# THE CURRENT TANGENTIAL TO THE SURFACE OF A METAL EMITTING ELECTRONS

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It is shown that field, thermionic, and thermionic-field emission (T-emission) from a single crystal are accompanied by the appearance of a current tangential to the emitting surface (or of an electric field which compensates this current) inside as well as outside the emitter. The ratio of this current to the emission current is estimated for some of the most important cases, and the quantity obtained in general is of the order of unity. Only for a certain symmetry of the crystal lattice of the emitter does the tangential current vanish.

LET a metal filling the half space $z < 0$ and placed in an electric field (Fig. 1) emit either sub-barrier electrons as a result of the tunnel effect (field emission) or above-barrier electrons (thermionic emission) or both (thermionic-field emission). If the electrons in the metal were free, then the emission would not affect the isotropy of the electron distribution with respect to the velocity components tangential to the surface, and the current density vector $\mathbf{j}$ would be directed along the z axis. Actually, for a single-crystal emitter (the surface of which will be assumed to be an atomic plane) no such isotropy takes place, and in general tangential currents are produced during emission both in the metal and in vacuum.

Under real conditions, when the dimensions of the emitter in the xy plane are bounded, there is produced in lieu of the tangential current an oppositely directed compensating electric field and a corresponding potential difference transverse to the emission current.

It is obvious that the effect under consideration will not occur if the z axis is a twofold symmetry axis for the emitter crystal lattice; the $p_z$ axis is here also a twofold symmetry axis for the equal-energy surfaces, and since the origin in $\mathbf{p}$-space is always a symmetry center, the plane $p_z = 0$ is a symmetry plane.[1)] If the plane $y = 0$ is a symmetry plane for the emitter lattice (and respectively the plane $p_y = 0$ is a symmetry plane for the equal-energy surfaces and the $p_y$ axis is a twofold symmetry axis), then $j_y$ vanishes.

1. We shall calculate the current in analogy with the calculation of $j_z$ for field[1] and thermionic[2] emissions. We use electron wave functions, each of which contains, when $z < 0$, one wave incident on the surface of the metal, a reflected wave, and a Bloch wave attenuating exponentially within the metal, and when $z > 0$—only waves emerging from the metal. In view of the conservation of the tangential quasimomentum of the electron, these wave functions are of the form (see [1], formulas (2a) and (2b))

$$\psi_{sp} + \sum_r a_r \psi_{r\tilde{p}_r} \qquad (z<0), \tag{1}$$

$$\sum_Q c_Q \exp\{i(\mathbf{P}/\hbar + \mathbf{Q})\mathbf{R}\} f_Q(z) \qquad (z>0). \tag{1a}$$

Here $\psi_{sp}$ is the incident ($v_z(\mathbf{p}) > 0$) Bloch wave (belonging to the s-th band and having a quasimomentum $\mathbf{p} = (\mathbf{P}, p_z)$ and an energy $E = \mathscr{E}_s(\mathbf{p})$; $\mathbf{p}_r = (\mathbf{P}, \tilde{p}_{zr})$, where $\tilde{p}_{zr}$ is the root of the equation $\mathscr{E}_r(\mathbf{P}, p_z) = E$, corresponding to the reflected Bloch wave ($v_z(\tilde{\mathbf{p}}_r) < 0$) or to the exponentially damped one ($\mathrm{Im}\, p_{zr} < 0$); $\mathbf{q}$ are the reciprocal-lattice vectors multiplied by $2\pi$; $f_Q(z)$ is the solution of the one-dimensional Schrödinger equation in the region $z > 0$ with potential energy $U(z)$ and energy $E_Q^{(z)} = E - (\mathbf{P} + \hbar\mathbf{Q})^2/2m_0$ ($m_0$—electron mass), corresponding to the motion of the electron from the metal; the capital letters denote the projections of the corresponding vectors on the xy plane. The coefficients $a_r$ and c are determined from the conditions for the continuity of $\psi$ and $\partial\psi/\partial z$ at $z = 0$ (see [1], Eqs. (4)). At sufficiently large value of z, the classical momentum

$$p_z(E_Q^{(z)}, z) = \sqrt{2m_0[E_Q^{(z)} - U(z)]}$$

is real, even if $E_Q^{(z)} < U_{max}$ ($U_{max} = -e^{3/2}F^{1/2}$, see Fig. 1), and the function $f_Q(z)$ in the quasiclassical region can be written in the form

$$f_Q(z) = p_z^{-1/2}(E_Q^{(z)}, z)\exp\left\{\frac{i}{\hbar}\int p_z(E_Q^{(z)}, z)\,dz\right\} \qquad (z > z_Q).$$

If the crystallographic indices of the $z = 0$ plane are not too large, and if we denote by $\mathbf{P}$ the reduced tangential quasimomentum, then the coefficients $c_Q$ with $\mathbf{Q} \neq 0$ are exponentially small compared with $c_0$, so that expression (1a) reduces in practice to

$$c_0 \exp\{(i/\hbar)\mathbf{PR}\} f_0(z) \qquad (z > 0); \tag{2}$$

in other words, the smallness of all the equivalent values of the tangential quasimomentum upon emergence of the electron from the crystal is actually transformed

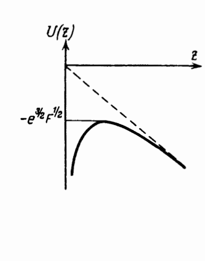


FIG. 1. Potential energy of electron outside the metal: Solid line – with allowance for the image force, dashed – without this force; –e – electron charge, F – intensity of external electric field.

1) Thus, if the effect is missing for any direction of the z axis relative to the crystallographic axes, the dispersion law is isotropic.

into the tangential momentum. This statement is proved in [1] for sub-barrier electrons ($E < U_{max}$), but it is easy to see that it remains in force also for above-barrier electrons with $E_0^{(z)} - U_{max} \lesssim 0.1$ eV, which are responsible for the thermionic emission.

We shall henceforth confine ourselves to the case when, in the energy interval essential for the type of emission of interest to us (we denoted by (E′, E″)), the equal-energy surfaces (with allowance for all the bands) have in the reciprocal-lattice unit cell not more than two common points with any straight line parallel to the $p_z$ axis; in other words, for fixed E and **P** there are either two undamped Bloch waves, one with $v_z > 0$ and the other with $v_z < 0$, or else only one with $v_z = 0$. Not being interested in surface effects, discarding accordingly the terms that attenuate exponentially inside the metal,[2)] and also using the continuity equation, we obtain for the averaged flux density in the state described by the wave function (1), (2)

$$\mathbf{i}^{(i)}(\mathbf{p}) = \mathbf{v}(\mathbf{p}) + \mathbf{v}(\tilde{\mathbf{p}})[1 - D(\mathbf{p})]v_z(\mathbf{p})/|v_z(\tilde{\mathbf{p}})| \qquad (z < 0) \tag{3a}$$

$$\mathbf{I}^{(e)}(\mathbf{p}, z) = \mathbf{P}|c_0|^2/m_0 p_z(E_0^{(z)}, z) = \mathbf{P}D(\mathbf{p})v_z(\mathbf{p})/p_z(E_0^{(z)}, z) \qquad (z > z_0),$$

$$i_z^{(e)}(\mathbf{p}) = i_z^{(i)}(\mathbf{p}) = v_z(\mathbf{p})D(\mathbf{p}). \tag{3b}$$

Here $\mathbf{v}(\mathbf{p}) = \nabla_p \mathscr{E}(\mathbf{p})$, and $D(\mathbf{p})$ is the coefficient of transmission of the electron through the surface of the emitter; the index of the band has been omitted. We note that even in total reflection (D = 0) the tangential components of the incident and reflected fluxes are in general not equal; equality takes place only in the case of symmetry of the equal-energy surfaces about the plane $p_z = 0$, when

$$\tilde{p}_z = -p_z, \quad v_z(\tilde{\mathbf{p}}) = -v_z(\mathbf{p}), \quad \mathbf{V}(\tilde{\mathbf{p}}) = \mathbf{V}(\mathbf{p}).$$

In view of the assumed limitation, the wave functions (1) and (2) for different s and **p** are orthogonal, and the current density is equal to

$$\mathbf{j} = -\frac{2e}{h^3}\sum_s \int_{(v_{zs}>0)} \mathbf{i}_s(\mathbf{p}) f(\mathscr{E}_s(\mathbf{p}))\, d^3\mathbf{p}, \tag{4}$$

where −e is the electron charge, f(E) is the Fermi distribution function, and the integration is carried out over half the reciprocal-lattice unit cell, corresponding to the states with $v_z > 0$.

2. The current in the metal is calculated from formulas (4) and (3a). Taking into account the relation

$$|v_{zs}(\tilde{\mathbf{p}}_s)|\,|d\tilde{p}_{zs}| = v_{zs}(\mathbf{p})\, dp_z,$$

which follows from the identity $\mathscr{E}_s(\tilde{\mathbf{p}}_s) = \mathscr{E}_s(\mathbf{p})$, and the presence of a symmetry center at the origin of **p**-space, we obtain

$$\mathbf{j}^{(i)} = \frac{2e}{h^3}\sum_s \int_{(v_{zs}>0)} \mathbf{v}_s(\tilde{\mathbf{p}}_s)\frac{v_{zs}(\mathbf{p})}{|v_{zs}(\tilde{\mathbf{p}}_s)|} D_s(\mathbf{p}) f(\mathscr{E}_s(\mathbf{p}))\, d^3\mathbf{p}$$

$$= \frac{2e}{h^3}\sum_s \int_{(v_{zs}<0)} \mathbf{v}_s(\tilde{\mathbf{p}}_s) D_s(\mathbf{p}) f(\mathscr{E}_s(\mathbf{p}))\, d^3\tilde{\mathbf{p}}_s. \tag{5}$$

We note that in the case of total reflection ($D \equiv 0$) both $j_z$ and the tangential component of the current $\mathbf{J}^{(i)}$ vanish.

Transformation to the integration variables **P** and E yields

$$\mathbf{J}^{(i)} = \frac{2e}{h^3}\int \Phi^{(i)}(E) f(E)\, dE, \quad \Phi^{(i)}(E) = \int_{\Sigma(E)} \frac{\tilde{\mathbf{v}}(E,\mathbf{P})}{|\tilde{v}_z(E,\mathbf{P})|} D(E,\mathbf{P})\, d^2\mathbf{P},$$

$$j_z = -\frac{2e}{h^3}\int \Phi(E) f(E)\, dE, \quad \Phi(E) = \int_{\Sigma(E)} D(E,\mathbf{P})\, d^2\mathbf{P}, \tag{6}$$

where Σ(E) is part of the projection of the equal-energy surface on the $p_x p_y$ plane, and lying in the central Brillouin zone of the plane lattice, and $\tilde{\mathbf{v}}(E, \mathbf{P}) = \mathbf{v}(\mathbf{P}, \tilde{p}_z(E, \mathbf{P}))$. A second change of variables makes it possible to represent $\Phi^{(i)}(E)$ also in the form

$$\Phi_x^{(i)}(E) = \int_{\Sigma_x^+(E)} D(E, p_y, p_z)|_{v_x>0}\, dp_y\, dp_z - \int_{\Sigma_x^-(E)} D(E, p_y, p_z)|_{v_x<0}\, dp_y\, dp_z$$

(and analogously for $\Phi_y^{(i)}(E)$), where $\Sigma_x^{\pm}(E)$ is the projection on the $p_y p_x$ plane of that part of the equal-energy surface, which lies in the unit cell corresponding to the central Brillouin zone of the flat lattice, and on which $v_z < 0$ and $v_x \gtrless 0$.

We shall use henceforth the following properties of the transmission coefficient: 1) D(E, **P**) can be represented in the form of a product of an odd function of **P** by a function that varies essentially only distances on the order of the reciprocal-lattice unit-cell dimensions; in the case of field emission, this is a rapidly varying exponential even function (see [1], formula (10)) and in the case of thermionic emission it is simply a constant;[2] 2) at the points p and −p at which $v_z = 0$, the transmission coefficient is the same (this follows from Eqs. (4) of [1]), that is, the function D(E, **P**) is an even function of **P** at the boundaries of the figure Σ(E).

Let us consider first small (compared with the dimensions of the unit cell) ellipsoidal groups, of the electron or hole type:

$$\tfrac{1}{2} m_{kl}^{-1}(\pm p_k - p_k^0)(\pm p_l - p_l^0) = |E - E_g|$$

($m_{kl}^{-1}$ is the tensor of the reciprocal effective masses with positive principal values, and the double signs take into account the existence of paired ellipsoids that are symmetrical with respect to the origin). In this case the ratio $\tilde{v}_\alpha(E, \mathbf{P})/|\tilde{v}_z(E, \mathbf{P})|$ is equal to the sum of the constant

$$\tilde{v}_\alpha(E, \mathbf{P}^0)/|\tilde{v}_z(E, \mathbf{P}^0)| = -m_{\alpha z}^{-1}/m_{zz}^{-1}$$

and an odd function of **P**. Taking into account the properties of the function D(E, **P**) and the symmetry of the figure Σ(E) about the origin, we get from (6)

$$\Phi_\alpha^{(i)}(E) \approx -(m_{\alpha z}^{-1}/m_{zz}^{-1})\Phi(E),$$

and if the latter takes place in the entire energy interval (E′, E″) that is of importance for the emission, then

$$j_\alpha^{(i)} \approx (m_{\alpha z}^{-1}/m_{zz}^{-1}) j_z. \tag{7}$$

If the ratio $m_{\alpha z}^{-1}/m_{zz}^{-1}$ is small, then the calculation should be carried out in a higher approximation, with due allowance for the odd part of the function D(E, **P**) and for the deviation of the shape of the equal-energy surfaces from ellipsoidal. We note in this connection that $m_{yz}^{-1}$ vanishes when the axes $p_x$ and $p_y$ are so directed that the sections of the ellipsoids with the planes $p_x$ = const are ellipses with symmetry axes parallel to the axes $p_y$ and $p_z$. If one of the principal axes of the ellipsoid is parallel to the $p_z$ axis, then $m_{xz}^{-1} = m_{yz}^{-1} = 0$.

2) Such terms are contained only in $\mathbf{I}^{(i)}$.

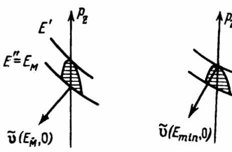


FIG. 2. p-space region giving an essential contribution to the current (shaded).

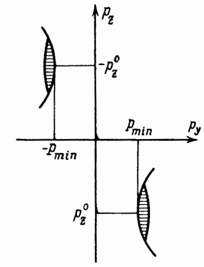


FIG. 3. Parts of equal-energy surface giving an appreciable contribution (shaded).

In order to obtain the estimate of $J^{(i)}/j_z$ for the case of large groups in general form, we shall assume that an appreciable contribution to the current is made by a region of p-space which is small compared with the unit cell. This is precisely the typical situation in the case of cold and thermionic emission, the energy interval (E′, E″) which includes the essential region of p-space having a width $E'' - E' \lesssim 0.1$ eV. In the case of cold emission, this interval is close to the energy $E_M$,[1] and in the case of thermionic emission to the energy $E_{min}$.[2]

If the equal energy surfaces in the interval (E′, E″) are crossed by the $p_z$ axis, then the essential region of p-space consists of small vicinities of the intersection point (Fig. 2). Expanding the equation for the equal-energy surface $p_z = \tilde{p}_z(E, \mathbf{P})$ near the point $\mathbf{P} = 0$ in powers of $p_x$ and $p_y$, and retaining turns up to quadratic inclusive, we obtain from (6), in the approximation of an even dependence of D on **P**,

$$\Phi^{(i)}(E) \approx [\tilde{\mathbf{V}}(E, 0)/|\tilde{v}_z(E, 0)|]\Phi(E)$$

and if $\tilde{\mathbf{V}}(E, 0)/|\tilde{v}_z(E, 0)|$ varies little in the interval (E′, E″), then

$$\mathbf{J}^{(i)} \approx -[\tilde{\mathbf{V}}(E^0, 0)/|\tilde{v}_z(E^0, 0)|]j_z; \tag{8}$$

Here $E^0$ is one of the quantities $E_M$ or $E_{min}$ (in this case $E_M = E_F$, and $E_{min} = E_F + w - e^{3/2}F^{1/2}$, where $E_F$ is the Fermi energy, w is the work function, and F the electric field applied to the metal). In this approximation the component of the current $\mathbf{J}^{(i)}$ in a direction perpendicular to the vector $\tilde{\mathbf{V}}(E^0, 0)$ vanishes, and if $\tilde{\mathbf{V}}(E^0, 0) = 0$, then $\mathbf{J}^{(i)} = 0$. A more accurate calculation of these quantities calls for allowance for the cubic terms in the equation of the equal-energy surface and for the odd part of the dependence of D on **P**.

If the equal-energy surfaces in the energy interval (E′, E″) do not cross the $p_z$ axis, then an appreciable contribution to the current is made by small parts of each surface near the pair of symmetrical points (relative to the origin), which lie closest to the $p_z$ axis, that is, the points at which $P = P_{min}(E)$. Drawing the plane $p_yp_z$ through these points (Fig. 3), we write the equation of the equal-energy surface in the region of interest to us approximately in the form

$$\pm p_y - P_{min} = {}^1\!/_2 A_{11}p_x^2 + A_{12}p_x(p_z \mp p_z^0) + {}^1\!/_2 A_{22}(p_z \mp p_z^0)^2 \tag{9}$$

(the double signs take into account the existence of a pair of symmetrical cavities). We should have here

$$A_{22} > 0 \text{ for } K \geqslant 0, \quad A_{22} \geqslant |K|P_{min} \text{ for } K < 0, \tag{10}$$

where $K = A_{11}A_{22} - A_{12}^2$ (the case when $A_{22} = A_{12} = 0$ and therefore $v_z \equiv 0$ requires an analysis in a higher approximation).

The coefficients $A_{\alpha\beta}$ are expressed in the following fashion in terms of the principal radii of curvature $R_{1,2}$ of the equal energy surface at the points $P = P_{min}$ and the angle $\gamma$ between the plane $p_xp_y$ and the plane of principal curvature $1/R_1$:

$$A_{11} = \frac{\cos^2\gamma}{R_1} + \frac{\sin^2\gamma}{R_2}, \quad A_{12} = \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\cos\gamma\sin\gamma,$$
$$A_{22} = \frac{\sin^2\gamma}{R_1} + \frac{\cos^2\gamma}{R_2}$$

(We assume that $R_\alpha$ is positive if the corresponding section of the surface is convex towards the $p_z$ axis.) Hence $K = 1/R_1R_2$, that is, K is the Gaussian curvature, and the conditions (10) denote that either $0 < R_{1,2} < \infty$, or $0 < R_1 < \infty$, $R_2 = \infty$, and $\gamma \neq 0$, or else $-\infty < R_1 \leq -P_{min}$, $0 < R_2 < \infty$, $|\tan\gamma| \leq [(|R_1| - P_{min})/(R_2 + P_{min})]^{1/2}$. Determining $v_x(E, \mathbf{P})/|\tilde{v}_z(E, \mathbf{P})|$ from (9), we obtain the sum of the constant

$$\tilde{v}_x(E, 0, p_y) \,/\, |v_z(E, 0, p_y)| = -A_{12}(E)\,/\,A_{22}(E)$$

and an out function of **P**, so that

$$\Phi_x^{(i)}(E) \approx -[A_{12}(E)/A_{22}(E)]\,\Phi(E).$$

If $A_{12}(E)/A_{22}(E)$ and the direction of the plane passing through the $p_z$ axis and the point $P = P_{min}$ vary little in the interval (E′, E″), then

$$j_x \approx \frac{A_{12}(E^0)}{A_{22}(E^0)}j_z = -\left[\frac{(1 - R_2/R_1)\,\mathrm{tg}\,\gamma}{1 + (R_2/R_1)\,\mathrm{tg}^2\gamma}\right]_{E=E^0} j_z. \tag{11}$$

The expression in the square brackets vanishes if the plane $p_yp_z$ is the plane of the principal curvature of the corresponding equal-energy surface; in this case the calculation must be carried out in the next higher approximation, with allowance for the cubic terms in the equation of the equal-energy surface and for the odd part of the function D(E, **P**).

The ratio $\tilde{v}_y(E, \mathbf{P})/|\tilde{v}_z(E, \mathbf{P})|$, obtained from (9), is an odd function of **P**, and in the approximation in which D has an even dependence on **P** we have $j_y = 0$; unlike the preceding cases, however, each of the mutually cancelling terms of the expression for $j_y$ is in this case large compared with $j_z$, so that the next higher approximation gives for $j_y/j_z$ not a small quantity, but a quantity on the order of unity.

We note that in all cases when $j_\alpha \approx 0$, there is a corresponding approximate symmetry in the essential region of p-space.

3. For the current outside the metal we get from (4) and (3b)

$$\mathbf{J}^{(e)}(z) = -\frac{2e}{h^3}\sum_s \int\limits_{(v_{zs}>0)} \frac{\mathbf{P}v_{zs}(\mathbf{p})D_s(\mathbf{p})f(\mathscr{E}_s(\mathbf{p}))}{p_z(\mathscr{E}_s(\mathbf{p}) - P^2/2m_0, z)}\,d^3\mathbf{p} \quad (z > z_0), \tag{12}$$

and after going over to integration variables P and E

$$\mathbf{J}^{(e)}(z) = -\frac{2e}{h^3}\int \Phi^{(e)}(E, z)f(E)\,dE,$$

$$\Phi^{(e)}(E, z) = \int_{\Sigma(E)} \frac{\mathbf{P} D(E, \mathbf{P})}{p_z(E - P^2/2m_0, z)} d^2\mathbf{P} \quad (z > z_0). \tag{13}$$

Thus, nonvanishing values of $\Phi^{(e)}(E, z)$ and $J^{(e)}(z)$ are obtained only when account is taken of the output of the dependence of D on P. In this connection, it is convenient to represent $\Phi^{(e)}(E, z)$ in the form

$$\Phi_\alpha^{(e)}(E, z) = \int_{\Sigma(E),\, p_\alpha > 0} \frac{p_\alpha [D(E, \mathbf{P}) - D(E, -\mathbf{P})]}{p_z(E - P^2/2m_0, z)} d^2\mathbf{P} \quad (z > z_0),$$

where the integration is carried out over half the figure $\Sigma(E)$, on which $p_\alpha > 0$. In the case of cold and thermionic emission, when an appreciable contribution to the current is made by a small region of **p**-space, the emitted electrons have tangential-momentum values close to $\pm P_{min}(E^0)$, with $P_{min}(E^0) = 0$ if the corresponding equal-energy surface is crossed by the $p_z$ axis.

The decrease of $J^{(e)}$ with increasing z is due to the decrease in the density of the electrons, resulting in turn from their acceleration in the external field, at the same time that the tangential momentum remains constant.

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Note added in Proof (August 8, 1967). It must be emphasized that inside the emitter, the tangential current is concentrated in a surface layer of thickness on the order of the mean free path of the electrons.

[1] F. I. Itskovich, Zh. Eksp. Teor. Fiz. **50**, 1425 (1966) [Sov. Phys.-JETP **23**, 945 (1966)].

[2] F. I. Itskovich, Zh. Eksp. Teor. Fiz. **51**, 301 (1966) [Sov. Phys.-JETP **24**, 202 (1967)].

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