DERIVATION OF THE CHEW-LOW EQUATIONS AND THE ORIGIN OF THE CUTOFF FUNCTION IN STATIC MODELS

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Submitted December 15, 1966

Zh. Eksp. Teor. Fiz. **53**, 175–182 (July, 1967)

A method for the derivation of Chew-low type equations from the Mandelstam representation is proposed. The meaning of the cutoff function, which takes into account the cut from the process III (18) in the partial wave, is elucidated for static models. It is shown that the function can also be regarded as simultaneous allowance for inelastic processes in the first two channels (18).

INTRODUCTION

THE Chew-Low equations were first established within the framework of a special πN scattering model describing the interaction with a fixed source [1,2]. These equations played an important role in the study of pion-nucleon interaction at low energies. Recently interest has been evinced in equations of the Chew-Low type in connection with the study of both the symmetries of strongly-interacting particles and the behavior of the scattering amplitudes at high energies [3]. It is therefore of interest to derive the Chew-Low equations on a more general basis than the special form of the interaction Hamiltonian [1,2]. The first steps in this direction were made by Oehme and by Chew et al. (CGLN) [5]. The authors of [5], starting with the dispersion relations with respect to energy at a fixed value of the momentum transfer, established the Chew-Low equations in the static limit. They were also the first to indicate the difficulties of such a derivation, one of which consists in the need for introducing a cutoff function connected with a Hamiltonian formulation with finite source dimensions. We shall discuss this question later.

SCATTERING OF NEUTRAL PIONS BY SPINLESS NUCLEONS

Let us consider for simplicity a model example, that of scattering of neutral pions (π) by spinless nucleons (N). The method of introducing into consideration the variables that assume discrete values (spin, isotopic spin, etc.) will be indicated later. The amplitude of the transition of this process is expressed in terms of the S-matrix as follows:

$$\langle q_2, p_2|S-1|q_1, p_1\rangle = i(2\pi)^4\delta(p_1+q_1-p_2-q_2)\frac{1}{(2\pi)^6} \times \frac{M}{(4q_1^0q_2^0p_1^0p_2^0)^{1/2}}T(p_1q_1;\ p_2q_2). \quad (1)$$

Here q_i (p_i) are the pion (nucleon) 4-momenta. For convenience in changing over to the case of real πN scattering, we have retained the fermion normalization for the nucleons. The Lorentz-invariant amplitude $T(p_1, q_1; p_2, q_2)$ depends on two variables, which can be chosen to be any two of the Mandelstam variables s, u, and t:

$$s=(p_1+q_1)^2,\ s=M^2+\mu^2+2q^2+2\sqrt{(M^2+q^2)(\mu^2+q^2)},$$
$$u=(p_1-q_2)^2,\quad t=-2q^2(1-\cos\theta),$$
$$t=(p_1-p_2)^2, \quad (2)$$

where q is the momentum and θ the scattering angle in the c.m.s.

For the scattering amplitude we have the expansion

$$T(s,t)=4\pi\frac{W}{M}\sum_{l\geqslant 0}(2l+1)f_l(s)P_l\left(1+\frac{t}{2q^2}\right), \quad (3)$$

where W is the total energy. The two-particle unitarity condition is of the form

$$\operatorname{Im} f_l(s)=q(s)\,|f_l(s)|^2. \quad (4)$$

We write down the dispersion relation (d.r.) in s for fixed t:

$$T(s,t)=g^2\left(\frac{1}{M^2-s}+\frac{1}{M^2-u}\right)+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}\operatorname{Im}T(s',t) \times\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right]ds'. \quad (5)$$

The d.r. (5) presupposes that $|T(s,t)|\to 0$ as $|s|\to\infty$. It will be shown later that this limitation is immaterial.

We obtain the Chew-Low equations from the d.r. (5) by the CGLN method [5]. To this end, we make a number of assumptions: first, we neglect in d.r. (5) all the inelastic processes; second, we confine ourselves to the lowest partial waves in the expansion (3), putting $f_l(s) = 0$, $l > 1$; third, in the equations for the partial waves we go over to the static limit ($\mu/M \to 0$, $s/M^2 \to 1$). As a result of the second assumption, it is sufficient to know the d.r. for $T(s, 0)$ and $T'_t(s, 0)$ in order to determine the equations for the S and P waves, since

$$T(s,t) = 4\pi \frac{W}{M}\left[f_0(s) + 3\left(1 + \frac{t}{2q^2}\right)f_1(s)\right],$$

$$T(s,0) = 4\pi \frac{W}{M}[f_0(s) + 3f_1(s)],$$

$$T_t'(s,0) = 4\pi \frac{W}{M}\frac{3}{2q^2} f_1(s). \tag{6}$$

Differentiating the d.r. (5) with respect to t, we readily obtain the d.r. for $T'_t(s, 0)$.

It is convenient to change over from the variable s to a new variable E - the meson energy in the laboratory frame:

$$s = M^2 + \mu^2 + 2ME, \quad q^2 = \frac{E^2 - \mu^2}{1 + (\mu/M)^2 + (E/M)^2}. \tag{7}$$

The d.r. for $T'_t(s, 0)$ must further be combined with d.r. (5) so as to obtain, in accordance with (6), the d.r. for the partial waves. After going over in the equations for the partial waves to the static limit, we get

$$f_0(\omega) = 2\frac{f^2}{\mu^2} + \frac{3}{\pi}\int_{\mu^2}^{\infty} \mathrm{Im}\frac{f_1(\omega')}{q'^2}d\omega'^2 + \frac{1}{\pi}\int_{\mu^2}^{\infty}\frac{\mathrm{Im} f_0(\omega')}{\omega'^2 - \omega^2}d\omega'^2, \tag{8}$$

$$f_1(\omega) = -\frac{2}{3}\frac{f^2}{\mu^2}\frac{q^2}{\omega} + \frac{q^2}{\pi}\int_{\mu^2}^{\infty}\frac{\mathrm{Im} f_1(\omega')}{q'^2(\omega'^2 - \omega^2)}d\omega'^2, \tag{9}$$

where $\omega = \lim_{M\to\infty} E$ and $f = g\mu/2M$.

Equations (8) and (9) have all the distinguishing features established in [5] for the πN scattering problem in the static limit. First, the crossing-symmetry condition $T(s, u, t) = T(u, s, t)$, which relates different partial waves, reduces to a number of uncoupled equations in the form

$$f_l(-\omega) = f_l(\omega). \tag{10}$$

Second, the coupling of the individual partial waves is effected by the pole term. Furthermore, the equation for the S-waves contains an additive constant that depends on $\mathrm{Im}\, f_1(\omega)$.

The equations (8) and (9) differ essentially from the usual Chew-Low equations in one respect. Thus, for the Hamiltonian of the interaction between neutral scalar mesons with the source

$$H_{int} = \sqrt{4\pi}\, g \int U(|\mathbf{r}|)\varphi(\mathbf{r})d\mathbf{r} \tag{11}$$

the Chew-Low equation is of the form

$$h_0(\omega) = \frac{1}{\pi}\int_{\mu^2}^{\infty}\frac{\mathrm{Im}\, h_0(\omega')}{\omega'^2 - \omega^2}d\omega'^2, \tag{12}$$

where $h_0(\omega) = f_0(\omega)/v(q^2)$ and $v(q^2)$ is the square of the Fourier transform of the source function $U(|\mathbf{r}|)$. Interaction (11) leads only to S-scattering. From the unitarity condition (4) follows the inequality

$$|h_0(\omega)| < 1/qv(q^2), \tag{13}$$

from which we see that for rapidly decreasing functions $v(q^2)$ the Chew-Low equation (12) must be written out with a number of subtractions.

We know very little about the function $v(q^2)$. It is usually assumed that $U(|\mathbf{r}|)$ differs from zero in a finite region $|\mathbf{r}| < R$. We can then say with regards to the function $v(q^2)$ that

$$v(q^2) \approx 0, \quad q^2 \gg 1/R^2. \tag{14}$$

Relation (14) says nothing about the analytic properties of the cutoff function. In concrete calculations, one prefers to use for it analytically-continuable functions. For example, it is assumed that [6,7]

$$v(q^2) = \frac{q^2_{max}}{q^2 + q^2_{max}}, \quad v(q^2) = \exp\left\{-\frac{q^2}{q^2_{max}}\right\},$$

$$v(q^2) = \left(\frac{q^2_{max}}{q^2 + q^2_{max}}\right)^n. \tag{15}$$

To be sure, a step-like cutoff is sometimes used [6]

$$v(q^2) = \begin{cases} 1 & q^2 \leqslant q^2_{max} \\ 0 & q^2 > q^2_{max}. \end{cases} \tag{16}$$

Comparing (8) with (12) we see that different functions, $f_0(\omega)$ and $h_0(\omega)$ respectively, have the same analytic properties with respect to ω. If we obtain the analytic properties of the partial wave $f_0(\omega)$ from the Chew-Low equations (12), then they can differ appreciably from those in (8) as a result of the cutoff function. Moreover, if the cutoff function is not analytic, then the partial wave $f_0(\omega)$ is likewise not analytic. To clarify the foregoing contradiction, which was first noted in [5], let us turn to the derivation of (8) and (9). They were obtained from the d.r. (5) on the basis of the analytic properties of the amplitude $T(s, t)$ with respect to s for fixed t.

The next step in the investigation of the analytic properties of the scattering amplitude is the Mandelstam representation [8], which postulates the position of the singularities of the amplitude as a function of two complex variables. It has not been proved as yet, although recently a number of important results have been obtained [9], which bring the analytic properties of T much closer to those proposed in [8]. For the case under consideration, the Mandelstam representation is

$$T(s,u,t)=g^2\left(\frac{1}{M^2-s}+\frac{1}{M^2-u}\right)+\frac{1}{\pi^2}\int\limits_{(M+\mu)^2}^{\infty} ds' \times \int\limits_{(M+\mu)^2}^{\infty} du'\frac{\rho(s',u')}{(s'-s)(u'-u)}+\frac{1}{\pi^2}\int\limits_{(M+\mu)^2}^{\infty} dx\int\limits_{4\mu^2}^{\infty} dt'\rho_1(x,t') \times\left[\frac{1}{(x-s)(t'-t)}+\frac{1}{(x-u)(t'-t))}\right]. \quad (17)$$

The double spectral representation (17) describes simultaneously three processes:

$$\left.\begin{array}{ll}\text{I} & \pi+N\to\pi'+N' \quad s\\ \text{II} & \bar{\pi}'+N\to\bar{\pi}+N' \quad u\\ \text{III} & \pi+\bar{\pi}'\to\bar{N}+N' \quad t\end{array}\right\}\text{energy variables.} \quad (18)$$

From the spectral representation (17) we readily obtain the d.r. (5). Here Im T(s, t) has, as a function of the variable t, cuts located outside the physical region of the process I:

$$\operatorname{Im} T(s,t)=\frac{1}{\pi}\int\limits_{4\mu^2}^{\infty}\frac{\rho_1(s,t')}{t'-t}dt' -\frac{1}{\pi}\int\limits_{\infty}^{(M-\mu)^2-s}\frac{\rho(s,2M^2+2\mu^2-s-t')}{t'-t}dt'. \quad (19)$$

In order to represent most clearly the consequences that follow from the presence of cuts with respect to the variable t in Im T(s, t), we shall use the Cini-Fubini approximate spectral representations [10]. Following Cini and Fubini, we represent the imaginary parts of the scattering amplitude in the form

$$\operatorname{Im} T(s,t)=\{\operatorname{Im} T(s,t)\}_{\rm el}+\{\operatorname{Im} T(s,t)\}_{\rm inel} \quad (20)$$

It follows from the spectral representation (17) that the first term in (20) has a cut in t, starting with $t = 16\mu^2$, and the second, which differs from zero only at $s > (M + 2\mu)^2$, starting with $t = 4\mu^2$.

We substitute (20) in d.r. (5) and neglect the dependence on s and u in the integrals of the second term. Such an approximation is justified in the region $|s| \sim |u| < (M + 2\mu)^2$:

$$T(s,u,t)=g^2\left(\frac{1}{M^2-s}+\frac{1}{M^2-u}\right)+\frac{1}{\pi}\int\limits_{(M+\mu)^2}^{\infty}\{\operatorname{Im} T(s',t)\}_{\rm el} \times\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right]ds'+\frac{1}{\pi}\int\limits_{4\mu^2}^{\infty}\frac{c(t')}{t'-t}dt'. \quad (21)$$

The presence of a cut in t, starting with $t = 4\mu^2$, is connected with the fact that the amplitude T describes also the process III. If the last goes via intermediate states with small lifetime, then the function c(t) - the imaginary part of the process III - can be reasonably approximated by a series of δ-functions:

$$c(t)=\pi\sum_i c_i\delta(t-t_i).$$

Even if c(t) is a smooth function, the influence of the last integral in (21) on processes I and II, for which $t < 0$, can be well accounted for by a system of poles. Therefore, with an aim at obtaining the Chew-Low equations for processes I and II, we transform (21) into

$$T(s,u,t)=g^2\left(\frac{1}{M^2-s}+\frac{1}{M^2-u}\right)+\frac{1}{\pi}\int\limits_{(M+\mu)^2}^{\infty}\{\operatorname{Im} T(s,t')\}_{\rm el} \times\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right]ds'+\sum_i\frac{c_i}{t-t_i}. \quad (22)$$

The separation of the partial waves from (22) will be carried out by a differential procedure proposed in [11], i.e., by combining the spectral representation (22) for forward and backward scattering

$$4\pi\frac{W}{M}f_0(s)=\frac{T(s,0)+T(s,-4q^2)}{2},$$

$$4\pi\frac{W}{M}f_1(s)=\frac{T(s,0)-T(s,-4q^2)}{6}. \quad (23)$$

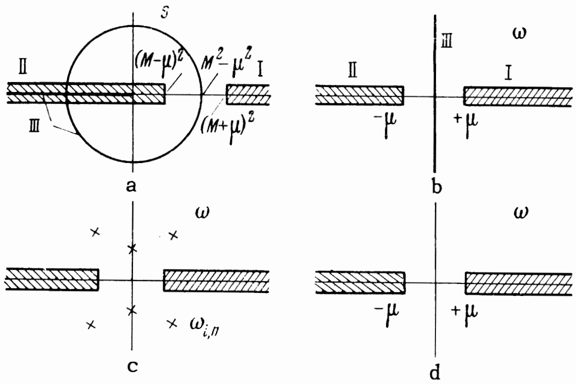
It is seen from formulas (23) that $f_0(s)$ and $f_1(s)$ have the same system of poles t_i. Going over then to the static limit in (23), we find that the functions $f_0(\omega)$ and $f_1(\omega)$ satisfy Eqs. (8) and (9), the right sides of which contain the same system of poles in ω. This system of poles is symmetrical with respect to the line $\operatorname{Re}\omega = 0$ (imaginary axis). If we were to take into account the higher waves, then the different partial waves could have different systems of poles, which originate, however, from the same set of poles in t of the amplitude T(s, u, t).

The last step in the derivation of the Chew-Low equations from the representations (17) consists in constructing the auxiliary function

$$V_l(q^2)=\prod_i\frac{1}{\omega-\omega_{l,i}},\qquad v_l(q^2)=\frac{V_l(q^2)}{V_l(-1)}. \quad (24)$$

Then we can consider in lieu of the partial wave $f_l(\omega)$ the function $f_l(\omega)/v_l(q^2) = h_l(\omega)$, the analytic properties of which coincide with those in the Chew-Low equation

We have started from d.r. (5) and the representation (17) in the so-called non-subtracted form. Since we do not know beforehand how many poles must be introduced on going from (21) to (22), it is clear that it is impossible to specify



beforehand the number of the subtractions in the Chew-Low equations. We shall therefore write them out without carrying out the subtractions (12), assuming that they can always be carried out.

SCATTERING OF MESONS BY NUCLEONS

The reasoning presented above can be generalized without difficulty to include the case of meson-nucleon scattering. With this, the amplitude T in formula (1) will have isotopic and spin structure

$$T = A + {}^1/_2(\hat{q}_1 + \hat{q}_2)B, \quad A = A^{(+)}\delta_{\alpha\beta}\delta_{t_1t_2} + A^{(-)1}/_2[\tau_\alpha\tau_\beta]_{t_1t_2} \tag{25}$$

and an analogous formula for B, where α, $\beta(t_1, t_2)$ are the isotopic indices of the pions (nucleons).

We shall not repeat the foregoing procedure for the amplitude specified by formula (25). We present a more formal exposition of the derivation of the Chew-Low equations from the Mandelstam double spectral representations. We deduce from the representation (17) the analytic properties of the partial waves $f_i(s)$ [12,13]. The arrangement of the cuts is shown in Fig. a.

We change over to a new variable

$$E = (s - M^2 - \mu^2)/2M \tag{26}$$

and take the limit

$$\omega = \lim_{M\to\infty} E. \tag{27}$$

The system of cuts of the partial wave $f_i(s)$ in the ω plane is shown in Fig. b. Further, we model the cut $-\infty i$, $+i\infty$ from process III by means of a system of poles $\omega_{i,n}$ (Fig. c). The poles must be symmetrically disposed relative to the imaginary axis. It is obvious that for different partial waves $f_i(\omega)$ the systems of poles $\omega_{i,n}$ will in general not coincide. We set up a union ω_n of all the $\omega_{i,n}$ whose partial waves are connected by the crossing-symmetry relation

$$f_i(-\omega) = A_{ij}f_j(\omega), \tag{28}$$

where A_{ij} is the crossing-symmetry matrix. In accordance with the aforementioned union, we construct the function

$$v(q^2) = \frac{V(q^2)}{V(-1)}, \quad V = \prod_n \frac{1}{\omega - \omega_n}. \tag{29}$$

It is then obvious that the functions $h_i(\omega) = f_i(\omega)/v(q^2)$ have the analytic properties shown in the Fig. d, i.e., they satisfy the Chew-Low equations

$$h_i(\omega) = \frac{\lambda_i}{\omega} + \frac{1}{\pi}\int_\mu^\infty \left[\frac{\mathrm{Im}\, h_i(\omega')}{\omega' - \omega} + \frac{A_{ij}\,\mathrm{Im}\, h_j(\omega')}{\omega' + \omega}\right] d\omega', \tag{30}$$

which have been written out, as usual, without subtractions. The necessary number of subtractions can always be easily introduced.

CONCLUSION

We have proposed a scheme for deriving the Chew-Low equations; this scheme is based on analytic properties, specified by the Mandelstam representation, of the scattering amplitude. It is a development of the CGLN paper [5], in which the Chew-Low equations are derived from d.r. at fixed t. During the course of the derivation, we have shown that the cut-off function $v(q^2)$ of the static model approximates the cut from the process III. The representation of the function $v(q^2)$ in the form (29) is probably one of the possible methods of taking into account the cut from process III, but not the only one.

On the other hand, inasmuch as the strong dependence of the scattering amplitude $T(s, u, t)$ on the variable t (cut in t from $t = 4\mu^2$) is due to allowance for the inelastic processes (20), it is clear that the cut-off function can be regarded also as a certain model of these inelastic processes. We note that inelastic processes were neglected in [5], so that the derivation presented in that paper of the Chew-Low equations had a more formal character.

The derivation of the Chew-Low equations is based essentially only on the analytic properties of the partial waves. The latter have been better substantiated [9] than the Mandelstam representations [8]. With this, in two of the channels described by the scattering amplitude $T(s, u, t)$, we can confine ourselves to several of the lower partial waves, and avoid violation of either analyticity or crossing symmetry. However, the rule for substitutions, of course, is violated here too, since the amplitude is represented by a polynomial in t, meaning that it has no cut, starting with the threshold of the remaining reaction III. The presence of this cut is taken into account within the framework of the static model by means of the cut-off function $v(q^2)$. It is clear from the foregoing that the proposed method for taking into account the cut from process III can probably be used not only within the framework of static models.

The author is deeply grateful to Academician N. N. Bogolyubov for interest in the work.

Note added in proof (15 June 1967). We note that in the derivation of (30), the concrete form of the scattering amplitude (25) is immaterial. Therefore, Eqs. (30) hold for any order of the matrix A_{ij}. Their solution for this case was presented by the author in Dokl. Acad. Nauk SSSR 174, No. 5 (1967) [Sov. Phys.-Dokl. 12, in press].

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Translated by J. G. Adashko
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