 2}\end{pmatrix}.$$

We change to a set of linear equations for the matrix elements of the unitary transformation:

$$A_\lambda U_\lambda = U_\lambda \varepsilon_\lambda. \tag{5}$$

We find from this the two branches of the spectrum:

$$\begin{aligned} \varepsilon_{\lambda 1} &= {}^1/_2(\mu_1+\mu_2)H + {}^1/_2[(\mu_1-\mu_2)^2H^2 + |J_1 + J_2 e^{i\lambda}|^2]^{1/2}, \\ \varepsilon_{\lambda 2} &= {}^1/_2(\mu_1+\mu_2)H - {}^1/_2[(\mu_1-\mu_2)^2H^2 + |J_1 + J_2 e^{i\lambda}|^2]^{1/2}. \end{aligned} \tag{6}$$

The solution U_λ of the set (5) has the form

$$U_\lambda = \begin{pmatrix} e^{i\varphi} & \dfrac{J_1 + J_2 e^{-i\lambda}}{2(\mu_1 H - \varepsilon_{\lambda 2})} e^{i\psi} \\ \dfrac{J_1 + J_2 e^{i\lambda}}{2(\mu_2 H - \varepsilon_{\lambda 1})} e^{i\varphi} & e^{i\psi} \end{pmatrix} \left[\frac{\mu_2 H - \varepsilon_{\lambda 1}}{\varepsilon_{\lambda 2} - \varepsilon_{\lambda 1}}\right]^{1/2}, \tag{7}$$

where φ and ψ are arbitrary phases which do not appear anywhere in the following. Noting that $\frac{1}{2}(\mu_1 + \mu_2)H = \frac{1}{2}(\epsilon_{\lambda 1} + \epsilon_{\lambda 2})$ we get finally

$$\mathcal{H} = \sum_{\lambda j} (b_{\lambda j}^+ b_{\lambda j} - {}^1/_2)\varepsilon_{\lambda j} \quad (j = 1, 2). \tag{8}$$

The two branches of the spectrum correspond to the fact that the chain consists of two sublattices. It is clear from Eqs. (6) that $\epsilon_{\lambda 1} > 0$ for all values of $H \geq 0$. As far as $\epsilon_{\lambda 2}$ is concerned, we must distinguish three ranges of magnetic field values. In the first range when $0 < H < H_1$, where

$$H_1 = |J_1 - J_2| / 2\sqrt{\mu_1\mu_2}, \tag{9}$$

$\epsilon_{\lambda 2}$ is negative for all λ. In the second range when $H_1 < H < H_2$, where

$$H_2 = (J_1 + J_2) / 2\sqrt{\mu_1\mu_2}, \tag{10}$$

$\epsilon_{\lambda 2} > 0$ when $|\lambda| > \lambda_{cr}$ and $\epsilon_{\lambda 2} < 0$ when $|\lambda| < \lambda_{cr}$, where

$$\lambda_{cr} = \arccos\frac{2H^2 - (H_1^2 + H_2^2)}{H_2^2 - H_1^2}. \tag{11}$$

Finally, in the third range when $H > H_2$, $\epsilon_{\lambda 2}$ is positive for all λ. In correspondence with such a behavior of the spectrum the ground state is re-ordered when the magnetic field changes from H_1 to H_2.

This re-ordering can be directly taken into account in the Hamiltonian (8) if we introduce the creation and annihilation operators of the "true" quasi-particles—the excitations above the ground state of the system. To do this we define new Fermi operators c_λ using the equations

$$c_{\lambda 1} = b_{\lambda 1}, \quad c_{\lambda 2} = \begin{cases} b_{\lambda 2} & \text{when } |\lambda| > \lambda_{cr} \\ b_{\lambda 2}^+ & \text{when } |\lambda| < \lambda_{cr} \end{cases}.$$

Using the new operators we can write the Hamiltonian (8) as follows:

$$\mathcal{H} = \sum_{\lambda j} (c_{\lambda j}^+ c_{\lambda j} - {}^1/_2)|\varepsilon_{\lambda j}|, \tag{12}$$

where the ground state corresponds to zero occupation numbers of the newly introduced quasiparticles, i.e.,

$$\mathcal{H}_0 = -\frac{1}{2}\sum_{\lambda j}|\varepsilon_{\lambda j}| = -N\frac{\mu_1+\mu_2}{2}H + \begin{cases} \sum_\lambda \varepsilon_{\lambda 2} & \text{when} \leqslant H \leqslant H_1 \\ \sum_{|\lambda|<\lambda_{cr}} \varepsilon_\lambda & \text{when } H_1 \leqslant H \leqslant H_2 \\ 0 & \text{when } H_2 \leqslant H \end{cases} \tag{13}$$

When $H < H_1$ it is clear from (6) and (12) that the energy spectrum has a gap,

$$\Delta(H) = {}^1/_2[(\mu_1-\mu_2)^2H^2 + (J_1 - J_2)^2]^{1/2} - {}^1/_2(\mu_1+\mu_2)H,$$

which is equal to $\frac{1}{2}|J_1 - J_2|$ when $H = 0$ and which tends to zero when $H = H_1$ (see Fig. 1). The existence of the gap is essentially connected with the fact that the interaction constants J_1 and J_2 are different.

From the classical point of view the system considered should have a gapless "acoustic" branch. To understand the absence of an "acoustic" branch it is simplest to consider the limiting case $J_2 = 0$. The chain then splits into non-interacting cells, each of which has four quantum states corresponding to the two spins in a cell. One checks easily that in that case there is a two-fold degenerate level corresponding to an elementary excitation and differing from the ground state by the "gap" $J_1/2$. When we switch to an arbitrarily weak interaction J_2 between the cells the corresponding level of the chain splits into two bands (which coincide at $H = 0$) and the gap remains.

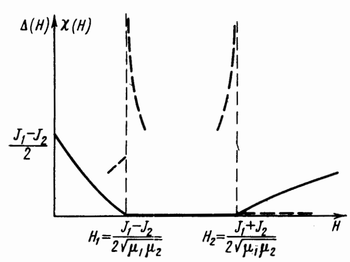

Fig. 1. Magnetic field dependence of the gap in the energy spectrum (full-drawn line) and of the susceptibility at $T = 0$ (dotted line). When $H < H_1$ the ground state is "antiferromagnetic" and the gap Δ decreases with increasing field. In the interval from H_1 to H_2, where $\Delta = 0$, a re-ordering of the ground state occurs. When $H < H_2$ the ground state is "ferromagnetic", and there is again a gap which increases with the field. In the points H_1 and H_2 the susceptibility has a square-root singularity at $T = 0$.

3. Using Eq. (12) for the ground state energy we can find the magnetic moment of the chain at zero temperature for different ranges of the magnetic field.

In the first range

$$M_0(H) = \frac{H}{2}\sum_\lambda \frac{(\mu_1-\mu_2)^2}{[(\mu_1-\mu_2)^2H^2 + |J_1 + J_2 e^{i\lambda}|^2]^{1/2}}, \tag{14}$$

or, changing from a sum to an integral,

$$M_0(H) = \frac{NH(\mu_1-\mu_2)^2 k}{2\pi\sqrt{J_1J_2}} K(k) \quad (0 \leqslant H \leqslant H_1), \tag{15}$$

$$k^2 = \frac{4J_1J_2}{(\mu_1-\mu_2)^2H^2 + (J_1+J_2)^2}$$

where $K(k)$ is the complete elliptical integral of the first kind.

It is clear from Eq. (14) that the moment $M_0 = 0$ for $H = 0$, i.e., there is no spontaneous magnetization. Moreover, $M_0(H) = 0$ if $\mu_1 = \mu_2$ in the whole of the first range. We note that if $J_1 = J_2 = J$, $\mu_1 \neq \mu_2$ the magnetization has a singularity at H = 0, viz.,

$$M_0(H) = -\frac{N|\mu_1-\mu_2|}{\pi}\frac{|\mu_1-\mu_2|H}{2J}\ln\frac{|\mu_1-\mu_2|H}{2J}.$$

In the second range

$$M_0(H) = \frac{N(\mu_1+\mu_2)}{2}\left(1-\frac{\lambda_{cr}}{\pi}\right)+\frac{NH(\mu_1-\mu_2)^2k}{2\pi\sqrt{J_1J_2}}F\left(\frac{\lambda_{cr}}{2},\ k\right), \quad (16)$$

where

$$F\left(\frac{\lambda_{cr}}{2},\ k\right)=\int_0^{\lambda_{cr}/2}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

is the incomplete elliptical integral of the first kind, and λ_{cr} and k are, respectively, defined by Eqs. (11) and (15). At the boundaries of the range $\lambda_{cr}(H_1) = \pi$, $\lambda_{cr}(H_2) = 0$. In these points the derivative $\partial\lambda_{cr}/\partial H$ can be seen from (11) to have a square root singularity. Because of this there occurs a singularity in the magnetic susceptibility. We write it in the form

$$\chi = \frac{2N\mu_0 H}{\pi\sqrt{(H_2^2-H^2)(H^2-H_1^2)}}+\chi_{reg} \quad (17)$$

where χ_{reg} is the part of the susceptibility which is non-singular, $\mu_0 = \mu_1\mu_2/(\mu_1+\mu_2)$ is the reduced magnetic moment. The occurrence of a singularity in χ can be considered to be a phase transition with respect to the magnetic field. It is here important that the transition takes place at zero temperature.

In the third range $M_0(H) = \frac{1}{2}N(\mu_1+\mu_2)$, i.e., the magnetization reaches saturation at $H = H_2$.

4. The magnetic properties of the system can be tracked in more detail by considering each of the sublattices separately. To do this we evaluate the mean value of the z-component of the spin in the ground state. Using Eq. (2) for s^z in terms of Fermi operators, and also Eq. (4), we get ($\langle s_j^z\rangle \equiv \langle s_{nj}^z\rangle$, j = 1, 2).

$$\langle s_1^z\rangle = \frac{1}{2}-\frac{1}{N}\sum_{|\lambda|\leqslant\lambda_{cr}}|u_{12}^\lambda|^2, \quad \langle s_2^z\rangle = \frac{1}{2}-\frac{1}{N}\sum_{|\lambda|\leqslant\lambda_{cr}}|u_{22}^\lambda|^2. \quad (18)$$

These formulae are valid for all three field ranges, if we take in the first range $\lambda_{cr} \equiv \pi$, and in the third $\lambda_{cr} \equiv 0$.

When $0 \le H \le H_1$, it follows from (18) and (7) that

$$\langle s_1^z\rangle = -\langle s_2^z\rangle = \frac{(\mu_1-\mu_2)Hk}{2\pi\sqrt{J_1J_2}}K(k). \quad (19)$$

From this it is clear that when $H \le H_1$ the average mechanical moments at both sites in the elementary cell are equal in magnitude and opposite in direction so that the total mechanical moment vanishes. At the same time the total magnetic moment is non-vanishing if $\mu_1 \neq \mu_2$ (see (14)). Furthermore, it follows from Eq. (19) that the average spin of each of the sublattices increases with the field while it is parallel to the field in the sublattice with the larger μ and antiparallel to the field in the other sublattice.

Such a behavior of the spin system is connected with the presence of a gap in the first region and with the character of the ground state. This can be followed in detail for the example when $J_2 = 0$. The ground state of the cell can then easily be seen to be a superposition of two "antiferromagnetic" states with opposite values of the z-component in the sites, while there are "ferromagnetic" states with the same values of the z-components which are split off from the ground state. For H = 0 both "antiferromagnetic" states are equally probable. When an arbitrarily weak field is switched on that "antiferromagnetic" state in which the spin with the larger μ is parallel to the field becomes more probable. The probability of that state acquires a correction proportional to $(\mu_1-\mu_2)H/J$ (the ratio of the Zeeman energy to the gap magnitude). The correction to the probability for the second state differs in sign through the different sign in the Zeeman energy. As a result the average spin values in both sites become different from zero and proportional to $\pm(\mu_1-\mu_2)H$. In the second region where there is no gap, the average spin $\langle s_1^z\rangle$ shows a monotonic increase reaching the maximum value ½ for $H = H_2$. The average spin $\langle s_2^z\rangle$ reaches a maximum negative value for $H = H_1$ and in the second region it increases monotonically, changes sign and becomes equal to ½ at $H = H_2$ (see Fig. 2). The corresponding analytical expressions have the form

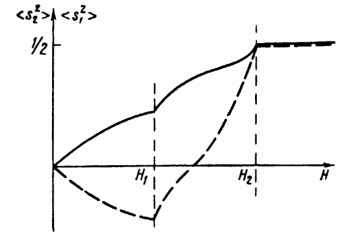

Fig. 2. Magnetic field dependence of the average sublattice spin. When $H < H_1$ the mechanical moments of the sublattices completely cancel one another while the moment of the sublattice with the smaller value of μ (dotted) is antiparallel to the field. In the second region both moments increase reaching saturation for $H = H_2$.

$$\langle s_{1,2}^z\rangle = \frac{1}{2}\left(1-\frac{\lambda_{cr}}{\pi}\right)\pm\frac{(\mu_1-\mu_2)Hk}{2\pi\sqrt{J_1J_2}}F\left(\frac{\lambda_{cr}}{2},\ k\right).$$

5. Let us now consider the magnetic properties of the system at finite temperatures. The free energy of the chain follows from (8) to be equal to

$$\mathcal{F} = -\frac{1}{\beta}\sum_{\lambda j}\ln\left(2\,\mathrm{ch}\frac{\beta\varepsilon_{\lambda j}}{2}\right), \quad \beta = \frac{1}{k_BT}. \quad (20)$$

Hence, the magnetization $M_z = -\partial\mathcal{F}/\partial H$ is determined by the equation

$$M_z = \frac{1}{2}\sum_{\lambda j}\frac{\partial\varepsilon_{\lambda j}}{\partial H}\,\mathrm{th}\frac{\beta\varepsilon_{\lambda j}}{2}. \quad (21)$$

We consider the behavior of the magnetization as function of the temperature in the field range $0 \le H \le H_1$. Using Eq. (14) we can write (21) in the form

$$M_z(H,T) = M_0(H)+\frac{\mu_1+\mu_2}{2}\sum_\lambda\left[\frac{1}{1+\exp(-\beta\varepsilon_{\lambda 2})}-\frac{1}{1+\exp(\beta\varepsilon_{\lambda 1})}\right]$$
$$-\frac{(\mu_1-\mu_2)^2H}{2}\sum_\lambda[(\mu_1-\mu_2)^2H^2+|J_1+J_2e^{i\lambda}|^2]^{-1/2}$$
$$\times\left[\frac{1}{1+\exp(\beta\varepsilon_{\lambda 1})}+\frac{1}{1+\exp(-\beta\varepsilon_{\lambda 2})}\right]. \quad (22)$$

The last term increases monotonically with temperature and as $T\to\infty$ cancels $M_0(H)$. When $\mu_1 = \mu_2$ there remains in (22) only the second term which tends to zero as $T\to 0$ or as $T\to\infty$, i.e., the magnetization has a maximum as function of the temperature (when $k_BT \sim J$). Such a temperature dependence of the magnetization is characteristic for antiferromagnetics in fields below the critical field (starting from which there is at zero temperature a non-vanishing moment[6]). We see thus that in the model considered the one-dimensional chain "simulates" antiferromagnetism. We note, however, that there is here no long range order and that the pair correlators tend to zero when the distance between the sites increases (see Sec. 6).

A finite temperature smears out the singularity in the susceptibility which occurs for T = 0 and for H, equal to H_1 or H_2. The general expression for χ has the form

$$\chi = \frac{1}{2}\sum_{\lambda j}\frac{\partial^2\varepsilon_{\lambda j}}{\partial H^2}\operatorname{th}\frac{\beta\varepsilon_{\lambda j}}{2} + \sum_{\lambda j}\left(\frac{\partial\varepsilon_{\lambda j}}{\partial H}\right)^2\frac{\beta}{4\operatorname{ch}^2(\beta\varepsilon_{\lambda j}/2)}. \quad (23)$$

The function $\delta(\epsilon, \beta) = \beta/4\cosh^2(\beta\epsilon/2)$ behaves as $\beta \to \infty$ as a δ-function of ϵ. One verifies easily that as $\beta \to \infty$ the term containing that δ-function gives the singular part of the susceptibility. When $\beta \gg J$ and $|H - H_{1,2}| \ll H$ the corresponding term gives the main contribution to the susceptibility. Using the steepness of the function $\delta(\epsilon, \beta)$ for large β, we get for H close to H_1

$$\chi(H, T)\,|_{H\approx H_1} = \frac{N\mu_0^{3/2}}{\pi}\left(\frac{2H_1}{H_2^2 - H_1^2}\right)^{1/2}\int\limits_{-\mu_0(H-H_1)}^{\infty} d\varepsilon\,\frac{\beta}{4\operatorname{ch}^2(\beta\varepsilon/2)}\cdot\frac{1}{[\varepsilon + \mu_0(H - H_1)]^{1/2}} + \chi_{\text{reg}}$$

As $\beta \to \infty$ the integral gives a square root singularity for $H \to H_1 + 0$. Introducing the dimensionless parameters

$$\xi_1 = \beta\mu_0(H - H_1)/2, \quad \xi_2 = \beta\mu_0(H_2 - H)/2,$$

we get the expression for the singular part of χ in the form

$$\chi_{\text{sing}}(H, T) = \frac{N\mu_0^{3/2}}{2\pi}\left(\frac{\beta H_1}{H_2^2 - H_1^2}\right)^{1/2}\int\limits_0^{\infty}\frac{dx}{\sqrt{x}\operatorname{ch}^2(x - \xi_1)}, \quad H - H_1 \ll H,$$
$$\chi_{\text{sing}}(H, T) = \frac{N\mu_0^{3/2}}{2\pi}\left(\frac{\beta H_2}{H_2^2 - H_1^2}\right)^{1/2}\int\limits_0^{\infty}\frac{dx}{\sqrt{x}\operatorname{ch}^2(x - \xi_2)}, \quad H_2 - H \ll H. \quad (24)$$

As $\xi_{1,2} \to \infty$ we get the zero temperature singularity in the susceptibility which was considered above, and as $\xi_{1,2} \to 0$ the square root singularity with respect to the temperature. There are thus two isolated singular points in the model. As $\xi \gg 1$ the susceptibility has narrow asymmetric maxima, the remnants of smeared-out singularities.

We note that if $J_2 = 0$, when $H_1 = H_2$, Eqs. (24) become inapplicable, since we assumed in the derivation that $\mu_0(H_2 - H_1) \gg 1/\beta$. It follows from the general Eq. (23) that in that case

$$\chi_{\text{sing}} = \frac{N\beta}{4}\left(\frac{\partial\varepsilon_2}{\partial H}\right)^2 \Big/ \operatorname{ch}^2\frac{\beta\varepsilon_2}{2} \quad (J_2 = 0).$$

Hence it is clear that for the critical values of the field when $\epsilon_2 = 0$ the magnetic susceptibility is inversely proportional to the temperature. At zero temperature χ has a δ-function-type singularity with respect to the field. It follows from (22) that the magnetic moment undergoes a jump equal to $N\mu_0$. This means that as $J_2 = 0$ there occurs a first order transition with respect to the field at zero temperature.

6. The degree of order in the chain can be characterized by the pair correlators:

$$\mathcal{P}_{mj,\,nk} = \langle s_{nj}^z s_{mk}^z\rangle - \langle s_{nj}^z\rangle\langle s_{mk}^z\rangle \quad (j, k = 1, 2). \quad (25)$$

Expressing the spin operators in terms of the Fermi operators, we find

$$\mathcal{P}_{m1,n1} = -\frac{1}{4\pi^2}\left|\int\limits_{-\pi}^{\pi} d\lambda\, e^{i(n-m)\lambda}[(u_{11}^\lambda)^2 n_{\lambda 1} + (u_{12}^\lambda)^2 n_{\lambda 2}]\right|^2,$$

$$\mathcal{P}_{m1,n2} = -\frac{1}{4\pi^2}\left|\int\limits_{-\pi}^{\pi} d\lambda\, e^{i(n-m)\lambda}(u_{11}^\lambda u_{21}^\lambda n_{\lambda 1} + u_{12}^\lambda u_{22}^\lambda n_{\lambda 2})\right|^2,$$

where $n_{\lambda j} = (e^{\beta\epsilon_{\lambda j}} + 1)^{-1}$. The values of the coefficients u_{jk}^λ were given in (7).

We restrict ourselves for the sake of simplicity to the case of zero temperature and $\mu_1 = \mu_2$. Then $n_{\lambda_1} = 0$ for all λ, and $n_{\lambda_2} = 0$ for $|\lambda| > \lambda_{cr}$ and $n_{\lambda_2} = 1$ for $|\lambda| < \lambda_{cr}$. As a result we get expressions for the correlators in the different magnetic field ranges.

When $H \le H_1$

$$\mathcal{P}_{m1,\,n1} = \mathcal{P}_{m2,\,n2} = 0,$$

$$\mathcal{P}_{m1,n2} = -\frac{1}{4\pi^2}\left|\int\limits_{-\pi}^{\pi} e^{i(n-m)\lambda}\frac{J_1 + J_2 e^{i\lambda}}{|J_1 + J_2 e^{i\lambda}|}d\lambda\right|^2.$$

i.e., in the first range both correlators are independent of the field, while the first one referring to spins on equivalent sites vanishes rigorously[1)] while the second one decreases inversely proportional to the square of the distance for large n − m.

When $H_1 \le H \le H_2$

$$\mathcal{P}_{m1,n1} = -\frac{1}{(2\pi)^2}\frac{\sin^2(n - m)\lambda_{cr}}{(n - m)^2},$$

$$\mathcal{P}_{m1,n2} = -\frac{1}{(4\pi)^2}\left|\int\limits_{-\lambda_{cr}}^{\lambda_{cr}} e^{i(n-m)\lambda}\frac{J_1 + J_2 e^{i\lambda}}{|J_1 + J_2 e^{i\lambda}|}d\lambda\right|^2.$$

For fixed n − m, the first correlator oscillates with the magnetic field vanishing for field values

$$H^{(k)} = 2^{-1/2}\left[H_1^2 + H_2^2 + (H_2^2 - H_1^2)\cos\frac{k\pi}{n - m}\right]^{1/2} \quad (k = 0, 1, \ldots, n - m)$$

in particular, when $H = H_1$ or $H = H_2$.

In the third range both correlators vanish. This is connected with the fact that a nominal magnetization is reached due to the orientation producing magnetic field.

7. Let us now consider the results to which a classical discussion leads. To do this we replace, as is usually done in such a case,[6] the spin operators by classical vectors of length s. The elementary excitation (spin wave) spectrum is classically determined as the small vibration spectrum of the spin system near its equilibrium position which is obtained from the minimum energy requirement. Minimizing the Hamiltonian (1) with respect to the spin orientation we find a (uniform) solution.

For a field $H < H_2 \equiv (J_1 + J_2)s/\sqrt{(\mu_1\mu_2)}$ the minimum is attained for

$$\cos\vartheta_1 = \frac{\mu_1 H}{(J_1 + J_2)s}\left[\frac{(\mu_2 H)^2 + s^2(J_1 + J_2)^2}{(\mu_1 H)^2 + s^2(J_1 + J_2)^2}\right]^{1/2}$$

$$\cos\vartheta_2 = \frac{\mu_2 H}{(J_1 + J_2)s}\left[\frac{(\mu_1 H)^2 + s^2(J_1 + J_2)^2}{(\mu_2 H)^2 + s^2(J_1 + J_2)^2}\right]^{1/2}$$

where ϑ_1 and ϑ_2 are the angles between the spin vectors of the first and the second sublattice and the z-axis, while the vectors of all spins lie in one plane.

For a field $H > H_2$ all spins are in the equilibrium position parallel to the z-axis. We draw attention to the

---
1)The lack of field dependence and the strict vanishing of the first correlator occurs only when $\mu_1 = \mu_2$.

fact that in a classical discussion the total spin of the lattice is non-vanishing in an arbitrarily weak field while in an exact quantal discussion the total spin vanishes up to a field $H_1 = (J_1 - J_2)s/\sqrt{(\mu_1\mu_2)}$ ($s = 1/2$).

Linearizing the equations of motion for the spins we find by standard methods the excitations which consist, as in the quantal case, of two branches. For a field $H > H_2$ for $s = 1/2$ the classical dispersion law is the same as the quantal one. However, for $H < H_2$ there is an essential difference between the classical and the quantal result which exists first of all in the occurrence of a gapless "acoustic" branch in the classical case. In the particular case $\mu_1 = \mu_2 = \mu$ the classical dispersion law has in this region the form

$$\varepsilon_{1,2}^2(\lambda) = \left(1 \pm \frac{|J_1 + J_2 e^{i\lambda}|}{J_1 + J_2}\right)\left[(J_1 + J_2)^2 s^2 \pm (\mu H^2)\frac{|J_1 + J_2 e^{i\lambda}|}{J_1 + J_2}\right].$$

There is thus a range of magnetic fields where the classical discussion leads to incorrect results. Moreover, when quantizing the classical oscillations we are led to Bose excitations while an exact treatment (for spin $1/2$) leads to Fermi statistics.

[1] E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. (N.Y.) 16, 407 (1961).
[2] S. A. Pikin and V. M. Tsukernik, Zh. Eksp. Teor. Fiz. 50, 1377 (1966) [Sov. Phys.-JETP 23, 914 (1966)].
[3] K. Meyer, Z. Naturf. 11a, 865 (1956).
[4] D. Frank, Z. Phys. 146, 615 (1956).
[5] S. Rodriguez, Phys. Rev. 116, 1474 (1959).
[6] A. I. Akhiezer, V. G. Bar'yakhtar, and M. I. Kaganov, Usp. Fiz. Nauk 71, 533 (1960) [Sov. Phys.-Usp. 3, 567 (1961)].

Translated by D. ter Haar
132