PHASE TRANSFORMATION OF SPHERICAL NUCLEI INTO NONSPHERICAL NUCLEI

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A phenomenological theory is developed for the phase transformation of spherical nuclei into nonspherical ones. It explains the sudden, discontinuous appearance of equilibrium deformation of the nucleus and also the asymmetric growth of the amplitudes of the zero deformation oscillations near the transition point as deduced from α-decay data. The theory predicts degeneracy of the nonspherical nucleus mass coefficient at the transition point in accordance with the law $(N - N_c)^2$, which agrees with the few data pertaining to octupole oscillations of heavy nuclei.

1. INTRODUCTION

According to modern notions (see, for example [1-3]), weakly-excited states of heavy even-even nuclei have a collective nature. The corresponding collective variables can be represented for the sake of clarity as the deformation parameters which enter in the equation

$$R(\mu, \varphi) = R_0 \left\{1 + \sum_{\lambda\nu} \alpha_{\lambda\nu} y_{\lambda\nu}(\mu, \varphi)\right\} \quad (1)$$

of the nuclear surface. Here $\mu = \cos\vartheta$, φ is the azimuth, and

$$y_{\lambda\nu}(\mu, \varphi) = (-1)^\nu \sqrt{\frac{(\lambda-\nu)!}{(\lambda+\nu)!}} P_\lambda^\nu(\mu) e^{i\nu\varphi} \quad (2)$$

are spherical functions. Such a point of view makes it possible in essence to speak of a function $E(\alpha)$ (we confine ourselves for simplicity, for the time being, to a single collective variable α, the role of which may be assumed, say, by the quadrupole deformation $\alpha_{20} \equiv \alpha_2$), that is, the energy of the nucleus in the ground state of its internal motion can be regarded as a function of its deformation. The equilibrium form of the nucleus is determined by the position of the minimum of $E(\alpha)$. The experimentally observed variation of this equilibrium value $\overline{\alpha}$ is usually explained as being connected with the more or less smooth change in the form of the function $E(\alpha)$ from nucleus to nucleus. Then, in principle, one can visualize two mechanisms under which a nonzero equilibrium deformation can arise. One of the possibilities (see, for example [4]) consists in the fact that the rigidity C of the spherical nucleus, defined as the coefficient of the expansion $E(\alpha) = E_0 + (1/2)C\alpha^2$ of the energy in powers of the deformation, which varies continuously as a function of the number of nucleons, goes to zero and then becomes negative. Schematically, such a sequence of $E(\alpha)$ curves is shown in Fig. 1. The dashed curve describes the continuous growth of the equilibrium deformation from a zero value, which is apparently an inseparable attribute of such a theory. However, this cannot be readily reconciled with the experimental data, which indicate more likely a sudden increase of the value of α, jumpwise, from zero to a finite value at the transition point. Let us consider, for example, the transition of spherical nuclei into nonspherical ones, observed between the emanation and radium. The experimental data [2] on the value of $\overline{\alpha}$ are shown qualitatively by one of the curves of Fig. 2. It is

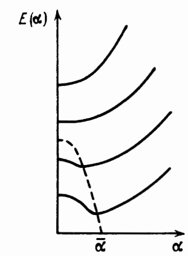

FIG. 1.

typical that immediately beyond radium, for the neighboring even-even nuclei, the difference in the deformation $\overline{\alpha}$ is only ~5% of the initial value which the deformation acquires jumpwise at the transition point; subsequently $\overline{\alpha}$ remains just as sluggish. Another conceivable mechanism is connected with the competition of the two minima on the $E(\alpha)$ curve. Here, as seen from Fig. 3, the deformation actually changes jumpwise at the transition point. However, such a physical picture ties in very poorly with the variation of the amplitudes of the zero-point oscillations of the deformations, which were recently calculated from α-decay data [5]. The results of these calculations are shown schematically by the corresponding curves in Fig. 2. Attention is called to the sharp rise of the zero-point amplitudes of the different deformations of the nonspherical nuclei in the vicinity of the transition point—on the other side of this point the quadrupole amplitude $(\overline{a_2^2})^{1/2}$ of the spherical nucleus behaves in much more stable fashion. Thus, it is precisely at the instant when both minimal values of $E(\alpha)$ coincide on Fig. 3, when the system replaces one potential well by another, that the amplitude of the zero-point oscillations in the second well increa-

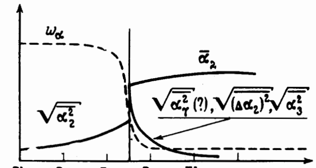

FIG. 2. Static and dynamic characteristics of the form of a heavy nucleus, shown qualitatively not to scale. Data on the amplitude of the α oscillations are essentially indirect and based on analogy with β oscillations [5]. No literal meaning should be attached to the joining of the two curves pertaining to nonspherical nuclei at the transition point – this joining merely expresses the fact that the amplitude of the zero-point oscillations increases here to a value of the order of the equilibrium deformation [5].

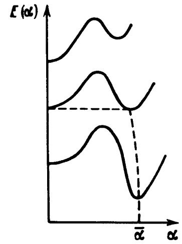

FIG. 3.

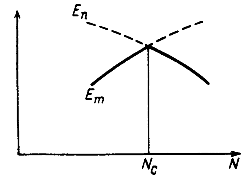

FIG. 4.

ses for some reason. Within the framework of the described theoretical scheme, this looks like a random superposition of two entirely different phenomena, which apparently cannot be regarded as satisfactory.

A common feature of the aforementioned theories is the fact that in final analysis they all operate only with a surface collective Hamiltonian $E(\alpha)$, without touching upon the internal state of the nucleus in any qualitative manner. In other words, it is assumed that the phase transition of spherical nuclei into nonspherical ones pertains, as it were, only to the collective Hamiltonian $E(\alpha)$ of the system. This is probably the reason for the already noted lack of agreement with experiment. Further evidence in favor of the existence, in this case, of a phase transformation also with respect to the internal state of the nucleus is the behavior of the internal probabilities w_α of α-particle production—this quantity increases by one order of magnitude on going from nonspherical nuclei to spherical ones[5] (see Fig. 2). Long ago, Landau[6] obtained with the aid of the detailed balancing principle the relation

$$w_\alpha \infty D, \qquad (3)$$

where D is a certain average distance between the energy levels of the nucleus. If such a quantity experiences so strong a change in the vicinity of the transition point, then the assumed qualitative invariance of the internal structure of the nucleus becomes worse than doubtful.

To understand the exposition that follows, it is necessary to bear in mind that from the phenomenological point of view the structure of the ground state, to which we are referring, becomes manifest in the properties of the spectrum of the weakly excited states of the nucleus. There are weighty reasons for assuming that nuclear matter as such possesses the energy spectrum of a Fermi liquid[7,8]. It is constructed in general in accordance with the same model as the spectrum of a degenerate Fermi gas. However, if we vary the number of nucleons in a real nucleus in such a way that it approaches one of the magic numbers, then the energy spectrum of this type experiences significant changes—it acquires a qualitatively different lower part. This part, roughly speaking, is due essentially to the excitation of the nucleons that are external relative to the magic "core." If we consider, for concreteness, the Pb^{210} nucleus, which has two neutrons beyond the doubly-magic Pb^{208}, then it is precisely their excitation which will cause the first energy levels of the nucleus. Only beyond a certain energy Δ, when a sufficiently large number of core nucleons can also become excited, will a return take place to the "normal" Fermi spectrum. In a less exaggerated example, when the outer shell has, say, four or six nucleons, they are capable, with increasing excitation of the nucleus, of producing quite rapidly a relatively large level density. But the spectral regions under consideration will differ even in their statistical properties, such as the decrement of the growth of the level density with energy. It should be expected that the boundary between them will be statistically sharp—roughly speaking, if the number of core nucleons capable of excitation is sufficiently large the density of the nuclear levels produced by them increases sharply in exponential fashion. The "collapse" of the energy gap Δ, which occurs with increasing distance from the magic numbers, can be visualized as being the result of a residual nuclear interaction between the outer nucleons and the core nucleons[1)]. It is difficult to assume that such a realignment of the energy spectrum and the nucleus occur jumpwise—we are more likely to deal here with a continuous phase transition.

2. PHENOMENOLOGICAL THEORY OF PHASE TRANSFORMATION IN HEAVY NUCLEI

The theory developed below starts from the assumption that the expression for the energy of one of the nuclear phases can be formally continued into the region of the existence of the other phase. We confine ourselves for the time being to consideration of only one sort of nucleons, the number of which in the nucleus will be denoted N. We assume also for concreteness that when the nucleus has a spherical configuration the region $N < N_c$ corresponds to the near-magic phase with nonzero gap Δ (m-phase), and when $N > N_c$, to the contrary, we have the normal state $\Delta = 0$ of nuclear matter (n-phase). This situation is concretized graphically in Fig. 4. Such a simple point of intersection of the phase energy curves is usually associated with the concept of first-order phase transitions which are, in particular, characterized by hysteresis phenomena. As applied to the nucleus this would mean that the nucleus is capable of existing in any of the two states under consideration, which for a given N differ from each other only in energy. It must be decisively emphasized, however, that in this case we have in mind an entirely different physical picture, which has the most essential features of continuous phase transitions—so-called second-order phase transitions[9]. The continuation of the curves for the energy (shown dashed in Fig. 4) into the region not belonging to the given phase corresponds to physically unrealizable states of the nucleus, which actually do not exist. We shall show below, for example, that the right-hand dashed curve, which describes formally the continuation of the m-phase into the region $N > N_c$, corresponds to physically meaningless imaginary values of the gap Δ. Similarly, continuation of the curve E_n to the left of the

1)We use a simplified terminology. In the language of modern Fermi-liquid theory it would be necessary to speak not of nucleon but of the corresponding elementary excitations in the nucleus.

Curie point $N = N_c$ pertains to a state for which the generalized mass B, corresponding to the collective degree of freedom α, is apparently negative (see Sec. 4 below). The state with negative mass coefficient B is completely unstable and cannot be realized in practice.

The energy of the nucleus in the ground state E will be assumed to depend on the quantity Δ, which is regarded as a phenomenological parameter. Since the gap Δ is indeed small in the direct vicinity of the Curie point, we expand here the general expression for the energy in a series:

$$E = E_n - A\Delta + {}^1/_2 D\Delta^2. \tag{4}$$

The coefficients A and D are, generally speaking, certain functions of the number of nucleons and of the deformation; we shall not need the higher terms of the expansion. The quadratic term, which restores the stability, should obviously be positive; the function $D(N, \alpha)$ can be replaced, with accuracy sufficient for our purposes, by its constant value $D > 0$ directly at the Curie point $N = N_c$; $\alpha = 0$.

The gap value which is physically realizable in the m-phase is obtained from the condition $\partial E/\partial\Delta = 0$ for the minimum of (4):

$$\Delta = A/D. \tag{5}$$

Thus, $A > 0$ in m-phase. Substitution in (4) yields

$$E_n - E_m = A^2/2D \tag{6}$$

for the difference between the energy values of the two phases.

Let us consider in greater detail the particular case $\alpha = 0$ of an undeformed nucleus, shown schematically in Fig. 4. Near the Curie point $N = N_c$ we obviously have

$$E_n - E_m = \frac{1}{2}\frac{a^2}{D}(N_c - N), \tag{7}$$

where $a > 0$ is a certain constant. Comparison with (6) leads to

$$A = a\sqrt{N_c - N}. \tag{8}$$

Substitution in (5) shows that when $N < N_c$ the gap

$$\Delta = \frac{a}{D}\sqrt{N_c - N} \tag{9}$$

is closed in accordance with a square-root law.

Let us turn now to the region $N > N_c$. In this region the general expression (4) becomes doubly valued because the function A(N) has a branch point (8). This ambiguity is physically insignificant, and the choice of the sign before the root depends on the arbitrarily chosen direction of time flow. If, say, we choose the form customarily employed in quantum mechanics [10] $\exp[-i(E - i\delta)t/\hbar] \backsim \exp[-\delta t/\hbar]$ for the time dependence of the amplitude of the decaying state, then the transition to the region under consideration must be executed in accordance with the rule:

$$\sqrt{N_c - N} \to +i\sqrt{N - N_c}. \tag{10}$$

Substitution in (8) and (4) shows that any positive Δ leads to the appearance of a negative imaginary addition to the energy, corresponding to decay, and only the value

$$\Delta = 0 \tag{11}$$

corresponds to a truly stationary state.

The inevitability of relation (11) for the n-phase can be explained also from a somewhat different point of view. Let us attempt to use in our region $N > N_c$ the old value of the gap (9) and let us see where this will lead us. The upper level, from which the normal Fermi spectrum begins, always exists; with respect to this level, the ground state of "near magic condensation" corresponds as it were to "elementary excitation" with energy $-\Delta$. Substituting here (9) and taking (10) into account we obtain a negative-imaginary value

$$-\Delta = -i\frac{a}{D}\sqrt{N - N_c}$$

for the energy of this "excitation." In other words, it corresponds to pure decay, of which there actually is none. By the same token, the initial upper level represents in fact a perfectly stable ground state of the nucleus in the case under consideration.

It is easy to see now how the nuclear deformation affects the physical picture described above. We can, for example, start in our reasoning from such a rough approximation as the independent-particle model. Within the framework of similar representations, the "magic" effect is the result of filling of a group of closely lying levels by the nucleons (see, for example, [10]). When the nucleus is deformed, they go into motion, and in final analysis, the levels belonging to different shells intersect[2)]. Inasmuch as Δ is, roughly speaking, a measure of the closeness of the nuclear structure to magic, we obtain, even when $N < N_c$, a transition to the n-phase $\Delta = 0$ if the deformation exceeds a certain definite value α_c.

The energy of the nucleus can depend only on combinations of deformations that are invariant with respect to rotations; the same holds true also with respect to the energy difference (6). Inasmuch as even a small deformation is sufficient to cause this difference to vanish in the vicinity of the Curie point $N = N_c$, we must choose the invariants of the lowest power in α. For any of the possible values $\lambda \geq 2$, such an invariant is

$$I_\lambda = \sum_{\nu=-\lambda}^{\lambda} |\alpha_{\lambda\nu}|^2. \tag{12}$$

If we bear in mind, say, the quadrupole case $\lambda = 2$, then we should actually mean by the square of the deformation α employed by us the analogous invariant

$$\alpha^2 = \sum_{\nu=-2}^{2} |\alpha_{2\nu}|^2.$$

On the other hand, if it is advantageous, for the sake of clarity, to specify concretely the "invariant deformation" α, we can visualize an axially symmetrical case $\alpha = \alpha_{20} \equiv \alpha_2$. Expanding the expression for the difference of the energies of the phases in powers of the deformation and taking the zeroth approximation (7) into account, we get

$$E_n - E_m = \frac{1}{2}\left\{\frac{a^2}{D}(N_c - N) - c\alpha^2\right\}. \tag{13}$$

2)The physical obviousness of the fact that the effect of crossing of single-particle levels upon deformation of the nucleus, leading to a destruction of its magic structure, is additional serious evidence in favor of the existence of the phase transition considered by us.

Since the deformation of the nucleus decreases this difference, bringing closer the instant of transition to the n-phase, it is clear that

$$c > 0. \tag{14}$$

Comparison with (6) and substitution in (5) yield

$$A = \sqrt{a^2(N_c - N) - cD\alpha^2},$$
$$\Delta = \frac{1}{D}\sqrt{a^2(N_c - N) - cD\alpha^2}. \tag{15}$$

These are the relations, so long as the radicand is positive (m-phase). On the other hand, when it is negative, reasoning perfectly analogous to that given above for the particular $\alpha = 0$ case shows that Eq. (11) is valid--we already have the n-phase. The Curie point is obtained from the deformation by equating to zero the radicand

$$\alpha_c^2 = \frac{a^2}{cD}(N_c - N). \tag{16}$$

It is convenient to express all the other quantities in terms of this one. We reduce, for example, our results for the gap to a similar form:

$$\Delta = \begin{cases} \sqrt{\frac{c}{D}}\sqrt{\alpha_c^2 - \alpha^2} & \text{for } \alpha^2 < \alpha_c^2 \text{(m-phase)} \\ 0 & \text{for } \alpha^2 > \alpha_c^2 \text{ (n-phase)} \end{cases} \tag{17}$$

The last inequality is always satisfied when $N > N_c$, for in accord with (16) α_c^2 is negative in this region. If $N \to N_c$, then $\alpha_c \to 0$, and any quantity pertaining to the Curie point $\alpha = \alpha_c$ becomes a characteristic of the spherical nucleus in this limit.

3. JUMP OF RIGIDITY

Differentiating (13) with respect to α twice, we obtain the following expression for the jump of the rigidity of the spherical nucleus at the Curie point $N = N_c$:

$$C_m - C_n = c. \tag{18}$$

In conjunction with (14), it acquires the character of a thermodynamic inequality that expresses, essentially, the simple fact that deformation of a nucleus destroys its magic structure (see the preceding section). The other inequality

$$c > C_m \tag{19}$$

does not follow from thermodynamic considerations, but does not contradict them. The experimental data give the impression that inequality (19) does not behold apparently in many cases, if not all.[3] Then substitution in (18) yields

$$C_n < 0, \tag{20}$$

that is, on going to the normal region $N > N_c$ the rigidity becomes negative jumpwise, making the spherical configuration of the nucleus unstable. As a result, the equilibrium deformation $\overline{\alpha}$ also acquires jumpwise a certain finite value, which agrees with the experimental data (see the Introduction).

It can be seen from (13) that when $N < N_c$ a jump takes place also in the first derivative of the energy

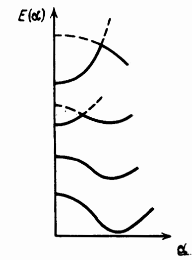

FIG. 5.

with respect to the deformation at the Curie point defined by formula (16):

$$\left.\frac{dE_m}{d\alpha}\right|_{\alpha=\alpha_c} - \left.\frac{dE_n}{d\alpha}\right|_{\alpha=\alpha_c} = c\alpha_c. \tag{21}$$

Thus, at the point $\alpha = \alpha_c$ the true $E(\alpha)$ curve has a kink; the thermodynamic inequality (14) shows the direction in which the tangent is rotated. Figure 5 shows qualitatively the variation of the character of the curves $E(\alpha)$ as a function of the number of nucleons. The dashed continuations of the curves correspond to physically unrealizable states.

4. MASS COEFFICIENT

To consider the properties of the kinetic energy corresponding to the time variation of the collective variables, it is necessary to specify, besides the value of the deformation α, also the generalized rate $\dot{\alpha}$. Differentiating relation (11) with respect to time, we get

$$\dot{\Delta} = 0. \tag{22}$$

In other words, within the limits of the n-phase a finite rate $\dot{\alpha}$ does not generate at all a proportional time variation $\dot{\Delta}$ of the gap.

An important universal feature of all second-order phase transitions is the following: at the phase transition point itself, the properties of the system are the same as on either side of this point[4]. Turning, for example, to relations (17), it is easy to see that in our case this pertains precisely to the region $\alpha > \alpha_c$. Consequently, when $N < N_c$ we should stipulate satisfaction of relations (11) and (22) in the entire region $\alpha \geq \alpha_c$, including the Curie point $\alpha = \alpha_c$. The fact that even at this point the finite rate $\dot{\alpha}$ is still incapable of generating a nonzero derivative can be interpreted physically as a certain unique "hindrance" against the occurrence of a gap Δ when the deformation of the nucleus is decreased.

Let us assume that during the course of such a change in the deformation the mechanical system under consideration goes from the region $\alpha > \alpha_c$ to the Curie point at a finite rate $\dot{\alpha}_n$. The system cannot go over into the region of the m-phase with a nonzero rate $\dot{\alpha}_m$, for this would inevitably involve violation of condition (22) at $\alpha = \alpha_c$.[5] We shall now apply the energy-conservation law directly to the transition point

[3]The latter remark pertains to quadrupole collective variables $\lambda = 2$.

[4]The corresponding theorem was proved by group methods [9] for the case when the transition is manifest in a change of the crystallographic symmetry of the body.

[5]As seen from (17), in this case the concrete character of the violation would consist of $\dot{\Delta}$ increasing jumpwise from zero to infinity.

$$\tfrac{1}{2} B_m(\alpha_c)\dot{\alpha}_m^2 + E_m(\alpha_c) = \tfrac{1}{2} B_n(\alpha_c)\dot{\alpha}_n^2 + E_n(\alpha_c). \quad (23)$$

The potential energy E experiences no jump anywhere--it was shown above (see the preceding section) that the $E(\alpha)$ curve has only a kink at the Curie point. Therefore the terms of (23) corresponding to the potential energy of the collective motion cancel out. Further, by virtue of the foregoing, $\dot{\alpha}_m = 0$, whereas $\dot{\alpha}_n$, by definition, differs from zero. Therefore

$$B_n(\alpha_c) = 0. \quad (24)$$

It follows hence that the tendency of the mass coefficient to approach zero as $N \to N_c$ in the region $N > N_c$ (n-phase), which is of greater direct physical interest, is also present. By the same token, this explains the growth of the amplitudes of the oscillations of the nonspherical nucleus near the Curie point, which is shown schematically in Fig. 2 (see the Introduction). Indeed, the amplitude of the zero-point oscillations of the deformation about a certain equilibrium position is connected with the rigidity C and the mass B of the corresponding oscillator by the well known relation $\overline{(\Delta\alpha)^2} = \hbar/2\sqrt{BC}$, that is, it increases with decreasing mass coefficient.

The law according to which the mass coefficient of the normal phase--as a function of the deformation or of the number of particles—tends to zero at the Curie point cannot be determined from purely phenomenological considerations. It can be established, however, by using the well known quantum-mechanical expression (see, for example, [4])

$$B_n = 2\hbar^2 \sum_{k\neq n} \left| \int \psi_k^* \frac{\partial\psi_n}{\partial\alpha} d\xi \right|^2 (E_k - E_n)^{-1} \quad (25)$$

for the mass coefficient in terms of the wave functions of the internal motion in the nucleus. Here ψ_k and E_k are the normalized wave functions and the corresponding eigenvalues of the energy of the "internal Hamiltonian" of the nucleus, which depends on the connective variable α as a parameter; $\int d\xi$ denotes integration over the configuration space of the internal degrees of freedom. We put now $\alpha = \alpha_c$ and compare (25) with (24). Since E_n is the energy of the ground state of the even-even nucleus, and consequently none of the terms of the sum (25) are negative, each of the integrals in the numerator vanishes separately:

$$\int \psi_k^* \frac{\partial\psi_n}{\partial\alpha} d\xi = 0, \quad k \neq n, \quad \alpha = \alpha_c. \quad (26)$$

In other words, at the Curie point the expansion of the derivative $\partial\psi_n/\partial\alpha$ in the eigenfunctions of the problem does not contain wave functions of other stationary states of the nucleus at all. Nor does this expansion contain ψ_n itself, since the ground state under consideration is nondegenerate, and its wave function can be chosen to be real.[6] Differentiating with respect to α the condition $\int \psi_n^2 d\xi = 1$ for its normalization, we get

$$\int \psi_n^* \frac{\partial\psi_n}{\partial\alpha} d\xi = 0. \quad (27)$$

As a result we must have

$$\left.\frac{\partial\psi_n}{\partial\alpha}\right|_{\alpha=\alpha_c} = 0. \quad (28)$$

Thus, $\alpha = \alpha_c$ is a kind of "point of indifference" of the internal state of the normal phase relative to a change of the collective variable α. This constitutes the microscopic manifestation of that "hindered" character of the occurrence of the gap Δ when the system moves from the region $\alpha > \alpha_c$, which was already discussed above from a different, phenomenological point of view. A condition of the type (28) for the internal wave function expresses in most essential fashion the effect of interaction between nucleons (elementary excitations) in the nucleus—no such relation could arise in the independent-particle model, nor, incidentally, the phase transition phenomenon itself. It would apparently be difficult to relate such an equality likewise with the simplified notion that the nucleus is a system of nucleons immersed in an infinitely deep potential well with sharply delineated boundaries. This equality, of course, does not denote complete loss of sensitivity of the internal state α_n to variation of the parameter α—the insensitivity, so to speak, takes place only in the linear approximation, and a relation between them does remain in the higher, second order in the deformation increment. Turning to (25), we see that at the Curie point, besides (24), we also have

$$\left.\frac{dB_n}{d\alpha}\right|_{\alpha=\alpha_c} = 0 \quad (29)$$

and only

$$\left.\frac{d^2B_n}{d\alpha^2}\right|_{\alpha=\alpha_c} = 4\hbar^2 \sum_{k\neq n} \left| \int \psi_k^* \frac{\partial^2\psi_n}{\partial\alpha^2} d\xi \right|^2 (E_k - E_n)^{-1} > 0. \quad (30)$$

Now, taking the derived relations into account, it is necessary to express the mass coefficient of the n-phase in terms of invariant combinations of the deformations (see the analogous reasoning in Sec. 2, pertaining to the potential energy $E(\alpha)$ of nuclear deformation). The expression satisfying these requirements is of the form

$$B_n = b(\alpha^2 - \alpha_c^2)^2. \quad (31)$$

Of fundamental physical interest is the dependence of the mass coefficient on the number of nucleons in the region $N > N_c$. Putting $\alpha = 0$ (the actual nonsphericity of the nuclei under consideration can be neglected here) and substituting (16) in (31), we get

$$B_n = \frac{ba^4}{c^2D^2}(N - N_c)^2 \quad (32)$$

We turn now to the region $N < N_c$ and consider the vicinity of the Curie point $\alpha = \alpha_c$. In its direct vicinity (31) goes over into

$$B_n \cong 4b\alpha_c^2(\alpha - \alpha_c)^2. \quad (33)$$

Since the mass coefficient B_n depends on the difference $\alpha - \alpha_c$ quadratically, an impression can be gained that on the other side of the transition point it will also assume reasonable positive values. In fact, however, expression (33) no longer has a physical meaning corresponding to the state ψ_n of the mass coefficient in the

[6] The presence of spin variables in the configuration ξ-space does not introduce any essential changes in our reasoning, since the concept of "reality" can also be extended to a spinor in our case. As is well known, such a possibility is closely related to the symmetry of the nondegenerate state relative to time reversal [10].

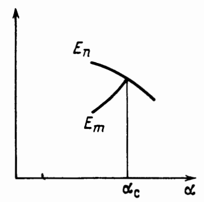

FIG. 6.

region $\alpha < \alpha_c$. Indeed, there appears here a new state—"near magic" ψ_m—which did not exist at all for $\alpha > \alpha_c$ (of course, the imaginary value of α given by the second formula of (15) does not correspond to any real state of the internal motion in the nucleus even for fixed α). From the microscopic point of view, when, in particular, E_n and E_m are regarded as "internal Hamiltonian" terms that depend on the parameter α, this situation is shown schematically in Fig. 6. The general formula (25) requires summation over all the existing states; therefore when $\alpha < \alpha_c$ we have

$$B_n = 2\hbar^2 \Big\{ -\Big| \int \psi_m^* \frac{\partial \psi_n}{\partial \alpha} d\xi \Big|^2 (E_n - E_m)^{-1} + \sum_{k \neq m,n} \Big| \int \psi_k^* \frac{\partial \psi_n}{\partial \alpha} d\xi \Big|^2 (E_k - E_n)^{-1} \Big\}. \quad (34)$$

Only the second term is given here by expression (33), and the smallness of the denominator $E_n - E_m$, which according to (13) and (16) is proportional to the difference $\alpha_c - \alpha$, shows that the first negative term predominates in (34).

However, a certain stipulation must be made with respect to the last conclusion: depending on the parameter α, the wave function ψ_m has, as a function of the parameter α, a singularity of unknown character at $\alpha = \alpha_c$. From this point of view, our last result admits, strictly speaking, of only the following more careful formulation: there are weighty grounds, both physical and mathematical, for assuming that the mass coefficient B becomes negative in the region $\alpha < \alpha_c$ where the m-phase exists. Therefore the state ψ_n is here completely unstable and cannot be in fact realized as a real stationary state of the nucleus. To ascertain the form of the singularity of the function α_m at the point $\alpha = \alpha_c$ it will probably be impossible to formulate a microscopic theory of the phenomenon considered by us with full mathematical rigor.[7)]

5. GENERALIZATION OF THE THEORY. COMPARISON WITH EXPERIMENT

Allowance for the simultaneous presence of collective degrees of freedom of the nucleus, corresponding to different λ in the right side of (1), entails no special difficulty. For each λ there exists its own invariant (12), and the invariant combination of the lowest (second) power in the deformation has obviously the following most general form:

$$J = \sum_{\lambda \geq 2} c_\lambda J_\lambda. \quad (35)$$

The energy difference of the phases in the direct vicinity of the Curie point $N = N_c$

$$E_n - E_m = \frac{1}{2} \Big\{ \frac{a^2}{D} (N_c - N) - J \Big\} \quad (36)$$

can depend only on this combination. The "Curie point" with respect to the deformation is given by the relation

$$J_c = \frac{a^2}{D} (N_c - N), \quad (37)$$

and all further generalizations are obvious. We present only the important formula for the jump in rigidity:

$$C_\lambda^{(m)} - C_\lambda^{(n)} = c_\lambda. \quad (38)$$

The thermodynamic inequality

$$c_\lambda > 0, \quad \lambda \geqslant 2, \quad (39)$$

which generalizes (14), corresponds physically to the fact that each of the deformations $\alpha_{\lambda\nu}$ destroys separately the magic structure in the nucleus, if the invariant (35) corresponding to it exceeds the value (37). To the contrary, an inequality of the type (19), which does not follow of necessity from thermodynamic considerations, does not call for any generalization. And indeed even for octupole deformation $\lambda = 3$ experiment apparently yields

$$c_3 < C_3^{(m)}, \quad C_3^{(n)} > 0,$$

which contradicts (19) and (20). In other words, insofar as can be judged from the experimental data (see, for example,[2]) even after exact transformation of spherical nuclei into nonspherical ones $\alpha_{30} \equiv \alpha_3 = 0$, that is, the equilibrium shape of the nucleus conserves the symmetry center.

The results of the preceding sections can likewise be generalized to arbitrary λ. Retaining where possible the earlier notation, we readily obtain

$$B_\lambda^{(n)} = (b_\lambda / c_\lambda^2)(J - J_c)^2, \quad b_\lambda > 0, \quad \lambda \geqslant 2. \quad (40)$$

Going over to the region of physical interest, $N > N_c$ we now express this quantity in terms of the number of particles:

$$B_\lambda^{(n)} = (b_\lambda a^4 / c_\lambda^2 D^2)(N - N_c)^2. \quad (41)$$

Thus, regardless of the number λ of the spherical harmonic in the right side of (1), the corresponding mass coefficient of the nonspherical nucleus near the Curie point tends to zero like $(N - N_c)^2$.

Essentially, the entire preceding exposition did not exclude the presence in the nucleus of two different sorts of nucleons. If we take N to mean the number of neutrons, then we can assume that we always had the case Z = const in mind. Such a line drawn on the Z—N plane will cross the line separating the two nuclear phases at a certain angle, the concrete value of which, reasoning abstractly, will not influence the correctness of the limiting quadratic behavior of the form (41). From the practical point of view, however, in order to exclude where possible the regular and smooth variation of the mass coefficient along the curve separating the phases, it is desirable to compare the law $B^{(n)}$ 11 $(N - N_c)^2$ with experiment along directions that are normal to this curve, or close to normal.

Experimental data on α decay of nonspherical heavy nuclei at the relatively low-lying level 1^- of octupole oscillations of the daughter nucleus have made it possible to determine the corresponding mass coefficient $B^{(3n)}$ in nine cases[5]. The results are listed in the

[7)]For similar reasons, it is still somewhat difficult to analyze from the microscopic point of view the limiting behavior of the true mass coefficient of the nucleus in the state ψ_m at the Curie point.

Decay	$\hbar^2/4B_3$, keV	Decay	$\hbar^2/4B_3$, keV
$Th^{224} \to Ra^{220}$	0,99	$U^{230} \to Th^{226}$	0,09
$Th^{226} \to Ra^{222}$	0,40	$U^{232} \to Th^{228}$	0,023
$Th^{228} \to Ra^{224}$	0,095	$Cm^{242} \to Pu^{238}$	0,035
$Th^{230} \to Ra^{226}$	0,042	$Cm^{244} \to Pu^{240}$	0,029
$U^{228} \to Th^{224}$	0,16		

table. The last two values pertain to regions that are quite far from the phase-transition line, and can hardly be useful for verifying the limiting relation (41). The remaining seven values of $\hbar^2/4B_3$ correspond to the points shown in Fig. 7; the axes represent the numbers of the nucleons, reckoned from the doubly magic Pb^{208}.[8)] Qualitative considerations give grounds for assuming that the line separating the phases is in the form of the "dashed" lobe, the ends of which correspond to the violation of the magic structure of lead by protons alone or by neutrons alone. The solid section of the curve lies between the four nonspherical isotopes of radium and the four isotopes of the emanation, all of which are known from experiment to be apparently spherical. Thus, the arbitrariness in the drawing of the transition line becomes practically insignificant here. On the lines drawn in the direction close to normal to the section of the separation curve under consideration there are located, as seen from Fig. 7, three pairs of points. Along each of them, within the limits of experimental accuracy, the quadratic law (41) holds for the variation of the mass coefficient of the octupole oscillations.

It would be advantageous to devise experiments capable of determining and refining the boundary between two nuclear phases on the Z—N plane in other sections of this plane.

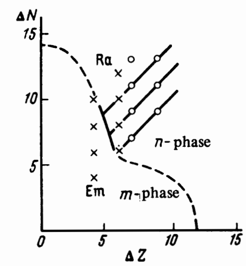

FIG. 7.

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8)The points correspond to odd numbers of nucleons because they are regarded as pertaining to a certain hypothetical nucleus, halfway between the daughter and the parent. See [5] for a more rigorous formulation of the law of averaging of the characteristics extracted from data on α decay between the parent and daughter nuclei.

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67