NONLINEARITY AND PARAMETRIC RESONANCE IN A PLASMA

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Submitted March 31, 1967

Zh. Eksp. Teor. Fiz. 53, 1053–1057 (September, 1967)

Superposition of the magnetic field of a wave on a constant external magnetic field is equivalent to modulating the external field. A study of the perturbed trajectories of the particles in the modulated field shows that the variation of the phase relations of the particle motion and the electric field can lead to energy absorption. This effect is of the nature of parametric resonance, and it appears at frequencies equal to the cyclotron frequency divided by an integer (undertones). The resonance width is inversely proportional to the relaxation time, and the height is proportional to the square of the relaxation time.

IN the theory of wave propagation in a plasma, the effects connected with the magnetic field of the wave are usually taken into account only in the higher approximations of perturbation theory. The Lorentz force due to the magnetic field of the wave is, as a rule, small. Nevertheless, important effects may be caused by the magnetic field of the wave.[1]

Superposition of the magnetic field of a wave on a constant external magnetic field is equivalent to modulating the latter field. The modulated magnetic field changes the phase relations of the particle motion and the electric field, and consequently, changes the absorption of energy by the particle from the electric field. As a result of this, parametric resonance may arise. Thus, investigation of the nonlinear effects connected with the magnetic field of the wave can be reduced to a linear problem of the parametric resonance type.

The motion of charged particles in modulated magnetic fields was examined in references [2-4] for simple field configurations, where a uniform but time-varying component $\tilde{H}\mathbf{z}\cos\gamma t$ (γ is the modulation frequency) is superimposed on a constant uniform magnetic field $H_0\mathbf{z}$ ($\mathbf{z}$ is the unit vector along the z-axis). For this case, the induced electric field increased linearly with distance from the z axis. In the present paper, we examine a more realistic case of a plane wave with wave vector $\mathbf{k}$ perpendicular to the external magnetic field $H_0\mathbf{z}$, and with the magnetic field of the wave directed along the external field. As is known, such a situation is realized, in particular, in the propagation of a direct magnetosonic wave in a plasma when $\omega \ll \omega_{ci}$ (ω_{ci} is the ion cyclotron frequency). We shall solve the nonrelativistic problem by studying the motion of the different charged particles in this wave. The analogous problem for the relativistic case, but only for small modulation intensity ($h \equiv \tilde{H}/H_0 \ll 1$), was solved in the first approximation in h by Trubnikov and Bazhanova,[5] where it was shown that at the higher harmonics of the cyclotron frequency, the electrons absorb energy from the wave. This effect, for waves propagating exactly transverse to the magnetic field, is of a purely relativistic nature (cf. for example, [6], p. 243) and will not be considered in the present paper.

The usual procedure consists of finding the unperturbed particle trajectories in the constant external magnetic field, and then calculating the corrections to these trajectories with the magnetic field of the wave regarded as a small perturbation. We, to the contrary, shall find the perturbed trajectories, or more precisely, the dependence of the particle velocities on time in the fields of the plane waves, taking into account completely their dependence on time but assuming their spatial variations to be small. The problem then reduces to solving the equations of motion for a particle with mass M and charge Ze:

$$M\frac{d\mathbf{v}}{dt}=Ze\left(\mathbf{E}+\frac{1}{c}[\mathbf{vH}]\right), \qquad (1)^*$$

where the fields $\mathbf{E}$ and $\mathbf{H}$ are given as functions of the argument $\mathbf{k}\cdot\mathbf{r}-\omega t$; $\mathbf{r}=\mathbf{R}_0+\delta\mathbf{r}$ is the position vector of the particle, and $\mathbf{R}_0$ denotes its initial position.

The problem can be solved in principle by successive approximations, if $\delta\mathbf{r}$ is found from the solution of (1). The approximation assumed in the present paper is that we consider long waves or small displacements $\delta\mathbf{r}$, such that the quantity $(\mathbf{k}, \delta\mathbf{r})$ is regarded as a small parameter. Under this hypothesis, we neglect spatial dependence of the fields completely, and describe them as

$$\mathbf{E}=\tilde{\mathbf{E}}\cos(\omega t-\beta),$$
$$\mathbf{H}=[H_0+\tilde{H}\cos(\omega t-\beta)]\mathbf{z},$$

where $\beta\equiv\mathbf{k}\cdot\mathbf{R}_0$ is the initial phase of the particle.

The proposed solution method is applicable when the initial phase can be assumed to be random. To this end, a random change in the particle trajectory must occur over the distance $\delta\mathbf{r}$, as a result of collisions or other relaxation processes. Thus, the condition of applicability of the proposed method is the fulfillment of the inequality:

$$\left(\mathbf{k}\frac{d\delta\mathbf{r}}{dt}\right)\tau\equiv(\mathbf{kv})\tau\ll 1, \qquad (2)$$

where τ is the relaxation time.

For the field configuration assumed, the component of the particle velocity along the magnetic field v_z will be conserved, and the field of the wave will act only on the perpendicular component of the velocity. If we put

$$\omega_c\equiv\frac{ZeH_0}{Mc},\quad \alpha\equiv\frac{Ze}{M}(\tilde{E}_x+i\tilde{E}_y),\quad h\equiv\frac{\tilde{H}}{H_0}$$

*$[\mathbf{vH}]\equiv\mathbf{v}\times\mathbf{H}$.

(h is called the intensity of the modulation of the magnetic field) and introduce $w \equiv v_x + iv_y$, then the motion of the charged particle in the plane perpendicular to the magnetic field $H_0\mathbf{z}$ will be described by one equation for the complex variable w:

$$\frac{dw}{dt} = a\cos(\omega t - \beta) - i\omega_c[1 + h\cos(\omega t - \beta)]w. \quad (3)$$

The general solution of (3) can be written in the form

$$w = e^{-i\varkappa}\int_0^t e^{i\varkappa}a\cos(\omega t - \beta)\,dt + (v_{0x} + iv_{0y})e^{-i\varkappa}, \quad (4)$$

where $(v)_x\big|_{t=0} \equiv v_{0x}$, $(v)_y\big|_{t=0} \equiv v_{0y}$, and if we put $\delta \equiv \omega_0/\omega$, then

$$\varkappa \equiv \omega_c t + \delta h\sin(\omega t - \beta) + \delta h\sin\beta.$$

Using the Jacobi–Anger formula

$$e^{iz\cos\varphi} = \sum_{n=-\infty}^{\infty} J_n(z)e^{in\varphi},$$

we readily see that

$$\cos\varkappa = \sum_{n=-\infty}^{\infty} J_n(\delta h)\cos[(\omega_c + n\omega)t - n\beta + \delta h\sin\beta],$$
$$\sin\varkappa = \sum_{n=-\infty}^{\infty} J_n(\delta h)\sin[(\omega_c + n\omega)t - n\beta + \delta h\sin\beta]. \quad (5)$$

Calculating in accordance with (4) with formula (5) taken into account, we obtain the following expression for w(t):

$$w = v_{0\perp}\cos(\varkappa - \psi) - iv_{0\perp}\sin(\varkappa - \psi) + \frac{Ze|E|}{M}[(Y_1 + iY_2)\cos(\varkappa - \eta) + (Y_2 - iY_1)\sin(\varkappa - \eta)],$$

where

$$Y_1 = -\frac{2}{\delta h}\sum_{n=-\infty}^{\infty} nJ_{-n}(\delta h)\frac{\sin[(\delta - n)\omega t/2]}{(\delta - n)\omega}\cos\frac{(\delta - n)\omega t + \gamma}{2},$$
$$Y_2 = -\frac{2}{\delta h}\sum_{n=-\infty}^{\infty} nJ_{-n}(\delta h)\frac{\sin[(\delta - n)\omega t/2]}{(\delta - n)\omega}\sin\frac{(\delta - n)\omega t + \gamma}{2}, \quad (6)$$

and the parameters have the following meaning:

$$v_{0\perp} \equiv \sqrt{v_{0x}^2 + v_{0y}^2}, \quad \operatorname{tg}\psi \equiv v_{0y}/v_{0x}, \quad \operatorname{tg}\eta \equiv E_y/E_x,$$
$$\gamma \equiv 2\delta h\sin\beta + 2n\beta.$$

Changing over from the complex variable w to the velocity components, we get

$$v_x = v_{0\perp}\cos(\varkappa - \psi) + \frac{Ze}{M}|E|[Y_1\cos(\varkappa - \eta) + Y_2\sin(\varkappa - \eta)],$$
$$v_y = -v_{0\perp}\sin(\varkappa - \psi) + \frac{Ze}{M}|E|[Y_2\cos(\varkappa - \eta) - Y_1\sin(\varkappa - \eta)]. \quad (7)$$

The velocity components v_x and v_y contain, according to (6), resonance terms at the values of

$$\delta \equiv \omega_c/\omega = n, \quad (8)$$

where n is an integer. It is precisely these resonance terms which determine the energy absorbed by the particle from the wave.

The change in the particle energy ΔE, resulting from the action of the wave field, is calculated from (7):

$$\Delta E \equiv M\frac{v_x^2 + v_y^2 - v_{0\perp}^2}{2} = \frac{Ze|E|}{2}\Big\{\frac{Ze|E|}{M}(Y_1^2 + Y_2^2) + 2v_{0\perp}[Y_1\cos(\psi - \eta) + Y_2\sin(\psi - \eta)]\Big\}. \quad (9)$$

The main contribution to ΔE is given by the resonance terms defined by the condition (8). Near resonance, this condition is replaced by

$$\delta - n = (\omega_c - n\omega)/\omega \equiv \varepsilon < 1, \quad (10)$$

where n is the integer closest to δ. Extracting from (9) only the resonance terms according to condition (10), we get

$$\Delta E \approx \frac{Ze|E|}{2}\Big\{\frac{Ze|E|}{M}\Big[\frac{2n}{\delta h}J_{-n}(\delta n)\frac{\sin(\varepsilon\omega t/2)}{\varepsilon\omega}\Big]^2 - \frac{4v_{0\perp}}{\delta h}nJ_{-n}(\delta h)\frac{\sin(\varepsilon\omega t/2)}{\varepsilon\omega}\cos\Big(\frac{\varepsilon\omega t}{2} + \delta h\sin\beta + \beta n + \eta - \psi\Big)\Big\}. \quad (11)$$

We assume the initial phases of the particles to be random, so that ΔE must be averaged over β. In the same way we can also average expression (11) over ψ, the azimuthal angle of the direction of the initial velocity. The averaged value of ΔE will be written as $\langle\Delta E\rangle_{\beta,\psi}$. Averaging over β, we get

$$\langle\Delta E\rangle_{\beta,\psi} \approx \frac{Ze|E|}{2}\Big\{\frac{Ze|E|}{M}\Big[\frac{2n}{\delta h}J_{-n}(\delta h)\frac{\sin(\varepsilon\omega t/2)}{\varepsilon\omega}\Big]^2 - \frac{4v_{0\perp}}{\delta h}nJ_n^2(\delta h)\frac{\sin(\varepsilon\omega t/2)}{\varepsilon\omega}\cos\Big(\frac{\varepsilon\omega t}{2} + \eta - \psi\Big)\Big\}. \quad (12)$$

For an isotropic initial distribution, the second term of (12) vanishes after averaging over ψ, and we get

$$\langle\Delta E\rangle_{\beta,\psi} \approx \frac{2}{M}\Big[\frac{Ze|E|n}{\delta h}J_{-n}(\delta h)\frac{\sin(\varepsilon\omega t/2)}{\varepsilon\omega}\Big]^2. \quad (12a)$$

This function is periodic, but if condition (10) is satisfied, its period is large; and if the period is significantly greater than the relaxation time τ, energy absorption takes place. At exact resonance ($\delta = n$) the expression (12) becomes

$$\langle\Delta E\rangle_{\beta,\psi} \approx \frac{t^2}{2M}\Big[\frac{Ze|E|}{h}J_{-n}(\delta h)\Big]^2. \quad (13)$$

In the limit as $h \to 0$, resonance takes place only for $\delta = 1$, i.e., it changes into ordinary cyclotron resonance, and formula (13) becomes the well known expression for energy absorbed in cyclotron resonance[4] (p. 383):

$$\langle\Delta E\rangle_{\beta,\psi} \approx \frac{(Ze|E|)^2}{8M}t^2.$$

At a finite modulation intensity h the resonance frequencies become equal to the cyclotron frequency divided by integers (undertones of the cyclotron frequency). This effect has exactly the same physical nature as parametric resonance in an externally modulated magnetic field. However, since in our case h is proportional to the amplitude of the wave, this effect has become nonlinear.

In the simplest field geometry,[2,3] the equation of particle motion reduces to Hill's equation with characteristic frequency $\omega_c/2$. Thus, the frequency spectrum of parametric resonance starts at the cyclotron frequency. In our case, resonance occurs at exactly the same frequencies, but results from the solution of the first order complex equation (3) and not from the solution of Hill's equation. The essential difference is that in our problem, the resonance width is determined by the relaxation time τ, while in the case of the simplest field geometry the resonance width is determined by the specific stability diagram.[3]

In expression (12) for the absorbed energy, the time t must be replaced by the relaxation time τ. In the absence of collisions ($\tau \to \infty$) the resonance is infinitely narrow and infinitely high. For finite values of τ, the resonance width can be estimated from the condition $\omega_c - n\omega < 1/\tau$. The resonance height is proportional to τ^2.

The conditions for the applicability of our results are the fulfillment of inequality (2) and the isotropic initial distribution of velocity, which is used in the averaging of (12) over ψ. Violation of the isotropy can only lead to the appearance in expression (13) for the absorbed energy of an additional term proportional to first power in time. For the fulfillment of inequality (2), the necessary conditions reduce obviously to the requirement

$$kR_c \ll 1, \quad k\tilde{v} / \omega \ll 1,$$

where R_c is the cyclotron radius and $\tilde{v}$ is the amplitude of the particle velocity in the wave. The first of these conditions requires a sufficiently strong constant magnetic field; the second is satisfied if the particle velocity is small in comparison with the phase velocity of the wave.

The results of this paper (the occurrence of resonance at the undertones of the cyclotron frequency) are the direct consequences of allowance for the magnetic field of the wave. They have a direct relation to the very general question of accelerating charged particles in high-frequency electromagnetic fields and to the heating of plasmas by such fields. A fugure goal of our work is the application of the results to a wide range of concrete physical phenomena of the type indicated; while the main problem of the present paper is the presentation of a procedure for taking into account the influence of the magnetic field of the wave on the motion of the charged particle. Essentially, the method presented forms the first step in the development of a more general problem—the solution of kinetic equations by the method of trajectories with exact allowance for the perturbations of the particle trajectories in electromagnetic fields.

In this paper, the problem was solved in an approximation in which the wave field is spatially independent. Further refinement of the method presented must be connected with an approximate allowance for the dependence of the wave field on the space coordinates. The inequality (2) may then be replaced by a less stringent condition.

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Translated by C. K. Chu
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