ASYMPTOTIC BEHAVIOR OF MANDELSTAM TYPE DIAGRAMS WHEN SPINS ARE TAKEN INTO ACCOUNT. I

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The asymptotic behavior of diagrams of the Mandelstam type (i.e., diagrams leading to the appearance of moving branch points in the complex j-plane) is studied by a straightforward calculation of the Feynman integrals with particle spins taken into account.

1. INTRODUCTION

THE asymptotic behavior of the so called Mandelstam diagram (that is, a diagram that leads to the appearance of moving branch points in the complex j-plane) was considered by Polkinghorne[1] with the aid of a method developed by him for directly calculating the Feynman integrals at $t \to \infty$, and later by Swift[2] by a similar method, in which the Mellin transform was used. However, the analysis in these papers did not take the particle spins into account. Allowance for the spins leads to an essential change in the asymptotic behavior, but also gives rise to considerable difficulties in the calculation. Thus, the appearance of a combination of Feynman parameters in the numerator of the integral does not make it possible to set this integral in correspondence with any definite reduced diagram, as was done in[1]. This in turn does not make it possible to sum over the number of the rungs of the ladder.

We shall use a somewhat modernized Polkinghorne method, in which the diagram is broken up into blocks, thus greatly simplifying the calculation and permitting the summation.

In Sec. 2 of this paper we apply the proposed method first to a scalar Mandelstam diagram. The result coincides with that of Polkinghorne[1]. In Sec. 3 we consider in detail a Mandelstam diagram with one internal ladder in the case of scattering of scalar particles with exchange of neutral massive vector mesons. The analysis is carried in renormalizable theory with conserved current. In Sec. 4 we discuss briefly the calculation of the asymptotic form of the Mandelstam diagram with two internal ladders.

Diagrams that lead to the appearance of moving branch points were considered earlier by a number of authors[3,4] by the method of complex angular momenta.

2. ANALYSIS WITHOUT ALLOWANCE FOR SPIN

We shall use the usual notation

$$s = p^2 = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2,$$

with $s + t + u = 4m^2$. We seek the asymptotic behavior of the diagram as $t \to \infty$ and with fixed s.

Let us consider that part of the denominator of the total Feynman integral for the diagram of Fig. 1, which pertains to the ladder insert. Introducing the Feynman parameters, we write this denominator in the form

$$M_3 = (3n)! \int_0^1 \prod_1^{n+1} d\alpha_i \prod_1^n d\sigma_i \, d\sigma_i' \delta\Big(\sum (\alpha_i + \sigma_i + \sigma_i') - 1\Big) \times \Psi_3^{-(3n+1)}, \tag{1}$$

where

$$\Psi_3 = \alpha_1 (k_1 - l_1)^2 + \alpha_{n+1} (k_n - l_2)^2 + \sum_2^n \alpha_i (k_i - k_{i-1})^2 + \sum_1^n \sigma_i k_i^2 + \sum_1^n \sigma_i' (k_i - l_3)^2 - m^2 = \sum_1^n \lambda_i \bar{k}_i^2 + \sum_1^3 A_{ij} l_i l_j - m^2. \tag{2}$$

The form Ψ_3 was reduced here by means of the change of variables $k_i \to \bar{k}_i$ to a canonical form. A_{ij} are functions of α, σ, and σ' only.

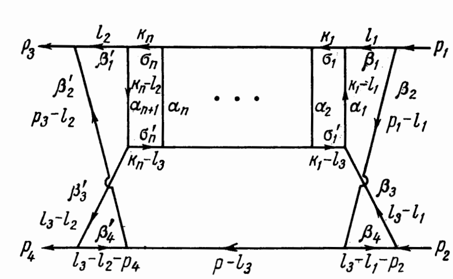

FIG. 1.

We now consider that part of denominator, which pertains to the side loops of the diagram:

$$M_1 = \{(l_1^2 - m^2)[(p_1 - l_1)^2 - m^2][(l_3 - l_1)^2 - m^2] \times [(p_2 + l_1 - l_3)^2 - m^2]\}^{-1} = 3! \int_0^1 \prod_1^4 d\beta \delta\left(\sum \beta - 1\right) \Psi_1^{-4}, \quad (3)$$

where

$$\Psi_1 = l_1^2 + l_3^2(\beta_3 + \beta_4) - 2l_1 l_3(\beta_3 + \beta_4) - 2l_1(\beta_2 p_1 - \beta_4 p_2) - 2\beta_4 l_3 p_2 + \beta_2 p_1^2 + \beta_4 p_2^2 - m^2. \quad (4)$$

Also

$$M_2 = 3! \int_0^1 \prod_1^4 d\beta' \delta\left(\sum \beta' - 1\right) \Psi_2^{-4} \quad (5)$$

where

$$\Psi_2 = l_2^2 + l_3^2(\beta_3' + \beta_4') - 2l_2 l_3(\beta_3' + \beta_4') - 2l_2(\beta_2' p_3 - \beta_4' p_4) - 2\beta_4' l_3 p_4 + \beta_2' p_3^2 + \beta_4' p_4^2 - m^2. \quad (6)$$

Now the total Feynman integral is written in the form

$$F_n = 3!3!(3n)!(g^2)^{n+4} \int_0^1 d\alpha \ldots d\beta' \int \frac{d^4 l_1 d^4 l_2 d^4 l_3 \prod_1^n d^4 \bar{k}_i}{\Psi_1^4 \Psi_2^4 \Psi_3^{3n+1} \Psi_4} \quad (7)$$

Here $\Psi_4 = (p - l_3)^2 - m^2$. The last integral (with respect to the momenta) in (7) is again transformed with the Feynman rule, and again we get for F_n

$$F_n = (3n+9)!(g^2)^{n+4} \int_0^1 d\alpha \ldots d\beta' \times \int_0^1 \prod_1^4 d\gamma_i \delta\left(\sum \gamma_i - 1\right) \gamma_1^3 \gamma_2^3 \gamma_3^{3n} \times \int d^4 l_1 d^4 l_2 d^4 l_3 \prod_1^n d^4 \bar{k}_i \cdot \Psi^{-(3n+10)}. \quad (8)$$

The form $\Psi = \sum_1^4 \gamma_i \Psi_i$ reduces to the canonical form

$$\Psi = \sum_1^3 \eta_i \bar{l}_i^2 + \gamma_3 \sum_1^n \lambda_i \bar{k}_i^2 + D/\Delta, \quad (9)$$

with

$$\Delta = \begin{vmatrix} a_{11} a_{12} a_{13} \\ a_{12} a_{22} a_{23} \\ a_{13} a_{23} a_{33} \end{vmatrix}, \qquad D = \begin{vmatrix} a_{11} a_{12} a_{13} b_1 \\ a_{12} a_{22} a_{23} b_2 \\ a_{13} a_{23} a_{33} b_3 \\ b_1 \; b_2 \; b_3 \; c \end{vmatrix}, \quad (10)$$

$$a_{11} = \gamma_1(\beta_1 + \beta_2 + \beta_3 + \beta_4) + \gamma_3 A_{11},$$
$$a_{22} = \gamma_2(\beta_1' + \beta_2' + \beta_3' + \beta_4') + \gamma_3 A_{22},$$
$$a_{33} = \gamma_1(\beta_3 + \beta_4) + \gamma_2(\beta_3' + \beta_4') + \gamma_3 A_{33} + \gamma_4, \quad a_{12} = \gamma_3 A_{12},$$
$$a_{13} = -\gamma_1(\beta_3 + \beta_4) + \gamma_3 A_{13}, \quad a_{23} = -\gamma_2(\beta_3' + \beta_4') + \gamma_3 A_{23},$$
$$b_1 = -\gamma_1(\beta_2 p_1 - \beta_4 p_2), \quad b_2 = -\gamma_2(\beta_2' p_3 - \beta_4' p_4),$$
$$b_3 = -(\gamma_1 \beta_4 p_2 + \gamma_2 \beta_4' p_4 + \gamma_4 p), \quad (11)$$
$$c = \gamma_1(\beta_2 p_1^2 + \beta_4 p_2^2) + \gamma_2(\beta_2' p_3^2 + \beta_4' p_4^2) + \gamma_4 s - m^2.$$

Following Polkinghorne[11], we can write

$$D = gt + f, \quad g = xyC + xP_1 + yP_2 + Q,$$
$$x = \beta_2\beta_3 - \beta_1\beta_4, \quad y = \beta_2'\beta_3' - \beta_1'\beta_4'. \quad (12)$$

In this notation

$$C = \gamma_1^2 \gamma_2^2,$$
$$P_1 = -\gamma_1^2 \gamma_2 \gamma_3 [\beta_4' A_{22} + (\beta_2' + \beta_4') A_{23}],$$
$$P_2 = -\gamma_1 \gamma_2^2 \gamma_3 [\beta_4 A_{11} + (\beta_2 + \beta_4) A_{13}], \quad (13)$$
$$Q - P_1 P_2 C^{-1} = -\gamma_1 \gamma_2 \gamma_3 A_{12} [(\beta_2 + \beta_4)(\beta_2' + \beta_4') a_{33} + \beta_4(\beta_2' + \beta_4') a_{13} + \beta_4'(\beta_2 + \beta_4) a_{23} + \beta_4 \beta_4' a_{12}].$$

Now

$$\Psi = \frac{1}{\Delta}(xyC + xP_1 + yP_2 + Q)t + \frac{f}{\Delta} + \sum_1^3 \eta_i \bar{l}_i^2 + \gamma_3 \sum_1^n \lambda_i \bar{k}_i^2. \quad (14)$$

The necessary conditions for the integration contour to encounter two coinciding singularities, expressed in terms of the new variables x and y ($\partial\Psi/\partial x = \partial\Psi/\partial y = \Psi = 0$), yield

$$x = -P_2 C^{-1}, \quad y = -P_1 C^{-1}, \quad (15a)$$

$$\frac{1}{\Delta}(Q - P_1 P_2 C^{-1})t + \frac{f}{\Delta} + \Sigma \eta_i \bar{l}_i^2 + \gamma_3 \Sigma \lambda_i \bar{k}_i^2 = 0. \quad (15b)$$

at the point of encounter with the contour. As $t \to \infty$, it follows from (15b) that

$$Q - P_1 P_2 C^{-1} \to 0. \quad (16)$$

since f is bounded and the integral in (8) is dominated by small $\bar{k}_i^2$ and $\bar{l}_i^2$ when the substitution $l_i \to \bar{l}_i$ is made. However, $A_2 = -\prod_1^{n+1} \alpha_i/\Delta_1$ (Δ_1 is a determinant similar to Δ, but for the internal ladder). It follows therefore from (13) and (16) that the integration region essential for the asymptotic form is the one in which

$$\alpha_1 \sim \ldots \sim \alpha_{n+1} \sim 0. \quad (17)$$

We then get from expression (2) for Ψ_3 that $A_{33}^0 = \sum_1^n \sigma_i \sigma_i'/(\sigma_i + \sigma_i')$, the remaining A_{ij}^0 vanish, and $\Delta_1^0 = \prod_1^n \lambda_i^0 = \prod_1^n (\sigma_i + \sigma_i')$. In addition, $P_1 \sim P_2 \sim 0$, and from (15a) we get

$$x = y = 0. \quad (18)$$

Now, under conditions (17) and (18), we have

$$\frac{1}{\Delta_0}(Q - P_1 P_2 C^{-1}) = \frac{\gamma_3 \beta_4 \beta_4' \prod_1^{n+1} \alpha_i}{(\beta_3 + \beta_4)(\beta_3' + \beta_4')} \frac{1}{\Delta_1^0}, \quad (19)$$

where the determinant Δ_0 was obtained from the determinant Δ under conditions (17) and (18). From (19) under condition (16) it follows also that $\beta_4 \sim \beta_4' \sim 0$. We note that since the parameter γ_3 enters in the numerator of the integral (8), it cannot be regarded as small in the asymptotic form.

This very important statement can be explained, in particular, in the following manner: The integral

$$\int_0^{\varepsilon_1}\ldots\int_0^{\varepsilon_n}\frac{d\alpha_1\ldots d\alpha_n}{(\alpha_1\ldots\alpha_n t+f)^q}\qquad(\varepsilon_i\geqslant 0)$$

in the asymptotic form does not depend on its upper limits when $t\to\infty$ and therefore the ϵ_i can be chosen sufficiently small. To the contrary, the integral

$$\int_0^{\varepsilon_1}\ldots\int_0^{\varepsilon_n}\frac{\alpha_1\ldots\alpha_m\, d\alpha_1\ldots d\alpha_n}{(\alpha_1\ldots\alpha_n t+f)^q}\qquad(m<n)$$

is proportional to $\epsilon_1, \ldots, \epsilon_m$ and reaches its maximum value at $\epsilon_i = 1$ $(i = 1, \ldots, m)$.

Integrating in (8) with respect to x and y, with respect to the small parameters α_i, β_4, and β_4', and also with respect to β_2 and β_2' with the aid of $\delta(\beta_2-\beta_1\beta_4/\beta_3)$ and $\delta(\beta_2'-\beta_1'\beta_4'/\beta_3')$, we get

$$F_n=(3n+7)!\,(g^2)^{n+4}\frac{2\pi i}{t}\frac{(\ln t)^{n+2}}{(n+2)!\,t}\times\int_0^1 d\sigma\, d\sigma'\Delta_1^0\delta\Big(\sum(\sigma+\sigma')-1\Big)\times\int_0^1 d\beta_1\, d\beta_3\,\delta(\beta_1+\beta_3-1)\int_0^1 d\beta_1'\, d\beta_3'\,\delta(\beta_1'+\beta_3'-1)\cdot\times\int d^4\bar l_1 d^4\bar l_2 d^4\bar l_3\prod d^4\bar k_i\int_0^1 d\gamma\,\delta\Big({\sum}'\gamma-1\Big)\times\gamma_1\gamma_2\gamma_3^{3n-1}\Delta_0\Psi_0^{-(3n+8)}. \tag{20}$$

It is easy to find that

$$\Delta_0=\gamma_1\gamma_2(a_{33}-\gamma_1\beta_3^2-\gamma_2\beta_3'^2)=\gamma_1\gamma_2(\gamma_1\beta_1\beta_3+\gamma_2\beta_1'\beta_3'+\gamma_3A_{33}^0+\gamma_4). \tag{21}$$

Therefore the last integral in expression (20) for F_n is written in the form

$$(3n+7)!\int_0^1\prod_1^4 d\gamma\,\delta\Big(\sum\gamma-1\Big)\times\gamma_1^2\gamma_2^2\gamma_3^{3n-1}\frac{\gamma_1\beta_1\beta_3+\gamma_2\beta_1'\beta_3'+\gamma_3A_{33}^0+\gamma_4}{\Big[\sum_1^4\gamma_i\Psi_i^0\Big]^{3n+8}}. \tag{22}$$

Each of the four terms in (22) can be integrated in accordance with the Feynman rule, after which we get the sum

$$\frac{3!\,2!\,(3n-1)!\,\beta_1\beta_3}{\Psi_1^4\Psi_2^3\Psi_3^{3n}\Psi_4}+\frac{2!\,3!\,(3n-1)!\,\beta_1'\beta_3'}{\Psi_1^3\Psi_2^4\Psi_3^{3n}\Psi_4}+\frac{2!\,2!\,(3n)!\,A_{33}^0}{\Psi_1^3\Psi_2^3\Psi_3^{3n+1}\Psi_4}+\frac{2!\,2!\,(3n-1)!}{\Psi_1^3\Psi_2^3\Psi_3^{3n}\Psi_4^2}. \tag{23}$$

Here

$$\Psi_1=\bar l_1^2+\beta_1\beta_3 l_3^2-m^2,\qquad \Psi_2=\bar l_2^2+\beta_1'\beta_3' l_3^2-m^2,\qquad \Psi_3=\sum_1^n\lambda_i^0\bar k_i^2+A_{33}^0 l_3^2-m^2. \tag{24}$$

We now substitute (23) and (24) in formula (20) and integrate over all the momenta k_i and l_i with the exception of l_3. As a result, F_n is expressed in terms of an integral with respect to l_3, which we write in a two-dimensional representation by introducing the new variables $s_1 = l_3^2$ and $s_2 = (p-l_3)^2$:

$$F_n=2\frac{(\ln t)^{n+2}}{(n+2)!\,t^2}\frac{g^4 i\pi^2}{2s}\int_{\lambda\leqslant 0} ds_1\, ds_2[\lambda(s,s_1,s_2)]^{1/2}\times\left(\frac{d}{ds_1}+\frac{d}{ds_2}\right)\frac{[I_0(s_1)]^{n+2}}{s_2-m^2}. \tag{25}$$

Here

$$I_0(s_1)=g^2 i\pi^2\int_0^1\frac{dx}{s_1x(1-x)-m^2}, \tag{26}$$

$$\lambda(s,s_1,s_2)=s^2+s_1^2+s_2^2-2(ss_1+ss_2+s_1s_2). \tag{27}$$

The limit of the region of integration is determined by the equation $\lambda = 0$.

Integrating in (25) by parts, we obtain ultimately the expression for F_n:

$$F_n=2\frac{(\ln t)^{n+2}}{(n+2)!\,t^2}g^4 i\pi^2\int_{\lambda\leqslant 0}\frac{ds_1\, ds_2}{[\lambda(s,s_1,s_2)]^{1/2}}\frac{[I_0(s_1)]^{n+2}}{s_2-m^2}. \tag{28}$$

This result coincides with the result obtained by Polkinghorne[1].

3. MANDELSTAM DIAGRAM WITH ALLOWANCE FOR SPIN

Let us consider the scattering of two scalar particles with exchange of neutral massive vector mesons in the s-channel. But before we proceed to the study of the Mandelstam diagram let us investigate briefly the asymptotic behavior of an ordinary ladder diagram with allowance for spin. This will help us to understand better the results obtained for the Mandelstam diagram.

Thus, let us consider the ladder of Fig. 2. The dashed lines correspond to massive particles with spin 1, and the solid lines to particles with spin 0. Vertices with two scalar particles and one vector particle correspond to the quantity $g(k_1 + k_2)_\mu$. In renormalized theory with conserved current, the

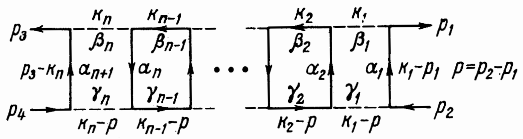

FIG. 2.

vector-particle propagator is proportional to $g_{\mu\nu}/(k^2 - m^2)$.

The case of a scalar ladder has been thoroughly investigated and it is known that the maximum contribution to the asymptotic form is obtained for all $\alpha_i \sim 0$. Let us see what results from allowance for the particle spins. The numerator of the Feynman integral differs from unity and is equal to

$$N = [(2p_1 + k_1)(2k_2 - k_1)][(2p_1 + k_1 + p)(2k_2 - k_1 - p)]$$
$$\times[(2p_3 - k_n)(2k_{n-1} - k_n)][(2p_3 - k_n - p)$$
$$\times(2k_{n-1} - k_n - p)]\bar{N},$$

$$\bar{N} = \prod_{i=3}^{n-2}[(2k_{i-1} - k_i)(2k_{i+1} - k_i)]$$
$$\times[(2k_{i-1} - k_i - p)(2k_{i+1} - k_i - p)]. \quad (29)$$

The index i assumes here odd values from 3 to $n - 2$ (n odd). On going over to new variable momenta in the Feynman integral, it is necessary also to make an appropriate substitution in the numerator of (29), the form of which for $\alpha_i = 0$ is simply

$$k_i = \bar{k}_i + \lambda_i p, \qquad \lambda_i = \gamma_i / (\gamma_i + \beta_i). \quad (30)$$

Substituting (30) in (29) and taking into account the subsequent integration over the momenta with odd indices, it is easy to see that the greatest contribution to the asymptotic form will be determined by the product of the "coupled" factors

$$N \sim 16(p_1\bar{k}_2)^2 \prod_{i=3}^{n-2} 16(\bar{k}_{i-1}\bar{k}_{i+1})^2 \cdot 16(p_3\bar{k}_{n-1})^2, \quad (31)$$

which leads, with allowance for the subsequent integration over the even momenta, to the expression

$$N = 3(4/3)^{(n+1)/2}\bar{k}_2^4 \ldots \bar{k}_{n-1}^4 t^2. \quad (32)$$

We see from the foregoing that the alternation of the vector and scalar loops in the ladder diagram causes the factors making up the numerator (29) to be

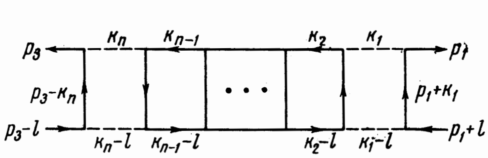

FIG. 3.

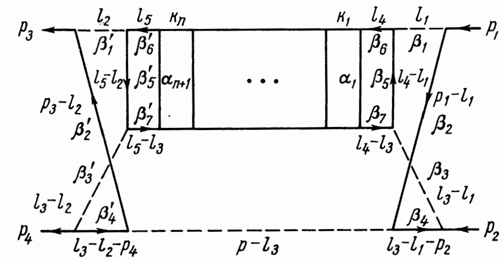

FIG. 4.

"coupled," and we get as a result of integration with respect to the k_i $N \sim (p_1p_3)^2 \sim t^2$. Thus, the presence of two vector lines in the s-channel (alternating with scalar lines) gives an additional factor t^2 in the asymptotic form.

Let us consider now the ladder shown in Fig. 3. A feature of this ladder is that each of its contours depends on the total momentum l, with respect to which we also integrate. We shall show that to obtain t^2 in the numerator in this case, it is not essential to have alternation of vector and scalar lines. In particular, let all the contours of the ladder, except the first and last, contain only scalar particles. Then

$$N = [(2p_1 + k_1)(2k_2 - k_1)][(2p_1 + k_1 + l)(2k_2 - k_1 - l)].$$
$$\times[(2p_3 - k_n)(2k_{n-1} - k_n)]$$
$$\times[(2p_3 - k_n - l)(2k_{n-1} - k_n - l)]. \quad (33)$$

For $\alpha_i = 0$, we make the substitution

$$k_i = \bar{k}_i + \lambda_i l. \quad (34)$$

We obtain

$$N = [(2p_1 + \bar{k}_1 + \lambda_1 l)(2\bar{k}_2 - \bar{k}_1 + [2\lambda_2 - \lambda_1]l)]$$
$$\times[(2p_1 + \bar{k}_1 + [1 + \lambda_1]l).$$
$$\times(2\bar{k}_2 - \bar{k}_1 + [2\lambda_2 - \lambda_1 - 1]l)][(2p_3 - \bar{k}_n - \lambda_n l)$$
$$\times(2\bar{k}_{n-1} - \bar{k}_n + [2\lambda_{n-1} - \lambda_n]l)] \quad (35)$$
$$\times[(2p_3 - \bar{k}_n - [\lambda_n + 1]l)(2\bar{k}_{n-1} - \bar{k}_n + [2\lambda_{n-1} - \lambda_n - 1]l)].$$

The factors contained in (35) are "coupled" via the common l. In the asymptotic expression we easily find

$$N = 16(p_1 l)^2(p_3 l)^2[\lambda_1 - 2\lambda_2][1 + \lambda_1 - 2\lambda_2]$$
$$\times[\lambda_n - 2\lambda_{n-1}][1 + \lambda_n - 2\lambda_{n-1}], \quad (36)$$

and $(p_1 l)^2(p_3 l)^2 \sim t^2$ when account is taken of the integration with respect to l. It is precisely the

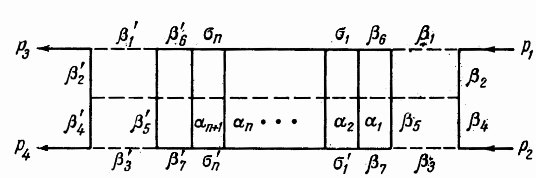

FIG. 5.

case of the last ladder which is encountered when the Mandelstam diagram is considered, and we now proceed to do so.

We start with the diagram shown in Fig. 4. The equivalent representation of this diagram is shown in Fig. 5, from which we see that the Mandelstam diagram is a ladder diagram with edges connected by an additional "non-planar" line. The presence of three vector particles in the s-channel leads, as we shall show, to t^3 in the numerator. However, since each contour of this ladder depends on the total momentum l_3, the appearance of t^3 does not depend on the particles from which the internal ladder of the Mandelstam diagram is constructed. We shall consider for simplicity a diagram whose internal ladder is made up of only scalar particles (Fig. 4).

According to the method developed in the preceding section, we break up the diagram into blocks. However, unlike the scalar case, it is necessary to separate also two more blocks—the two extreme contours of the internal ladder—l_4 and l_5. Indeed, we obtain in the numerator of the asymptotic expression an equation similar to (36), and this does not allow us to factor the ladder (that is, to represent it in the form I_0^n). After separating the contours l_4 and l_5, the factorization of the remaining (scalar) part of the ladder is performed in elementary fashion. As in the preceding section, we introduce the following notation: M_1, Ψ_1, M_2, and Ψ_2 for the side contours (see formulas (3)—(6) of the preceding section), and also

$$\Psi_3 = (p - l_3)^2 - n^2, \tag{37}$$

$$M_4 = 2! \int_0^1 d\beta_5\, d\beta_6\, d\beta_7\, \delta\left(\sum \beta - 1\right) \Psi_4^{-3}, \tag{38}$$

$$\Psi_4 = l_4^2 - 2l_4(\beta_5 l_1 + \beta_7 l_3) + \beta_5 l_1^2 + \beta_7 l_3^2 - m^2, \tag{39}$$

$$M_5 = 2! \int_0^1 d\beta_5'\, d\beta_6'\, d\beta_7'\, \delta\left(\sum \beta' - 1\right) \Psi_5^{-3}, \tag{40}$$

$$\Psi_5 = l_5^2 - 2l_5(\beta_5' l_2 + \beta_7' l_3) + \beta_5' l_2^2 + \beta_7' l_3^2 - m^2, \tag{41}$$

$$M_6 = (3n)! \int_0^1 d\alpha\, d\sigma\, d\sigma'\, \delta\left[\sum (\alpha + \sigma + \sigma') - 1\right] \Psi_6^{-(3n+1)}, \tag{42}$$

$$\Psi_6 = \sum_1^n \lambda_i \bar{k}_i^2 + \sum_3^5 A_{ij} l_i l_j - m^2. \tag{43}$$

We set up the product

$$M_0 = M_4 M_5 M_6 = 2!\,2!\,(3n)! \int_0^1 d\beta \ldots d\sigma' \frac{1}{\Psi_4^3 \Psi_5^3 \Psi_6^{3n+1}} \tag{44}$$

and apply the Feynman rule to the denominator:

$$M_0 = (3n+6)! \int_0^1 d\beta \ldots d\sigma' \int_0^1 d\gamma_4\, d\gamma_5\, d\gamma_6\, \delta\left(\sum \gamma - 1\right) \times \gamma_4^2 \gamma_5^2 \gamma_6^{3n} \Psi_0^{-(3n+7)}. \tag{45}$$

The form $\Psi_0 = \sum_4^6 \gamma_i \Psi_i$ can be reduced, by means of the change of variables $l \to \bar{l}$, to the form

$$\Psi_0 = \lambda_4' \bar{l}_4^2 + \lambda_5' \bar{l}_5^2 + \gamma_6 \sum_1^n \lambda_i \bar{k}_i^2 + D_2/\Delta_2, \tag{46}$$

where

$$\Delta_2 = \begin{vmatrix} a_{44} & a_{45} \\ a_{45} & a_{55} \end{vmatrix}, \qquad D_2 = \begin{vmatrix} a_{44} & a_{45} & b_4 \\ a_{45} & a_{55} & b_5 \\ b_4 & b_5 & c_2 \end{vmatrix}, \quad \lambda_4' \lambda_5' = \Delta_2, \tag{47}$$

$$a_{44} = \gamma_4 + \gamma_6 A_{44}, \quad a_{55} = \gamma_5 + \gamma_6 A_{55}, \quad a_{45} = \gamma_6 A_{45},$$
$$b_4 = -\gamma_4(\beta_5 l_1 + \beta_7 l_3) + \gamma_6 A_{34} l_3,$$
$$b_5 = -\gamma_5(\beta_5' l_2 + \beta_7' l_3) + \gamma_6 A_{35} l_3, \tag{48}$$
$$c_2 = \gamma_4(\beta_5 l_1^2 + \beta_7 l_3^2) + \gamma_5(\beta_5' l_2^2 + \beta_7' l_3^2) + \gamma_6 A_{33} l_3^2 - m^2.$$

In other words, we can write

$$D_2/\Delta_2 = \sum_1^3 B_{ij} l_i l_j - m^2. \tag{49}$$

We now set up the product

$$M = M_0 M_1 M_2 M_3 = 3!\,3!\,(3n+6)! \int_0^1 d\beta \ldots d\sigma' \times \int_0^1 d\gamma_4\, d\gamma_5\, d\gamma_6\, \delta\left(\sum \gamma - 1\right) \gamma_4^2 \gamma_5^2 \gamma_6^{3n} \frac{1}{\Psi_0^{3n+7} \Psi_1^4 \Psi_2^4 \Psi_3} \tag{50}$$

Applying Feynman's rule to the denominator, we obtain for the amplitude

$$F_n = (g^2)^{n+6} (3n+15)! \int_0^1 d\beta \ldots d\sigma' \int_0^1 d\gamma_4\, d\gamma_5\, d\gamma_6 \times \delta\left(\sum \gamma - 1\right) \gamma_4^2 \gamma_5^2 \gamma_6^{3n} \times \int_0^1 d\gamma_0\, d\gamma_1\, d\gamma_2\, d\gamma_3\, \delta\left(\sum \gamma - 1\right) \gamma_0^{3n+6} \gamma_1^3 \gamma_2^3 \int \frac{d^4\bar{l}_1 \ldots d^4\bar{l}_5}{\Psi^{3n+16}} \times \prod_1^n d^4 k_i N. \tag{51}$$

Here N is the numerator, which reflects the dependence of the amplitude on the spins; in the denominator we have

$$\Psi = \sum_{i=0}^3 \gamma_i \Psi_i = \gamma_0 \left(\lambda_4' \bar{l}_4^2 + \lambda_5' \bar{l}_5^2 + \gamma_6 \sum \lambda_i \bar{k}_i^2\right) + \sum_1^3 \lambda_i'' \bar{l}_i^2 + \frac{D}{\Delta}, \tag{52}$$

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}, \qquad D = \begin{vmatrix} & \Delta & & b_1 \\ & & & b_2 \\ & & & b_3 \\ b_1 & b_2 & b_3 & c \end{vmatrix}, \tag{53}$$

$$a_{11} = \gamma_1(\beta_1+\beta_2+\beta_3+\beta_4)+\gamma_0 B_{11},$$
$$a_{22} = \gamma_2(\beta_1'+\beta_2'+\beta_3'+\beta_4')+\gamma_0 B_{22},$$
$$a_{33} = \gamma_1(\beta_3+\beta_4)+\gamma_2(\beta_3'+\beta_4')+\gamma_3+\gamma_0 B_{33},$$
$$a_{12} = \gamma_0 B_{12}, \quad a_{13} = -\gamma_1(\beta_3+\beta_4)+\gamma_0 B_{13},$$
$$a_{23} = -\gamma_2(\beta_3'+\beta_4')+\gamma_0 B_{23}, \; b_1 = -\gamma_1[(\beta_2+\beta_4)p_1-\beta_4 p],$$
$$b_2 = -\gamma_2[(\beta_2'+\beta_4')p_3-\beta_4' p],$$
$$b_3 = \gamma_1\beta_4 p_1+\gamma_2\beta_4' p_3-(\gamma_1\beta_4+\gamma_2\beta_4'+\gamma_3)p, \tag{54}$$
$$c = \gamma_1(\beta_2 p_1^2+\beta_4 p_2^2)+\gamma_2(\beta_2' p_3^2+\beta_4' p_4^2)+\gamma_3 s-m^2.$$

In perfect analogy with the scalar case, we get

$$Q-P_1P_2C^{-1} = -\gamma_0\gamma_1\gamma_2 B_{12}[(\beta_2+\beta_4) \times (\beta_2'+\beta_4')a_{33}+\beta_4(\beta_2'+\beta_4')a_{13} + \beta_4'(\beta_2+\beta_4)a_{23}+\beta_4\beta_4' a_{12}], \quad C=\gamma_1^2\gamma_2^2. \tag{55}$$

We obtain

$$B_{12} = \gamma_4\gamma_5\beta_5\beta_5'\frac{a_{45}}{\Delta_2} = -\gamma_4\gamma_5\gamma_6\beta_5\beta_5'\prod_1^{n+1}\alpha_i\frac{1}{\Delta_1\Delta_2}. \tag{56}$$

With allowance for the encounter of the integration contour with two coinciding singularities in the variables $x = \beta_2\beta_3 - \beta_1\beta_4$ and $y = \beta_2'\beta_3' - \beta_1'\beta_4'$, we find in the limit as $t \to \infty$, from expressions (55) and (56), that $\alpha_i \sim \beta_5 \sim \beta_5' \sim 0$, and then (55) can be rewritten in the form

$$\frac{1}{\Delta_0}(Q-P_1P_2C^{-1}) \to \gamma_0\gamma_6\beta_4\beta_4'\beta_5\beta_5'\prod_1^{n+1}\alpha_i\frac{1}{\beta_3\beta_3'\Delta_1^0}. \tag{57}$$

It follows therefore that $\beta_4 \sim \beta_4' \sim 0$, too. In fact, as already mentioned, the smallness of the parameters in the coefficient of t in the denominator is determined by the form of the numerator. Thus, γ_0 and γ_6 cannot be regarded as small, since they are included in the integral of (51). It will be seen from what follows that our assumptions concerning the smallness of α_i, β_4, β_4', β_5, and β_5' are valid.

Integrating in (51) with respect to x and y, with respect to the small parameters α_i, β_4, β_4', β_5, and β_5', and also with respect to β_2 and β_2' with the aid of $\delta(\beta_2 - \beta_1\beta_4/\beta_3)$ and $\delta(\beta_2' - \beta_1'\beta_4'/\beta_3')$, we get

$$F_n = (g^2)^{n+6}(3n+13)!\,\frac{2\pi i}{t}\frac{(\ln t)^{n+4}}{(n+4)!\,t} \times \int_0^1 d\beta\ldots d\sigma'\Delta_1^0\int_0^1 d\gamma_4\,d\gamma_5\,d\gamma_6 \times \delta\Big(\sum\gamma-1\Big)\gamma_4^2\gamma_5^2\gamma_6^{3n-1}\int_0^1 d\gamma_0\,d\gamma_1\,d\gamma_2\,d\gamma_3 \times \delta\Big(\sum\gamma-1\Big)\gamma_0^{3n+5}\gamma_1\gamma_2\int\frac{d^4\bar{l}_i d^4\bar{k}_i\Delta_0 N}{(\Psi^0)^{3n+14}}. \tag{58}$$

Here

$$\Delta_0 = \gamma_1\gamma_2(\gamma_1\beta_1\beta_3+\gamma_2\beta_1'\beta_3'+\gamma_3+\gamma_0 B_{33}^0), \tag{59}$$

with

$$B_{33}^0 = \gamma_4\beta_6\beta_7+\gamma_5\beta_6'\beta_7'+\gamma_6 A_{33}^0. \tag{60}$$

We substitute (59) and (60) in the integral (58) and integrate in accordance with the Feynman rule first with respect to $\gamma_0 \ldots \gamma_3$ and then with respect to γ_4, γ_5, and γ_6. As a result we obtain a sum of six terms (similar to the sum (23) of Sec. 1).

Before we integrate over the internal momenta, let us examine the numerator N and separate in it the dependence on the asymptotic parameter t

$$N = [(2p_1-p-2l_1+l_3)(2p_3-p-2l_2+l_3)] \times[(2p_1-2p-l_1+l_3)(2l_4-l_1-l_3)] \times[(2p_3-2p-l_2+l_3)(2l_5-l_2-l_3)] \times[(2p_1-l_1)(2l_4-l_1)][(2p_3-l_2)(2l_5-l_2)]. \tag{61}$$

Compared with the numerator for the ladder diagram (33), we have added here the product in the first square brackets, due to the additional vector line $p - l_3$.

In reducing the denominator of the Feynman integral to the form (52), we used substitutions that can be described, if $\alpha_i \sim \beta_4 \sim \beta_4' \sim \beta_5 \sim \beta_5' \sim 0$, by means of the formulas

$$k_i = \bar{k}_i + \frac{\sigma_i'}{\sigma_i+\sigma_i'}l_3, \quad l_1 = \bar{l}_1+\beta_3 l_3, \quad l_2 = \bar{l}_2+\beta_3' l_3, \quad l_4 = \bar{l}_4+\beta_7 l_3, \quad l_5 = \bar{l}_5+\beta_7' l_3, \tag{62}$$

with

$$l_3 = \bar{l}_3 + \frac{\gamma_1\gamma_2\gamma_3}{\Delta_0}p. \tag{63}$$

Substituting in (61) in accordance with (62) (inasmuch as in the integration with respect to γ_i we have then over from $\bar{l}_3$ back to l_3, we retain l_3 in the numerator without using (63)), we can readily show that when $t \to \infty$ we get

$$N \to -32t(p_1l_3)^2(p_3l_3)^2(1-2\beta_7) \times(1+\beta_3-2\beta_7)(1-2\beta_7')(1+\beta_3'-2\beta_7'). \tag{64}$$

The additional t^2 is due to the factor $(p_1 l_3)^2(p_3 l_3)^2$ after integration with respect to l_3.

Let us carry out the integration over all the momenta k_i and l_i with the exception of l_3. As a result F_n is expressed in terms of the integral with respect to l_3, which we write in two-dimensional form. To this end, we go over to the c.m.s. of particles 1 and 2, in which $\mathbf{p}_1 + \mathbf{p}_2 = 0$. Then $s \equiv p^2 = p_0^2$, $t = -(\mathbf{p}_1 - \mathbf{p}_3)^2$, and $(p_1 l_3)(p_3 l_3) \sim (\mathbf{p}_1\mathbf{l}_3)(\mathbf{p}_3\mathbf{l}_3)$, for only these scalar products will yield t. We use the well known formulas

$$\cos\nu = \cos\theta\cos\bar{\theta}+\sin\theta\sin\bar{\theta}\cos(\bar{\varphi}-\varphi),$$
$$\cos\nu' = \cos\theta\cos\bar{\theta}'+\sin\theta\sin\bar{\theta}'\cos(\varphi-\bar{\varphi}'),$$
$$\cos\alpha = \cos\bar{\theta}\cos\bar{\theta}'+\sin\bar{\theta}\sin\bar{\theta}'\cos(\bar{\varphi}-\bar{\varphi}'), \tag{65}$$

where ν is the angle between $\mathbf{l}_3$ and $\mathbf{p}_1$, θ and φ are the polar angles of the vector $\mathbf{l}_3$, $\overline{\theta}$ and $\overline{\varphi}$ are the polar angles of the vector $\mathbf{p}_1$, ν' is the angle between $\mathbf{l}_3$ and $\mathbf{p}_3$, $\overline{\theta'}$ and $\overline{\varphi'}$ are the polar angles of the vector $\mathbf{p}_3$, and α is the angle between $\mathbf{p}_1$ and $\mathbf{p}_3$. In this notation

$$(\mathbf{p}_1\mathbf{l}_3)^2(\mathbf{p}_3\mathbf{l}_3)^2 = \mathbf{p}_1^2\mathbf{p}_3^2\mathbf{l}_3^4(\cos\nu\cos\nu')^2. \tag{66}$$

Substituting here the values of $\cos\nu$ and $\cos\nu'$ from (65) and integrating with respect to θ and φ, we obtain

$$(\mathbf{p}_1\mathbf{l}_3)^2(\mathbf{p}_3\mathbf{l}_3)^2 \to \frac{2\pi}{15}t^2\mathbf{l}_3^4. \tag{67}$$

Thus

$$F_n = \frac{1}{15}t\frac{(\ln t)^{n+4}}{(n+4)!}(g^2)^2\frac{i\pi^2}{s^3}\int\limits_{\lambda\leqslant 0} ds_1\,ds_2[\lambda(s,s_1,s_2)]^{5/2} \times\left(\frac{d}{ds_1}+\frac{d}{ds_2}\right)\frac{[K(s_1)]^2[I_0(s_1)]^{n+4}}{s_2-m^2}. \tag{68}$$

After integrating in (68) by parts, we obtain finally the expression

$$F_n = \frac{2}{3}t\frac{(\ln t)^{n+4}}{(n+4)!}(g^2)^2\frac{i\pi^2}{s^2} \times\int\limits_{\lambda\leqslant 0} ds_1\,ds_2\frac{[\lambda(s,s_1,s_2)]^{3/2}}{s_2-m^2}[K(s_1)]^2[I_0(s_1)]^{n+4}, \tag{69}$$

where

$$K(s_1) = K_0(s_1)\,/\,I_0^2(s_1),$$

$$I_0(s_1) = g^2 i\pi^2\int\limits_0^1\frac{dx}{s_1x(1-x)-m^2}, \tag{70}$$

$$K_0(s_1) = (g^2 i\pi^2)^2\int\limits_0^1 dx\,dy\frac{(1-2y)(1+x-2y)}{[s_1x(1-x)-m^2][s_1y(1-y)-m^2]} = \frac{s_1-4m^2}{s_1}I_0^2(s_1)-\frac{4}{s_1}g^2 i\pi^2 I_0(s_1). \tag{71}$$

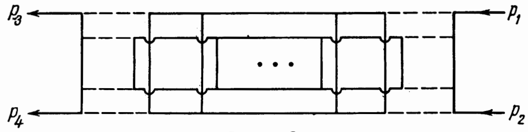

FIG. 6.

4. MANDELSTAM DIAGRAM WITH TWO LADDERS

We shall now dwell in detail on the Mandelstam diagram with two ladders, since it is a quite trivial generalization of the diagram with one ladder. This diagram is shown in Fig. 6 and comprises two ladders, which are superimposed on each other in "non-planar" fashion and are interconnected at the ends. The quantity $Q - P_1P_2C^{-1}$, which appears in the denominator of the Feynman integral, depends additively on α_i and α_i', which belong to the two respective ladders. As a result of integration with respect to x and y, and then with respect to α_i and α_i', we get t^3 in the denominator. In the numerator, as can be readily seen from Fig. 6, there will appear t^4. The final two-dimensional integral for the diagram with n rungs of one ladder and m rungs of the other ladder will be proportional to $t(\ln t)^{n+m}/n!m!$.

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