CONDITION FOR FLUTE INSTABILITY OF A TOROIDAL-GEOMETRY PLASMA

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The Suydam criterion is extended to the case of toroidal geometry. The problem is solved in two ways. In the first flute disturbances that correspond to large azimuthal numbers (m ≫ 1) are considered. In the second, use is made of a general criterion, due to Solov'ev, for instability of disturbances localized in the direction of the normal to the magnetic surfaces. Both methods yield the same criterion which is simpler and more compact than the criterion previously obtained by Kadomtsev and Pogutse.

1. INTRODUCTION

It is well known that the condition for the stability of an equilibrium plasma ($\nabla p = \mathbf{j} \times \mathbf{B}$, $\mathbf{j} = \operatorname{curl} \mathbf{B}$) against flute disturbances is expressed in cylindrical geometry by the Suydam criterion[1]

$$\frac{1}{4}\left(\frac{q'}{q}\right)^2 + \frac{2p'}{rB_s^2} > 0. \quad (1)$$

Here p(r) is the plasma pressure, $B_s(r)$ the longitudinal magnetic field, $q(r) = \mathrm{const} \cdot rB_s/B_\theta$, and B_θ is the azimuthal field produced by the longitudinal current. In a closed plasma pinch of length L it is convenient to put const = $2\pi/L$, and then q is the coefficient of stability margin against helical disturbances.[2] To stabilize helical disturbances it is necessary to satisfy the condition q > 1. Here, as noted by Kadomtsev and Pogutse,[3] the curvature of the torus is comparable with the curvature of the force lines, and additional terms appear in the criterion (1). The stability condition obtained in [3] is

$$\frac{1}{4}\left(\frac{q'}{q}\right)^2 + \frac{2p'}{rB_s^2} + \frac{p'}{B_\theta^2}\frac{U'}{U} - \frac{1}{2}\left(\frac{2p'}{B_\theta^2}\right)^2\frac{r^2}{R^2} > 0, \quad (2)$$

where $R = L/2\pi$ is the radius of the torus. The term quadratic in the pressure corresponds to the balloon instability mode. The term containing the derivative of the specific volume $U = dV/d\Phi$ of the longitudinal flux Φ plays the stabilizing role, since there is a "magnetic well"[4] in the toroidal plasma pinch ($U'/U < 0$).

We show in this paper that besides the effects allowed in [3] and caused by the curvature of the torus, it is necessary to take into account one more effect, connected with the azimuthal variation of the pitch of the force line. It turns out that in the case of a round torus all the three components connected with the curvature combine and the plasma stability condition takes the simple form

$$\frac{1}{4}\left(\frac{q'}{q}\right)^2 + \frac{2p'}{rB_s^2}(1 - q^2) > 0. \quad (3)$$

We see that to stabilize the flute disturbance of a round toroidal plasma pinch it is sufficient to satisfy the condition

$$q^2 > 1. \quad (4)$$

This result was obtained earlier by Ware[5] on the basis of the energy principle. The condition $q^2 > 1$ was obtained in [6,7] by using a general criterion for local instability in the vicinity of the magnetic axis. It follows from the criterion (3) that the condition $q^2 > 1$ ensures stability in the entire region of the plasma pinch.

The difference between the criteria obtained in [3] and [5] did not exclude, in principle, the possibility that the local disturbances, investigated on the basis of the energy principle, and the flute disturbances investigated by the small-oscillation method, are not fully equivalent. In this connection, we use both methods of analyzing the instability. In Sec. 3 we investigate by the small-oscillation method the flute instability corresponding to large azimuthal numbers m ≫ 1. In Sec. 4 we use a general criterion for the stability of radially localized disturbance; this criterion was obtained by Solov'ev by greatly simplifying the criterion obtained by Mercier, Bineau, and also by Green and Johnson. Both methods lead to the same criterion (3).

2. COORDINATE SYSTEM

In both calculation methods, we use a curvilinear coordinate system a, θ, φ in which the magnetic surfaces coincide with the coordinates a = const, and the azimuthal angle variable θ is chosen such that the force lines on the surface a = const are "straight," i.e., the ratio of the contravariant components B^2/B^3 on a given surface is independent of θ and φ.[8] In such a coordinate system, the contravariant components of the magnetic field and of the current density are given by the formulas

$$B^1 = 0, \quad B^2 = \chi'(a)/2\pi\sqrt{g}, \quad B^3 = \Phi'(a)/2\pi\sqrt{g}, \quad (5)$$

$$j^1 = 0, \quad j^2 = \left[I'(a) - \frac{\partial\nu}{\partial\varphi}\right]\Big/ 2\pi\sqrt{g}, \quad j^3 = \left[J'(a) + \frac{\partial\nu}{\partial\theta}\right]\Big/ 2\pi\sqrt{g}. \quad (6)$$

Here χ and Φ are the transverse and longitudinal fluxes of the magnetic field, I and J are the transverse and longitudinal currents bounded by the given magnetic surface a = const, and g is the determinant of the metric tensor, $g = \det g_{ik}$. The currents I and J are connected with the fluxes χ and Φ by the relations

$$J = \int_0^{2\pi} B_2 d\theta, \quad I = -\int_0^{2\pi} B_3 d\varphi, \quad (7)$$

where $B_2 = g_{22}B^2 + g_{23}B^3$ and $B_3 = g_{23}B^2 + g_{33}B^3$ are the covariant components of the magnetic field. Besides the longitudinal and azimuthal currents, it is possible to define the average azimuthal and longitudinal magnetic fields

$$B_\theta \equiv J / 2\pi a, \quad B_s \equiv I / 2\pi R. \tag{8}$$

The conditions for the plasma equilibrium $\nabla p = \mathbf{j} \times \mathbf{B}$ reduces in this coordinate system to the two relations

$$p'V' = I'\Phi' - J'\chi', \tag{9}$$

$$\chi' \frac{\partial v}{\partial \theta} + \Phi' \frac{\partial v}{\partial \varphi} = p'(V' - 4\pi^2 \sqrt{g}), \tag{10}$$

where V(a) is the volume bounded by the magnetic surface a = const. By definition,

$$V'(a) = \int_0^{2\pi}\int_0^{2\pi} \sqrt{g}\, d\theta\, d\varphi. \tag{11}$$

To find the metric coefficients of the coordinate system, we shall use the method of expansion in the curvature. We shall assume that the plasma torus is axisymmetric ($\partial/\partial\varphi = 0$). We start from a quasicylindrical coordinate system, in which the square of the element of length is

$$dl^2 = d\rho^2 + \rho^2 d\omega^2 + R^2(1 - k\rho\cos\omega)^2 d\varphi^2, \tag{12}$$

where k = 1/R is the curvature of the magnetic axis. We denote by $\xi(a)$ the displacement of the center of the cross section of the magnetic surface a = const relative to the magnetic axis, and change over to polar coordinates ρ_0, ω_0 connected with the displaced center:

$$\rho\cos\omega = \rho_0\cos\omega_0 + \xi(a), \quad \rho\sin\omega = \rho_0\sin\omega_0. \tag{13}$$

The equation of the magnetic surface in the new coordinate system, $\rho_0 = \rho_0(a, \omega_0)$ and the azimuthal angle ω_0 are presented in the form of the expansion

$$\rho_0 = a + \alpha(a)\cos 2\theta + \ldots, \quad \omega_0 = \theta + \lambda(a)\sin\theta + \mu(a)\sin 2\theta + \ldots. \tag{14}$$

Substituting the foregoing expressions into the formula for dl^2, we get:

$$dl^2 = g_{11}da^2 + 2g_{12}da\,d\theta + g_{22}d\theta^2 + g_{33}d\varphi^2.$$

In the approximation quadratic in the curvature, the coefficients g_{ik} and $\sqrt{g}$ are equal to

$$\begin{aligned}
g_{11} &= 1 + 2\xi'\cos\theta + \xi'^2 + (a^2\lambda'^2 - 2\xi'\lambda - 2\xi'\lambda' a)\sin^2\theta + 2\alpha'\cos 2\theta,\\
g_{12} &= (a^2\lambda' - a\xi')\sin\theta + (a^2\mu' - 2\alpha - a\xi'\lambda + a^2\lambda\lambda'/2)\sin 2\theta,\\
g_{22} &= a^2[1 + 2\lambda\cos\theta + \lambda^2/2 + (2\alpha/a + 4\mu + \lambda^2/2)\cos 2\theta],\\
g_{33} &= R^2(1 - ka\cos\theta - k\xi + ka\lambda\sin^2\theta)^2,\\
g_{13} &= g_{23} = 0,\\
\sqrt{g} &= aR[1 - k\xi - ka\xi'/2 + (\lambda + \xi' - ka)\cos\theta\\
&\quad + (\alpha' + \lambda\xi' + 2\mu + \alpha/a - ka\lambda - ka\xi'/2)\cos 2\theta].
\end{aligned} \tag{15}$$

We present also the following useful formulas for the coefficients $g_{ik}/\sqrt{g}$:

$$\begin{aligned}
\frac{g_{11}}{\sqrt{g}} &= \frac{1}{aR}\Big\{1 + k\xi + \frac{ka\xi'}{2} + (\xi' - \lambda + ka)\cos\theta + (\lambda - ka)^2\cos^2\theta\\
&\quad + [(a\lambda' - \xi')^2 - 2\lambda\xi']\sin^2\theta + \Big(\alpha' - \lambda\xi' - 2\mu - \frac{\alpha}{a}\\
&\quad + ka\lambda + \frac{ka\xi'}{2}\Big)\cos 2\theta\Big\},\\
\frac{g_{12}}{\sqrt{g}} &= \frac{1}{R}\Big[(a\lambda' - \xi')\sin\theta\\
&\quad + \Big(a^2\mu' - 2\alpha - a\xi'\lambda + \frac{a^2\lambda'\lambda}{2} + ka^3\lambda' - ka^2\xi'\Big)\sin 2\theta\Big],\\
\frac{g_{22}}{\sqrt{g}} &= \frac{a}{R}\Big[1 + k\xi - \frac{ka\xi'}{2} + \frac{\xi'^2}{2} + \frac{k^2a^2}{2} + (\lambda - \xi' + ka)\cos\theta\\
&\quad + \Big(\frac{\alpha}{a} + 2\mu + \frac{\xi'^2}{2} + \frac{k^2a^2}{2} - \frac{\xi' ka}{2} - \alpha' - \lambda\xi' + ka\lambda\Big)\cos 2\theta\Big],\\
\frac{g_{33}}{\sqrt{g}} &= \frac{R}{a}\Big\{1 + ka\lambda + \frac{ka\xi'}{2} + \frac{(\lambda + \xi')^2}{2} - (ka + \lambda + \xi')\cos\theta\\
&\quad - \Big[\alpha' + \lambda\xi' + 2\mu + \frac{\alpha}{a} - \frac{ka\xi'}{2} - \frac{(\lambda + \xi')^2}{2}\Big]\cos 2\theta\Big\}.
\end{aligned} \tag{16}$$

To determine the coefficients λ and μ, we shall use relations (7). Assuming axial symmetry and recognizing that $g_{23} = 0$, we obtain from (7)

$$J(a) = \chi'(a)(g_{22}/\sqrt{g})_0, \quad I(a) = -\Phi'(a)(g_{33}/\sqrt{g}). \tag{17}$$

Here and henceforth, the zero subscript denotes averaging with respect to θ:

$$\left(\frac{g_{22}}{\sqrt{g}}\right)_0 \equiv \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{g_{22}}{\sqrt{g}}\right)d\theta. \tag{18}$$

No such averaging is performed in the second formula, which relates I with Φ'. It follows hence that in the chosen coordinate system the ratio $g_{33}/\sqrt{g}$ does not depend on θ:

$$g_{33}/\sqrt{g} = (g_{33}/\sqrt{g})_0. \tag{19}$$

Consequently, λ is determined from the condition that the coefficient of $\cos\theta$ in the expression for $g_{33}/\sqrt{g}$ vanish:

$$\lambda = -\xi' - ka, \tag{20}$$

and μ is determined from the condition that the coefficient of $\cos 2\theta$ in the same expression vanish. We note that the parameters λ, μ, ... in the considered axial-symmetry case could be chosen such that the coordinate system would be orthogonal. However, such a choice would offer no noticeable advantage.

The characteristics of the magnetic surfaces ξ, α, ... are best obtained by comparing the expression derived for j^3 from formulas (6) and (10) with the expression for j^3 from the equation curl $\mathbf{B} = \mathbf{j}$:

$$j^3 = \frac{1}{\sqrt{g}}\left(\frac{\partial B_2}{\partial a} - \frac{\partial B_1}{\partial\theta}\right) = \frac{1}{2\pi\sqrt{g}}\left\{\left(\frac{g_{22}}{\sqrt{g}}\chi'\right)' - \chi'\frac{\partial}{\partial\theta}\left(\frac{g_{12}}{\sqrt{g}}\right)\right\}. \tag{21}$$

Equating these expressions, we get

$$\left[J\frac{g_{22}/\sqrt{g}}{(g_{22}/\sqrt{g})_0}\right]' - J' - J\frac{\partial(g_{12}/\sqrt{g})/\partial\theta}{(g_{22}/\sqrt{g})_0} = \frac{4\pi^2 p'}{J}\left(\frac{g_{22}}{\sqrt{g}}\right)_0[(\sqrt{g})_0 - \sqrt{g}]. \tag{22}$$

In the first approximation in the curvature, this yields an expression for ξ [9]

$$\xi'' + \xi'\left(\frac{1}{a} + \frac{2B_\theta'}{B_\theta}\right) = k\left(1 - \frac{2ap'}{B_\theta^2}\right). \tag{23}$$

In the quadratic approximation, equating the coefficients of $\cos 2\theta$, we can obtain an equation for α. The integration constants of the obtained equations are determined from the boundary conditions. We shall assume that the plasma pinch is in a conducting jacket of round cross section. In this case $\alpha \sim k^2$. The terms of the type $\alpha\cos 2\theta$ drop out, as a result of averaging, in the derivation of the stability criterion that takes into account effects quadratic in the curvature. We shall therefore not need α.

We present also an expression for the specific volume $U = V'/\Phi'$, which we shall need later on. According to (11) and (17), we have for V' and Φ'

$$U=\frac{V'}{\Phi'}=\frac{4\pi^2(\sqrt{g})_0}{I}\left(\frac{g_{33}}{\sqrt{g}}\right)_0=\frac{4\pi^2R^2}{I}(1-\varkappa), \tag{24}$$

where $\varkappa$ is the relative depth of the magnetic well, equal by definition to

$$\varkappa=1-\left(\frac{g_{33}}{\sqrt{g}}\right)_0\frac{(\sqrt{g})_0}{R^2}. \tag{25}$$

In the approximation quadratic in the curvature we have

$$\varkappa=2k\xi+ka\xi'+k^2a^2/2. \tag{26}$$

The stability criterion contains not the relative depth of the well, but its derivative $\varkappa'$, which contains the second derivative of the displacement ξ. We can eliminate ξ'' with the aid of (23). We then get

$$\varkappa'=-\frac{2p'}{B_\theta^2}k^2a^2+2k^2a+2ka\xi'\left(\frac{q'}{q}-\frac{B_s'}{B_s}\right). \tag{27}$$

3. DERIVATION OF THE STABILITY CRITERION BY THE METHOD OF SMALL OSCILLATIONS

To obtain the stability criterion by the method of small oscillations, it is sufficient to consider, as was done in [2], the equilibrium equations at the start of the disturbance. The possibility of production of a new equilibrium state with a disturbed magnetic field corresponds, within the framework of ideal magnetohydrodynamics, to instability of the initial state.

We choose the equilibrium-equation system in the form[3]

$$\mathbf{B}\nabla\alpha+\left[\nabla\left(\frac{1}{B^2}\right)\mathbf{B}\right]\nabla p=0, \tag{28}$$

$$\mathbf{B}\nabla p=0, \tag{29}$$

$$\alpha B^2=\mathbf{B}\,\mathrm{rot}\,\mathbf{B}, \tag{30}$$

$$\mathrm{div}\,\mathbf{B}=0. \tag{31}$$

The initial state corresponds to an axisymmetrical toroidal plasma pinch. We assume the longitudinal field to be sufficiently strong, $B_s/B_\theta \sim 1/ka \gg 1$, which corresponds to the conditions that exist in toroidal systems of the Tokamak type. The perturbed longitudinal field $B_s^* \sim B_\theta^* B_\theta/B_s$ is in this case small compared with the disturbed azimuthal field, and can be neglected in the equation $\mathrm{div}\,\mathbf{B}^* = 0$. Consequently, the disturbance of the magnetic field can be expressed in terms of one function ψ:

$$B^{*1}=\frac{1}{\sqrt{g}}\frac{\partial\psi}{\partial\theta};\quad B^{*2}=-\frac{1}{\sqrt{g}}\frac{\partial\psi}{\partial a}. \tag{32}$$

From the linearized Eq. (30) we obtain the following expression for the perturbed function α^*:

$$\begin{aligned}\alpha^*=-\frac{B_s}{\sqrt{g}|\mathbf{B}|^2}\Big\{&\frac{\partial}{\partial a}\left(\frac{g_{22}}{\sqrt{g}}\frac{\partial\psi}{\partial a}\right)+\frac{g_{11}}{\sqrt{g}}\frac{\partial^2\psi}{\partial\theta^2}-2\frac{g_{12}}{\sqrt{g}}\frac{\partial^2\psi}{\partial a\partial\theta}\\&+\left[\frac{\partial}{\partial\theta}\left(\frac{g_{11}}{\sqrt{g}}\right)-\frac{\partial}{\partial a}\left(\frac{g_{12}}{\sqrt{g}}\right)\right]\frac{\partial\psi}{\partial\theta}-\frac{\partial\psi}{\partial a}\frac{\partial}{\partial\theta}\left(\frac{g_{12}}{\sqrt{g}}\right)-k\frac{B_\theta}{B_s}\frac{\partial^2\psi}{\partial\varphi\,\partial\theta}\Big\}.\end{aligned} \tag{33}$$

Substituting the perturbed quantities B^{*1}, B^{*2}, and α^* into the linearized equations (28) and (29) we obtain a system of two differential equations for the functions ψ and p^*.

We seek the solution of this system in the form

$$\psi=\sum_l\psi_l\exp[i(m+l)\theta-in\varphi],$$

assuming that the harmonics ψ_l decrease with the number l, and represent the disturbance of the pressure in similar form

$$p^*=\sum_l p_l^*\exp[i(m+l)\theta-in\varphi].$$

Eliminating the harmonics p_l^* from the equations, we write the equation for the determination of ψ_l in the form

$$\begin{aligned}&\gamma_l[\Delta_l\psi_l-\xi'(\Delta_{l-1}\psi_{l-1}+\Delta_{l+1}\psi_{l+1})]-\left[\left(\frac{U'}{U}+\frac{B_s'}{B_s}+\frac{2B_\theta^2}{aB_s^2}\right)\frac{\mu np'}{\gamma_lB_\theta^2}\right.\\&\quad\left.-\left(\frac{p'+B_sB_s'}{B_\theta}\right)'\frac{\mu}{B_\theta}\right](m+l)\psi_l-\left\{\frac{1}{B_s}\left[\left(\frac{b\gamma^l}{\gamma_{l-1}}\right)'-\frac{bn}{a\gamma_{l-1}}\right]\right.\\&\quad\left.+\gamma_l\left[\frac{ka(m+l-1)-h_3}{a^2}\right]\right\}(m+l-1)\psi_{l-1}\\&-\left\{\frac{1}{B_s}\left[\left(\frac{b\gamma_l}{\gamma_{l+1}}\right)'-\frac{bn}{a\gamma_{l+1}}\right]+\gamma_l[ka(m+l+1)+h_3]\frac{1}{a^2}\right\}(m+l+1)\psi_{l+1}\\&\quad-\left\{\frac{1}{B_s}\frac{bn}{\gamma_{l-1}}+\frac{\gamma_l}{a}\left[h_1-\left(\frac{3}{2}h_2+\frac{1}{2}\xi'\right)(m+l-1)\right]\right\}\psi_{l-1}'\\&-\left\{-\frac{1}{B_s}\frac{bn}{\gamma_{l+1}}+\frac{\gamma_l}{a}\left[h_1+\left(\frac{1}{2}h_2-\frac{1}{2}\xi'\right)(m+l+1)\right]\right\}\psi_{l+1}'=0.\end{aligned} \tag{34}$$

Here

$$\Delta_l=\frac{1}{a}\frac{d}{da}a\frac{d}{da}-\frac{(m+l)^2}{a^2},\qquad \mu=\frac{B^2}{B^3}=\frac{\chi'}{\Phi'}$$

$$\gamma_l=(m+l)\mu-n,\qquad b=p'/B_\theta,$$

$$h_1={}^1/_2(a\xi''+\xi'+ka),\qquad h_2=a\xi''+ka,$$

$$h_3={}^1/_2a^2\xi'''+a\xi''-\xi'-{}^1/_2ka.$$

Equation (34) differs from the corresponding equation of [5] in the presence of a number of additional terms with harmonics ψ_{l-1} and ψ_{l+1}. These terms appear when the toroidality is taken into account in the first term of (28). In order to obtain in (34) a term containing U' it is necessary to take into account in the second term of (28) the quantities that are quadratic in the curvature, and to use formula (26). The result of the calculations then coincides with the corresponding expression obtained in [3] by integrating along the force line.

To investigate local flute-type disturbances ($m \gg 1$), which can develop near the singular point $a = a_0$, where the longitudinal wave number $\gamma_0 = m\mu - n$ vanishes, we use the system (34) with $l = 0$ and ± 1. Let $x = a_0 - a$ be the distance from the singular point; we can then write for small x

$$\gamma_0=m\mu'x,\quad \gamma_{-1}=-\mu.$$

In the case of practical interest, $2p/B_\theta^2 \lesssim 1$, to take into account the quantities quadratic in the curvature it is sufficient to retain only the fundamental and second harmonics. We shall solve this system near the singular point. To this end we apply the operator Δ_0 to Eq. (34) for the fundamental frequency. We write out the resultant expression, neglecting the terms that are insignificant when $m \gg 1$ and when x is small:

$$\begin{aligned}&\Delta_0x\Delta_0x\zeta_0-\left(\frac{U'}{U}+\frac{B_s'}{B_s}+\frac{2B_\theta^2}{aB_s^2}\right)\frac{p'}{B_\theta^2}\left(\frac{\mu}{\mu'}\right)^2\Delta_0\zeta_0\\&\quad-k\frac{p'}{B_sB_\theta}\frac{\mu'-\mu/a}{\mu'^2}\Delta_0(\zeta_{-1}+\zeta_{+1})\\&\quad-k\frac{p'}{B_sB_\theta}\frac{\mu}{m(\mu')^2}\frac{d}{dx}[\Delta_0(\zeta_{-1}-\zeta_{+1})]=0.\end{aligned} \tag{35}$$

We have introduced here the notation $\zeta_l = \psi_l/\gamma_l$. With the aid of Eq. (34) for the second harmonics we can obtain the following relations:

$$\Delta_0(\zeta_{-1}+\zeta_{+1}) = -k\frac{2p'}{B_sB_\theta}\frac{m^2}{\mu a}\zeta_0, \quad (36)$$

$$\Delta_0(\zeta_{-1}-\zeta_{+1}) = -k\frac{2p'}{B_sB_\theta}\frac{m}{\mu}\zeta_0 - 2\xi'\frac{m\mu'}{\mu}x\Delta_0\zeta_0. \quad (37)$$

The presence of a term proportional to ξ' in (37) takes into account an effect connected with the dependence of the pitch of the force line on the azimuthal angle ω_0 on the magnetic surface. It can be shown that the pitch of the force-line helix is $h(\omega_0) = 2\pi(B_S/B_\theta) \times (1+\xi'\cos\omega_0)$. Eliminating the functions ζ_{-1} and ζ_{+1} from (35) with the aid of expressions (36) and (37), we obtain an equation for the function ζ_0

$$\Delta_0 x\Delta_0 x\zeta_0 - \Big[\Big(\frac{U'}{U}+\frac{B_s'}{B_s}+\frac{2B_\theta^2}{aB_s^2}\Big)\frac{p'}{B_\theta^2}\Big(\frac{\mu}{\mu'}\Big)^2 + \frac{1}{2}k^2a^2\Big(\frac{2p'}{B_\theta^2}\Big)^2\Big(\frac{\mu}{\mu'}\Big)^2 + \xi' ka\frac{2p'}{B_\theta^2}\frac{\mu}{\mu'}\Big]\Delta_0\zeta_0 = 0. \quad (38)$$

Equation (38) differs from the analogous equation of [3] in a term proportional to κ'. Therefore the stability criterion can be written directly in the form

$$\frac{1}{4}\Big(\frac{\mu'}{\mu}\Big)^2 + \frac{2p'}{aB_s^2} + \frac{p'}{B_\theta^2}\Big(\frac{U'}{U}+\frac{B_s'}{B_s}\Big) - \frac{1}{2}\Big(\frac{2p'}{B_\theta^2}\Big)^2k^2a^2 - \xi' ka\frac{2p'}{B_\theta^2}\frac{\mu'}{\mu} > 0. \quad (39)$$

Since $U'/U + B_S'/B_S = -\kappa'$ and $\mu \equiv 1/q$ we obtain, substituting the expression (27) for κ' in the criterion (39), the plasma stability condition in the form (3).

4. USE OF GENERAL STABILITY CRITERION

The general criterion for plasma stability against local disturbances obtained by Solov'ev,[7] can be written in the form

$$\frac{1}{4}\Big(\frac{\mu'}{\mu}\Big)^2 + \frac{V'^3}{\chi'^2\Phi'^2}[\langle\alpha\rangle(J'\chi''-I'\Phi'+p'V'') - \langle\gamma\rangle\mu'\Phi'^2] + \frac{V'^6}{\chi'^2\Phi'^2}(\langle\gamma\rangle^2 - \langle\alpha\rangle\langle\beta\rangle) > 0. \quad (40)$$

where the prime denotes differentiation with respect to a. We have used here the notation:

$$\mu = \frac{\chi'}{\Phi'}, \quad \alpha = \frac{\mathbf{B}^2}{|\nabla V|^2}, \quad \beta = \frac{\mathbf{j}^2}{|\nabla V|^2}, \quad \gamma = \frac{\mathbf{jB}}{|\nabla V|^2}. \quad (41)$$

The angle brackets denote averaging over the volume of the layer between two close magnetic surfaces

$$\langle f\rangle \equiv \frac{2\pi}{V'}\int_0^{2\pi} f\sqrt{g}\,d\theta \equiv \frac{4\pi^2}{V'}(f\sqrt{g})_0. \quad (42)$$

Substituting the expressions (5) and (6) into the expressions for α, β, and γ, and recognizing that $|\nabla V|^2 = g^{11}V'^2 = V'^2 g_{22}g_{33}/g$, we obtain after transformations, using Eq. (9),

$$\frac{1}{4}\Big(\frac{\mu'}{\mu}\Big)^2 + \frac{p'V'}{4\pi^2}\Big[\Big\langle\frac{1}{g_{33}}\Big\rangle\frac{\chi'}{\Phi'^2}\Big(\frac{V'}{\chi'}\Big)' + \Big\langle\frac{1}{g_{22}}\Big\rangle\frac{\Phi'}{\chi'^2}\Big(\frac{V'}{\Phi'}\Big)'\Big] - \Big\langle\frac{1}{g_{22}}\frac{\partial\nu}{\partial\theta}\Big\rangle\frac{\mu'\Phi'V'}{4\pi^2\chi'^2} + \frac{V'^2}{16\pi^4\chi'^2\Phi'^2}\Big\{-\Big\langle\frac{1}{g_{22}}\Big\rangle\Big\langle\frac{1}{g_{33}}\Big\rangle p'^2V'^2 + 2\Big\langle\frac{1}{g_{22}}\frac{\partial\nu}{\partial\theta}\Big\rangle\Big\langle\frac{1}{g_{33}}\Big\rangle\chi'p'V' + \Big\langle\frac{1}{g_{22}}\frac{\partial\nu}{\partial\theta}\Big\rangle^2\Phi'^2 - \Big\langle\frac{1}{g_{22}}\Big\rangle\Big\langle\frac{1}{g_{22}}\Big(\frac{\partial\nu}{\partial\theta}\Big)^2\Big\rangle\Phi'^2 - \Big\langle\frac{1}{g_{33}}\Big\rangle\Big\langle\frac{1}{g_{22}}\Big(\frac{\partial\nu}{\partial\theta}\Big)^2\Big\rangle\chi'^2\Big\} > 0. \quad (43)$$

Let us transform this expression. We note first that the first, fourth, and seventh terms represent the square of the difference of two terms. In the fifth term, we express the square of the derivative of the pressure with the aid of formula (9) in the form

$$-p'^2V' = p'(J'\chi' - I'\Phi'). \quad (44)$$

We combine the first part of this term with the second term of (43), and the second part with the third term. Taking into account the connections between J and χ and between I and Φ, as given by (17), as well as condition (19), we obtain a criterion that has the same structure as in (2):

$$\Big[\frac{\mu'}{2\mu} - \Big\langle\frac{1}{g_{22}}\frac{\partial\nu}{\partial\theta}\Big\rangle\frac{V'}{4\pi^2\chi'}\Big]^2 + \frac{p'V'}{\Phi'^2}\Big(\frac{\sqrt{g}}{g_{33}}\Big)_0\Big\{\frac{V''}{V'} + \Big(\frac{\sqrt{g}}{g_{22}}\Big)_0\Big(\frac{g_{22}}{\sqrt{g}}\Big)_0' + \frac{\chi''}{\chi'}\Big[\Big(\frac{\sqrt{g}}{g_{22}}\Big)_0\Big(\frac{g_{22}}{\sqrt{g}}\Big)_0 - 1\Big]\Big\} + \frac{p'V'}{\chi'^2}\Big(\frac{\sqrt{g}}{g_{22}}\Big)_0\Big(\frac{U'}{U}+\frac{I'}{I}\Big) - \frac{1}{4\pi^2}\Big\langle\frac{1}{g_{22}}\Big(\frac{\partial\nu}{\partial\theta}\Big)^2\Big\rangle\Big[\frac{V'}{\chi'^2}\Big(\frac{\sqrt{g}}{g_{22}}\Big)_0 + \frac{V'}{\Phi'^2}\Big(\frac{\sqrt{g}}{g_{33}}\Big)_0\Big] + \frac{4\pi^2}{V'}\Big(\frac{\sqrt{g}}{g_{33}}\Big)_0\Big\langle\frac{1}{g_{22}}\frac{\partial\nu}{\partial\theta}\Big\rangle\frac{V'^2}{4\pi^2\Phi'^2}\frac{2p'V'}{4\pi^2\chi'^2} > 0. \quad (45)$$

It is sufficient to introduce the toroidal corrections only in the terms containing $\partial\nu/\partial\theta$. The derivative $\partial\nu/\partial\theta$ needs to be known here only in first approximation. From (10) we obtain directly

$$\frac{\partial\nu}{\partial\theta} = \frac{p'}{\chi'}8\pi^2a^2\cos\theta = 4\pi\frac{a^2}{R}\frac{p'}{B_\theta}\cos\theta. \quad (46)$$

Using the definition of the mean value (42) and the expression (16) for $g_{22}/\sqrt{g}$, we obtain for the mean values contained in the criterion (45)

$$\Big\langle\frac{1}{g_{22}}\frac{\partial\nu}{\partial\theta}\Big\rangle \approx 4\pi^2\xi'\frac{2p'}{a\chi'} = \frac{2\pi\xi'}{R}\frac{2p'}{aB_\theta}, \quad (47)$$

$$\Big\langle\frac{1}{g_{22}}\Big(\frac{\partial\nu}{\partial\theta}\Big)^2\Big\rangle \approx \frac{2\pi^2a^2}{R^2}\Big(\frac{2p'}{B_\theta^2}\Big)^2. \quad (48)$$

The expression in the brackets is equal to 2/a in our approximation. The last term of (45) is of the same order of magnitude as the second part in the term with $(\partial\nu/\partial\theta)^2$. Their ratio to the first part of this term is $\sim B_\theta^2/B_S^2$. As noted in the introduction, toroidal corrections are essential only when $B_\theta^2/B_S^2 \lesssim k^2a^2$. Therefore the last two terms in the criterion (45) constitute small additions and can be omitted. The criterion then takes the form

$$\Big(\frac{\mu'}{2\mu} - \frac{2p'}{B_\theta^2}ka\xi'\Big)^2 + \frac{2p'}{aB_s^2} - \frac{p'}{B_\theta^2}\kappa' - \frac{1}{2}\Big(\frac{2p'}{B_\theta^2}\Big)^2k^2a^2 > 0. \quad (49)$$

It differs from the criterion (2) in the term with ξ'. In our approximation we should neglect the square of this term. Thus, when $\mu' = 0$ the local disturbances are stabilized as a result of the magnetic well (the term with κ'). When $\mu' \neq 0$ the product of the terms with μ' and ξ' is comparable with the remaining terms. Substituting in (40) the expression (27) for κ' and recognizing that when $B_\theta/B_S \sim ka$ we can neglect the ratio B_S'/B_S compared with μ'/μ, we obtain the criterion for the stability of a toroidal plasma pinch against local disturbances.

$$\frac{1}{4}\Big(\frac{q'}{q}\Big)^2 + \frac{2p'}{aB_s^2}(1-q^2) > 0. \quad (50)$$

This criterion is a generalization of Suydam's criterion to include the case of toroidal geometry.

5. CONCLUSIONS

The analysis presented in Secs. 3 and 4 shows that radially localized disturbances, investigated on the basis of the energy principle, correspond to flute disturbances with large azimuthal numbers $m \gg 1$. Flute disturbances certainly are stabilized when $q^2 > 1$, i.e., under the condition required for the stabilization of a helical disturbance.

An essential feature of this stability condition is the fact that it does not limit the plasma pressure, in spite of the estimate usually derived from a comparison of the effect of the balloon mode with the magnetic-well effect (see [10, 11]). It must be noted, however, that the criterion $q^2 > 1$ is apparently not exhaustive even within the framework of ideal magnetohydrodynamics, since disturbances with small azimuthal numbers have been disregarded. The value of the parameter q which is critical for the stability must be determined with allowance for disturbances with $m = 1, 2, 3, \ldots$, which do not have a narrow region of localization.

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