

DYNAMICS OF A SUPERFLUID FERMION GAS AT FINITE TEMPERATURES

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Keldysh's diagram technique for non-stationary processes at finite temperatures is generalized to the case of a superfluid Fermi system. We obtain in the linear approximation an equation for a gap Δ which varies in space and time. We find an expression for the energy of the elementary excitations up to second spatial derivatives of Δ . In the case where Δ varies slowly in space and in time we obtain a set of equations consisting of a kinetic equation and an equation for the gap which in the quasi-classical approximation is the same as the set of Khalatnikov for a superfluid Bose liquid.

1. SOLUTION OF GOR'KOV'S EQUATIONS AT FINITE TEMPERATURES

IN the present paper we continue the study of superfluid Fermi systems for the case when the gap Δ varies slowly in space and time. We showed in a previous paper^[1] that in that case the problem can be reduced to a differential equation for Δ . At finite temperatures the problem is considerably complicated by the presence of excitations which must be described by a collisionless kinetic equation for their distribution function. Our aim is to find in the lowest approximation that equation and the equation determining the gap in terms of the distribution function and its derivatives. Pokrovskii and Savvinykh^[2] (see also^[3]) were the first to consider a kinetic equation for the excitations in superconductors.

As in^[1] we first obtain the equation for Δ in the linear approximation when the deviations of Δ from its equilibrium value are small. We shall again start from the Gor'kov equations which are valid also at finite temperatures:

$$\begin{aligned} \left(i \frac{\partial}{\partial t} + \frac{\nabla^2}{2m}\right)G(x, x') - i\Delta(x)F^+(x, x') &= \delta(x - x'), \\ \left(i \frac{\partial}{\partial t} - \frac{\nabla^2}{2m} - 2\mu\right)F^+(x, x') + i\Delta^*(x)G(x, x') &= 0, \\ \Delta^*(x) &= gF^+(x, x), \end{aligned} \tag{1}$$

where the functions G and F^+ are expressed in terms of ψ -operators in the second quantization representation:

$$\begin{aligned} \langle T(\psi_\alpha^+(x)\psi_\beta^+(x')) \rangle &= e^{2i\mu t} I_{\alpha\beta} F^+(x, x'), \\ -i \langle T(\psi_\alpha(x)\psi_\beta^+(x')) \rangle &= \delta_{\alpha\beta} G(x, x'), \end{aligned} \tag{1a}$$

I is the matrix of the form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The analytical properties of the Green functions are, however, different for $T \neq 0$ and this leads to a change in the boundary conditions in t for these equations. Putting

$$\begin{aligned} \Delta &= \Delta_0 + \Delta_1(\mathbf{r}, t), \quad G = G_0(x - x') + G_1(x, x'), \\ F^+ &= F_0^+(x - x') + F_1^+(x, x'), \end{aligned} \tag{2}$$

we find that G_1 and F_1^+ satisfy the linearized equations

$$\begin{aligned} \left(i \frac{\partial}{\partial t} + \frac{\nabla^2}{2m}\right)G_1(x, x') - i\Delta_0(x)F_1^+(x, x') &= i\Delta_1(x)F_0^+(x - x'), \\ \left(i \frac{\partial}{\partial t} - \frac{\nabla^2}{2m} - 2\mu\right)F_1^+(x, x') + i\Delta_0^*(x)G_1(x, x') &= -i\Delta_1^*(x)G_0(x - x'). \end{aligned} \tag{3}$$

Finding from these F_1^* and putting $gF_1^+(x, x) = \Delta_1^*(x)$, we find the equation for Δ_1 .

The exact solution of Eq. (3) can be written in the form

$$\begin{aligned} F_1^+(x, x') &= i \int d^4y [G_0(y - x)\Delta_1^*(y)G_0(y - x') \\ &\quad + F_0^+(x - y)\Delta_1(y)F_0^+(y - x')]. \end{aligned} \tag{4}$$

Such a solution is, however, not unique; we can add to it any solution of the homogeneous Eqs. (1). In fact, the solution must be chosen such that the analytical behavior of all functions be regular. Keldysh^[4] established a method for choosing the solutions of the equations of the quantum theory of fields at finite temperatures for normal systems. The gist of this method, when applied to solving equations similar to (3), consists in adding to each

integral term on the right-hand side of (4) the same term with the opposite sign in which, however, the operators occurring in the definition of the functions G and F and which depend on the integration variable y stand on the left. In other words, one can say that the point y is formally considered to be later than x and x' in each interval where its time component varies. As a result we have instead of (4) the equation

$$\begin{aligned}
 F_{1+}(x, x') &= i \int d^4y [G_0(y-x)\Delta_1^*(y)G_0(y-x') \\
 &\quad - G_{0+}(y-x)\Delta_1^*(y) \\
 &\quad \times G_{0+}(y-x')] + i \int d^4y [F_{0+}^+(x-y)\Delta_1(y)F_{0+}^+(y-x') \\
 &\quad - F_{0+}^+(x-y)\Delta_1(y)F_{0+}^+(y-x')], \quad (5)
 \end{aligned}$$

where in accordance with what we have just said G_0 is the usual Green function defined according to (1a) for $\Delta = \Delta_0 = \text{const}$, while

$$\begin{aligned}
 \delta_{\alpha\beta} G_{0+}(x, x') &= -i \langle \psi_\alpha(x) \psi_\beta^+(x') \rangle, \\
 \delta_{\alpha\beta} G_{0-}(x, x') &= i \langle \psi_\beta^+(x') \psi_\alpha(x) \rangle. \quad (6)
 \end{aligned}$$

The functions G_{0+-} and G_{0-+} have no discontinuity at $t = t'$ and F_{0+-}^+ and F_{0-+}^+ can be expressed in terms of G_{0+-} and G_{0-+} through the "homogeneous" relation

$$\left(i \frac{\partial}{\partial t} + \frac{\nabla^2}{2m} \right) G_{0+-} - i \Delta_0 F_{0+-}^+ = 0 \quad (7)$$

and the same relation for G_{0-+} and F_{0-+}^+ .

We note that since G_{0-+} and F_{0-+}^+ satisfy the homogeneous Eqs. (1), (5) is a solution of Eqs. (3), just as (4). The method explained here can directly be applied also to other problems of the problem of superconductivity, in particular, to superconductors with impurities. Each diagram is then in the form of a sum of several expressions in which each vertex occurs twice—as a normal one and as an anomalous one with a minus sign.

We now turn to the determination of the functions G and F . For this we perform calculations for an arbitrary distribution function for the excitations, $n(\mathbf{p})$. We perform in the usual expressions for the Schrödinger operators ψ_α and ψ_β^+

$$\psi_\alpha = \sum_p a_{\alpha p} e^{i\mathbf{p}\mathbf{r}}, \quad \psi_\beta^+ = \sum_p a_{\beta p}^+ e^{-i\mathbf{p}\mathbf{r}} \quad (8)$$

the Bogolyubov transformation

$$\begin{aligned}
 a_{\alpha p} &= u_p b_{\alpha p} - I_{\alpha\sigma} v_p b_{\sigma-p}^+, \\
 a_{\alpha p}^+ &= u_p b_{\alpha p}^+ - I_{\alpha\sigma} v_p b_{\sigma-p}. \quad (9)
 \end{aligned}$$

At the same time we change to the interaction representation which involves the substitution

$$b_{\alpha p} \rightarrow b_{\alpha p} e^{-i\varepsilon_p t}, \quad b_{\alpha p}^+ \rightarrow b_{\alpha p}^+ e^{i\varepsilon_p t}.$$

As a result we have

$$\begin{aligned}
 \psi_\alpha &= \sum_p [u_p b_{\alpha p} \exp\{i(\mathbf{p}\mathbf{r} - \varepsilon_p t)\} \\
 &\quad - I_{\alpha\sigma} v_p b_{\sigma-p} \exp\{i(\mathbf{p}\mathbf{r} + \varepsilon_p t)\}], \quad (10)
 \end{aligned}$$

$$\begin{aligned}
 \psi_{\alpha}^+ &= \sum_p [u_p b_{\alpha p}^+ \exp\{-i(\mathbf{p}\mathbf{r} - \varepsilon_p t)\} \\
 &\quad - I_{\alpha\sigma} v_p b_{\sigma-p} \exp\{-i(\mathbf{p}\mathbf{r} + \varepsilon_p t)\}].
 \end{aligned}$$

We now substitute (10) into (1a) and average over the state of the system, bearing in mind that

$$\langle b_{\alpha p}^+ b_{\beta p'} \rangle = \delta_{\alpha\beta} \delta_{pp'} n_p, \quad \langle b_{\alpha p} b_{\beta p'}^+ \rangle = \delta_{\alpha\beta} \delta_{pp'} (1 - n_p).$$

We put for $t > t'$

$$\begin{aligned}
 G(x-x') &= -i \sum_p \exp\{i[\mathbf{p}(\mathbf{r}-\mathbf{r}') - \varepsilon(t-t')]\} u_p^2 (1 - n_p) \\
 &\quad + v_p^2 n_p \exp\{i[\mathbf{p}(\mathbf{r}-\mathbf{r}') + \varepsilon(t-t')]\} \quad (11a)
 \end{aligned}$$

and for $t < t'$

$$\begin{aligned}
 G(x-x') &= i \sum_p \exp\{i[\mathbf{p}(\mathbf{r}-\mathbf{r}') - i\varepsilon(t-t')]\} u_p^2 n_p \\
 &\quad + v_p^2 (1 - n_p) \exp\{i[\mathbf{p}(\mathbf{r}-\mathbf{r}') + \varepsilon(t-t')]\} \quad (11b)
 \end{aligned}$$

or, changing to Fourier transforms

$$\begin{aligned}
 G_0(\omega, \mathbf{p}) &= \frac{u_p^2}{\omega - \varepsilon_p + i\delta} + \frac{v_p^2}{\omega + \varepsilon_p - i\delta} \\
 &\quad + 2\pi i [n_p \delta(\omega - \varepsilon_p) u_p^2 - n_p v_p^2 \delta(\omega + \varepsilon_p)]. \quad (12)
 \end{aligned}$$

Similarly

$$G_{0+}(\omega, \mathbf{p}) = 2\pi i [(n_p - 1) u_p^2 \delta(\omega - \varepsilon_p) - n_p v_p^2 \delta(\omega + \varepsilon_p)], \quad (13)$$

$$G_{0-}(\omega, \mathbf{p}) = 2\pi i [n_p u_p^2 \delta(\omega - \varepsilon_p) + (1 - n_p) v_p^2 \delta(\omega + \varepsilon_p)]. \quad (14)$$

We now use the first Gor'kov equation to determine the corresponding functions F^+ (we recall that we must take that equation without the right-hand side for the determination of F_{0+-}^+ and F_{0-+}^+). Simple calculations give

$$\begin{aligned}
 F_{0+}^+(\omega, \mathbf{p}) &= - \frac{i\Delta_0^*}{(\omega - \varepsilon_p + i\delta)(\omega + \varepsilon_p - i\delta)} \\
 &\quad + \frac{\pi\Delta_0^*}{\varepsilon} [n_p \delta(\omega - \varepsilon_p) + n_p \delta(\omega + \varepsilon_p)], \\
 F_{0-+}^+(\omega, \mathbf{p}) &= \frac{\pi\Delta_0^*}{\varepsilon} [(n_p - 1) \delta(\omega - \varepsilon_p) + n_p \delta(\omega + \varepsilon_p)], \\
 F_{0+-}^+(\omega, \mathbf{p}) &= \frac{\pi\Delta_0^*}{\varepsilon} [n_p \delta(\omega - \varepsilon_p) + (n_p - 1) \delta(\omega + \varepsilon_p)]. \quad (15)
 \end{aligned}$$

Substituting into (5)

$$\Delta_1(x) = \Delta_1 \exp\{i\mathbf{k}\mathbf{r} - i\omega_0 t\},$$

$$\Delta_1^*(x) = \Delta_1^* \exp\{i\mathbf{k}\mathbf{r} - i\omega_0 t\}$$

and putting $x = x'$ we get the final equation for Δ_1 and Δ_1^* :

$$\begin{aligned} \Delta_1^*(\omega_0, \mathbf{k}) &= i \frac{g}{(2\pi)^4} \Delta_1^*(\omega_0, \mathbf{k}) \\ &\times \left\{ \int d\omega d^3p \frac{(-\omega_+ + \xi_+)(\omega_- + \xi_-)}{(\omega_+^2 - \xi_+^2 - \Delta^2)(\omega_-^2 - \xi_-^2 - \Delta^2)} \right. \\ &+ 2\pi i \int d^3p \left[\frac{2n_- u_-^2 v_+^2}{-\omega_0 - \varepsilon_+ - \varepsilon_- - i\delta} - \frac{2n_+ v_+^2 v_-^2}{-\omega_0 + \varepsilon_+ + \varepsilon_- - i\delta} \right. \\ &\left. \left. + \frac{2(n_- - n_+) u_-^2 v_+^2}{-\omega_0 + \varepsilon_+ - \varepsilon_- - i\delta} \right] \right\} + i \frac{g}{(2\pi)^4} \Delta_1(\omega_0, \mathbf{k}) \\ &\times \left\{ \int d\omega d^3p \frac{-\Delta_0^{*2}}{(\omega_+^2 - \xi_+^2 - \Delta^2)(\omega_-^2 - \xi_-^2 - \Delta^2)} \right. \\ &- i\pi \Delta_0^{*2} \int d^3p \frac{1}{\varepsilon_+ \varepsilon_-} \left[-\frac{n_-}{\omega_0 + \varepsilon_+ + \varepsilon_- + i\delta} \right. \\ &\left. \left. - \frac{n_+}{-\omega_0 + \varepsilon_+ + \varepsilon_- - i\delta} + \frac{n_+ - n_-}{\varepsilon_+ - \varepsilon_- - \omega_0 - i\delta} \right] \right\}. \quad (16) \end{aligned}$$

The indices \pm indicate here that the corresponding functions are taken with the argument $\mathbf{p} \pm \mathbf{k}/2$.

When transforming Eq. (16) we use the well-known formula

$$\frac{1}{\omega - \varepsilon \pm i\delta} = \text{P} \frac{1}{\omega - \varepsilon} \mp i\pi\delta(\omega - \varepsilon).$$

It is clear from (16) that ω_0 everywhere in the denominators occurs in the combination $\omega + i\delta$. As should be the case, all expressions turn thus out to be analytical in the upper halfplane of the complex variable ω_0 .

We are interested in the equation for Δ for small ω_0 and \mathbf{k} . The right-hand side of (16) can, however, not be expanded directly in ω_0 and \mathbf{k} . Because of the presence of denominators of the form $\varepsilon_+ - \varepsilon_- - \omega_0$ there arise non-local terms containing the ratio ω_0/\mathbf{k} . The origin of these terms is connected with the fact that (16) is the equation for only one function Δ . The correction to the distribution function has already been eliminated from this equation. If, however, we formulate the equation in the form of a set of one equation for Δ and a kinetic equation both equations turn out to be local. We can then immediately go to the limit $\omega_0 \rightarrow 0$, $\mathbf{k} \rightarrow 0$, $\omega/\mathbf{k} \rightarrow \text{const.}$, i.e., when neglecting quantum corrections. The kinetic equation for the distribution function has in that case the usual form:

$$\frac{\partial n}{\partial t} + \frac{\partial \varepsilon}{\partial \mathbf{p}} \frac{\partial n}{\partial \mathbf{r}} - \frac{\partial \varepsilon}{\partial \mathbf{r}} \frac{\partial n}{\partial \mathbf{p}} = 0, \quad \varepsilon = \sqrt{\xi^2 + |\Delta|^2}. \quad (17)$$

Moreover, in the spatially uniform case Δ is connected with n through the equation

$$\frac{(2\pi)^3}{g} = 4\pi m p_0 \ln \frac{2\tilde{\omega}}{\Delta} - \int \frac{n_p}{\varepsilon} d^3p. \quad (18)$$

If we now put

$$\Delta = \Delta_0 + \Delta_1 \exp \{i(\mathbf{k}\mathbf{r} - \omega_0 t)\}, \quad n = n_0 + n_1, \quad (19)$$

linearize Eqs. (17) and (18), and eliminate n_1 we get

$$\begin{aligned} \frac{(2\pi)^3}{g} \Delta_1^* &= \left[4\pi m p_0 \ln \frac{2\tilde{\omega}}{\Delta_0} - \int \frac{n}{\varepsilon_0} d^3p \right. \\ &- |\Delta_0|^2 \int \frac{1}{2\varepsilon_0^2} \frac{\partial n}{\partial \mathbf{p}} \frac{\mathbf{k}}{\mathbf{k}\mathbf{v} - \omega} d^3p + |\Delta|^2 \int \frac{n}{2\varepsilon_0^3} d^3p \left. \right] \Delta_1^* \\ &- \Delta_0^* \Delta_1 \left[\int \frac{1}{2\varepsilon_0^2} \frac{\partial n}{\partial \mathbf{p}} \frac{\mathbf{k}}{\mathbf{k}\mathbf{v} - \omega} d^3p - \int \frac{n}{2\varepsilon_0^3} d^3p \right]. \quad (20) \end{aligned}$$

On the other hand, we go in Eq. (16) to the limit $\omega_0 \rightarrow 0$, $\mathbf{k} \rightarrow 0$, taking ω_0/\mathbf{k} to be finite. One verifies easily that in that limit (16) is, indeed, the same as (20). This derivation very clearly demonstrates the structure of Eq. (16) and the origin of the non-local terms in it.

The set of Eqs. (18) and (17) does not contain the equations of the hydrodynamics of the superfluid part. This is connected with the fact that (18) is one real equation. On the other hand, as is clear from [11], the equation giving the conservation of the number of particles, equivalent to the hydrodynamic equation, is contained in the imaginary part of the equation for Δ . It is therefore necessary for us to write down the equation for Δ with greater accuracy, retaining terms with derivatives of Δ with respect to the coordinates and the time. This will be done in the next section.

2. EQUATION OF MOTION OF THE SUPERFLUID PART

As in previous papers, we start from a variational principle. We assume that we can obtain the equation for Δ^* by putting equal to zero the variational derivatives with respect to Δ of some functional. In accordance with the fact that $n(\mathbf{p}, \mathbf{r})$ was assumed to be arbitrary, we can perform the variation for a given distribution function, which corresponds in fact to varying the energy at constant entropy. We note that just such a minimization method is applied in the BCS theory to determine the relation between the parameters u and v and Δ (see, e.g., [5], Sec. 80). In the case when Δ is independent of \mathbf{r} and t , the functional has the form

$$\begin{aligned} \bar{S} &= -\frac{1}{(2\pi)^3} \int d^3r dt \\ &\times \left\{ \left[\frac{(2\pi)^3}{g} - 4\pi m p_0 \ln \frac{2\tilde{\omega}}{\Delta} \right] |\Delta|^2 + 2 \int n_p \varepsilon d^3p \right\}. \quad (21) \end{aligned}$$

Indeed, varying (21) with respect to Δ for constant n_p and putting the variation equal to zero, we get Eq. (18). The common coefficient in (21) is

chosen such that the terms depending on n_p are the same as the thermodynamic potential Ω (see [6]). If Δ depends on the coordinates and the time, we must add to (21) terms containing derivatives of Δ . Those terms in \bar{S} which are independent of n_p are the same as the corresponding terms at $T = 0$ and were investigated in [11]. That part of the functional which depends on n_p and contains terms with derivatives of Δ of first order can be determined from general considerations.

We note first of all that in the system of coordinates in which the superfluid part moves we must add to ϵ the quantity

$$\mathbf{p}\mathbf{v}_s = i(\mathbf{p}\nabla)(\varphi^* - \varphi).$$

On the other hand, bearing in mind that a change in the chemical potential $\mu \rightarrow \mu + \delta\mu$ involves a change in the quantity Δ according to the rule $\Delta \rightarrow \Delta e^{-2it\delta\mu}$, we realize easily that μ must occur in the equation only in the combination

$$\mu + \frac{i}{4} \frac{\partial}{\partial t} (\varphi - \varphi^*) \quad (22)$$

This discussion shows that up to derivatives of first order ϵ_0 in (21) must be replaced by

$$\epsilon_0 + \frac{i}{4} \frac{\partial \epsilon}{\partial \mu} \frac{\partial}{\partial t} (\varphi^* - \varphi) + \frac{i}{4m} (\mathbf{p}\nabla)(\varphi^* - \varphi). \quad (23)$$

Substituting (23) into (21) and varying with respect to Δ we can obtain the equation for Δ^* . We are, however, interested not so much in that equation as in the conservation law for the number of particles which follows from it:

$$\frac{\partial}{\partial t} \delta\rho + \text{div } \mathbf{j} = 0. \quad (24)$$

To determine $\delta\rho$ and \mathbf{j} we introduce into the functional a vector potential \mathbf{A} and a scalar potential Φ , through the substitution

$$\nabla\varphi^* \rightarrow \nabla\varphi^* + 2ie\mathbf{A}, \quad \frac{\partial\varphi^*}{\partial t} \rightarrow \frac{\partial\varphi^*}{\partial t} - 2ie\Phi. \quad (25)$$

After this, varying S with respect to Φ and \mathbf{A} we can find $\delta\rho$ and \mathbf{j} from the equation

$$\delta S = \int d^3r \left[\frac{e}{m} (\delta\rho) \delta\Phi - \frac{e}{m} \mathbf{j} \delta\mathbf{A} \right].$$

Here, $e\mathbf{j}/m$ is the electrical current density and $e\delta\rho/m$ the deviation of the charge density from the equilibrium. Taking the variation, we find

$$\mathbf{j} = \mathbf{j}_0 + \int d^3p n \mathbf{p}, \quad (26a)$$

$$\delta\rho = \delta\rho_0 - m \int d^3p n \frac{\partial \epsilon}{\partial \mu}, \quad (26b)$$

where \mathbf{j}_0 and $\delta\rho_0$ are the values of these quantities at zero temperature, found in [11]. In the semiclassical approximation we have

$$\delta\rho_0 = - \frac{mp_0}{\pi^2} \left[\frac{i}{4m} \frac{\partial(\varphi^* - \varphi)}{\partial t} + \frac{v_s^2}{2} \right]. \quad (27)$$

Substituting $\delta\rho_0$ from (27) into (26b), multiplying the resultant equation by $\partial\mu/\partial\rho$ and taking its gradient, we get

$$\frac{\partial v}{\partial t} + \nabla \left[\frac{v_s^2}{2} + \mu(\rho) + \int d^3p n \frac{\partial \epsilon}{\partial \rho} \right], \quad (28)$$

where $\mu(\rho)$ is the chemical potential as function of ρ at absolute zero. As should be the case, this equation is the same as the one obtained by Khalatnikov [7] for the case of a Bose liquid. The second term in the current is clearly the usual extra term connected with the momentum of the elementary excitation.

Equations (17), (18), and (28) form a complete set of equations to describe our system. These equations are, however, semiclassical. To obtain the equations which are exactly equivalent to the Ginzburg-Landau equations we must take into account quantum corrections of order \hbar^2 . In the next section we shall write down a kinetic equation with that accuracy.

3. ELEMENTARY EXCITATIONS AND KINETIC EQUATION

In the case where the gap Δ is independent of \mathbf{r} and t the dispersion law for the elementary excitations has the form

$$\epsilon_0(p) = \sqrt{\xi^2 + |\Delta|^2}, \quad (29)$$

where $\xi = p^2/m - \mu$. In the general case when $\Delta = \Delta(\mathbf{r}, t)$ the energy depends not only on Δ but also on its derivatives with respect to time and coordinates. If Δ varies slowly, we can write ϵ as a series in these derivatives. We must find an expression for the energy operator of the elementary excitations up to terms of second order in the spatial derivatives of Δ .

The dependence of the derivatives of φ and φ^* can be determined from general considerations. Using (22) and the requirement of Galilean invariance (see [11]) we can write ϵ in the form

$$\epsilon = \epsilon_0 + \frac{i}{4m} (\mathbf{p}\nabla)(\varphi^* - \varphi) - \frac{1}{32m} \frac{\xi}{\epsilon} [\nabla(\varphi - \varphi^*)]^2 + \beta_{ik} \nabla_i(\varphi + \varphi^*) \nabla_k(\varphi + \varphi^*) + \gamma_{ik} \nabla_i \nabla_k(\varphi + \varphi^*), \quad (30)$$

where β_{ik} and γ_{ik} are unknown functions of \mathbf{p} and $|\Delta|^2$.

To determine them we turn to Eqs. (1). The energy of the excitations is determined by the poles

of the functions G and F . When determining the position of the poles we can neglect the right-hand side of Eqs. (1) so that the excitation spectrum is determined by the eigenvalues of homogeneous equations of the form

$$(\omega - \hat{\xi})\chi_1 - i\Delta(x)\chi_2 = 0, \quad (\omega + \hat{\xi})\chi_2 + i\Delta^*(x)\chi_1 = 0. \quad (31)$$

These equations are the same as the set of equations of Bogolyubov.^[8]

The set (31) has the form of a Schrödinger equation for a two-component wave function $\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$. The fact that the function has two components is connected with the fact that it describes at the same time quasiparticles and holes. To determine the Hamiltonian of an elementary excitation we must perform a transformation which separates particle and hole states. This procedure is conceptually similar to the Foldy-Wouthuysen transformation for the Dirac equation.^[9] We must perform this transformation in the approximation which leads us to ϵ in lowest approximation.

In (31) we change to the momentum representation. We have

$$\begin{aligned} (\omega - \xi)\chi_{1p} - i\hat{\Delta}(\mathbf{r})\chi_{2p} &= 0, \\ (\omega + \xi)\chi_{2p} + i\hat{\Delta}^*(\mathbf{r})\chi_{1p} &= 0. \end{aligned} \quad (32)$$

If Δ is independent of \mathbf{r} :

$$\Delta = \bar{\Delta} = \text{const},$$

the separation of the hole states can be performed by a Bogolyubov transformation

$$\chi_{1p} = u\chi_{+} - iv\chi_{-}, \quad \chi_{2p} = u\chi_{-} - iv\chi_{+}, \quad (33)$$

where, as always,

$$\begin{aligned} u^2 - v^2 &= \bar{\xi}/\epsilon, \quad uv = \bar{\Delta}/2\bar{\epsilon}, \\ u^2 + v^2 &= 1, \quad \bar{\epsilon} = (\bar{\xi}^2 + |\bar{\Delta}|^2)^{1/2}. \end{aligned}$$

These transformations change (32) to the form

$$(\omega - \bar{\epsilon})\chi_{+} = 0, \quad (\omega + \bar{\epsilon})\chi_{-} = 0.$$

Let now Δ be dependent on the coordinates. To simplify the further calculations we shall assume that Δ be real. We expand Δ in a series in the coordinates near the point $\mathbf{r} = 0$:

$$\Delta(\mathbf{r}) = \bar{\Delta} + a_i x_i + b_{ik} x_i x_k + \dots, \quad (34)$$

where

$$a_i = \left. \frac{\partial \Delta}{\partial x_i} \right|_{\mathbf{r}=0}, \quad b_{ik} = \left. \frac{1}{2} \frac{\partial^2 \Delta}{\partial x_i \partial x_k} \right|_{\mathbf{r}=0}.$$

The problem is now to determine the dependence of ϵ on a_i and b_{ik} . In the p -representation $\Delta(\mathbf{r})$ changes to the operator

$$\hat{\Delta}(\mathbf{r}) = \bar{\Delta} + \hat{\delta}, \quad (35)$$

where

$$\hat{\delta} = ia_i \frac{\partial}{\partial p_i} - b_{ik} \frac{\partial^2}{\partial p_i \partial p_k}.$$

Substituting now (35) into (32) and performing the same transformation (33) we get the equations for χ_{+} and χ_{-} :

$$(\omega - \bar{\epsilon})\chi_{+} = (u\hat{\delta}v + v\hat{\delta}u)\chi_{+} + i(u\hat{\delta}u - v\hat{\delta}v)\chi_{-}, \quad (36a)$$

$$(\omega + \bar{\epsilon})\chi_{-} = i(v\hat{\delta}v - u\hat{\delta}u)\chi_{+} - (v\hat{\delta}u + u\hat{\delta}v)\chi_{-}. \quad (36b)$$

We must now eliminate approximately χ_{-} from these equations. To do that it is necessary to determine χ_{-} from (36b) up to terms $\sim a$. We have

$$\begin{aligned} \chi_{-} &\approx \frac{i}{\omega + \bar{\epsilon}} (v\hat{\delta}v - u\hat{\delta}u)\chi_{+} \\ &\approx a_i \left[\frac{\xi}{2\epsilon^2} \frac{\partial}{\partial p_i} + \frac{\Delta^2 - \xi^2}{4\epsilon^3} v_i \right] \chi_{+} = \hat{\eta}\chi_{+}. \end{aligned} \quad (37)$$

Substituting (37) into (36a) we get for χ_{+} an equation of the form

$$(\bar{\epsilon} + D_1 - \omega)\chi_{+} = 0, \quad (38)$$

where D_1 is a differential operator, the form of which we shall not write down here.

It is, however, no longer possible to consider Eq. (38) as a Schrödinger equation for the elementary excitations, since the operator \bar{D} is not self-adjoint. This is connected with the fact that the initial Eqs. (32) guaranteed the conservation of an integral of the form

$$\int (|\chi_1|^2 + |\chi_2|^2) d^3p = \int (|\chi_{+}|^2 + |\chi_{-}|^2) d^3p.$$

To obtain a self-adjoint equation we must thus introduce instead of χ_{+} a new function $\Psi = (1 + \hat{\mu})\chi_{+}$ and choose the operator $\hat{\mu}$ such that

$$\int |\Psi|^2 d^3p = \int (|\chi_{+}|^2 + |\chi_{-}|^2) d^3p.$$

One sees easily that the operator μ must satisfy the relation

$$\mu + \mu^+ = \eta^+ \eta, \quad (39)$$

where the cross indicates as usual Hermitian conjugation. The function Ψ will satisfy the equation

$$(\hat{\epsilon} - \omega)\Psi = 0, \quad (40)$$

$$\hat{\epsilon} = \bar{\epsilon} + \mu\bar{\epsilon} - \bar{\epsilon}\mu + D_1, \quad (41)$$

where the operator will be Hermitian if condition (39) is satisfied.

The next stage in the calculations consists in eliminating the quantity ϵ from the equations for D_1 and $\hat{\epsilon}$. The quantity ϵ does not have a direct physical meaning since it is expressed in terms of

Δ —the value of the gap in the point $\mathbf{r} = 0$ which has no particular significance. A physical meaning is possessed by the quantity

$$\varepsilon_0 = (\hat{\xi}^2 + \hat{\Delta}^2)^{1/2} = [\hat{\xi}^2 + (\bar{\Delta} + a_i \hat{x}_i + b_{ik} \hat{x}_i \hat{x}_k)^2]^{1/2}, \quad (42)$$

which occurs in (30). We shall express ε_0 approximately in terms of $\bar{\varepsilon}$ and eliminate ε . One checks easily that

$$\varepsilon_0 = \bar{\varepsilon}(\mathbf{p}) + \varepsilon_1(\mathbf{p}, \mathbf{r}); \quad (43)$$

$$\begin{aligned} \varepsilon_1 = & \frac{\Delta}{\varepsilon} a_i \hat{x}_i - i \frac{\Delta \hat{\xi}}{2\varepsilon^3} a_i v_i + a_i a_k \left[\frac{\hat{\xi}^2}{2\varepsilon^3} \hat{x}_i \hat{x}_k \right. \\ & - i \hat{\xi} \frac{\xi^2 - 2\Delta^2}{2\varepsilon^5} v_i \hat{x}_k - \frac{2\Delta^4 + 2\xi^4 - 11\xi^2 \Delta^2}{8\varepsilon^7} v_i v_k \\ & \left. + \frac{\xi^2 - \Delta^2}{4\varepsilon^5} \frac{\xi}{m} \delta_{ik} \right] \\ & + b_{ik} \left[\frac{\Delta}{\varepsilon} \hat{x}_i \hat{x}_k - i \frac{\Delta}{\varepsilon^3} \xi v_i \hat{x}_k + \frac{\Delta}{2\varepsilon^3} \frac{\xi}{m} \delta_{ik} \right. \\ & \left. + \Delta \frac{\Delta^2 - \xi^2}{\gamma \varepsilon^5} v_i v_k \right]. \end{aligned} \quad (44)$$

To check Eq. (43) we must square it and use the commutation relations for the operators \hat{x} and \hat{p} .

In the representation realized by the functions χ_{1p} and χ_{2p} the operator \hat{x} has the form $x_i = i\partial/\partial p_i$ so that we can write ε_0 in that representation in the form

$$\varepsilon_0 = \bar{\varepsilon}(\mathbf{p}) + \varepsilon_1(\mathbf{p}, i\partial/\partial \mathbf{p}). \quad (43a)$$

We must, however, change to the representation realized by the function Ψ . In the required approximation we have in that representation

$$\begin{aligned} \varepsilon_0 = & \bar{\varepsilon} - \mu^+ \bar{\varepsilon} - \bar{\varepsilon} \mu + \eta^+ \bar{\varepsilon} \eta + u \varepsilon_1 u + v \varepsilon_1 v \\ & + i(v \varepsilon_1 u - u \varepsilon_1 v) \eta + i \eta^+(v \varepsilon_1 u - u \varepsilon_1 v). \end{aligned} \quad (45)$$

We now subtract (45) from (41) and find an expression for $\mu + \mu^+$ using (39). We finally obtain

$$\begin{aligned} \hat{\varepsilon} = & \varepsilon_0 + \eta^+(\eta \bar{\varepsilon} - \bar{\varepsilon} \eta) - (u \varepsilon_1 u + v \varepsilon_1 v) \\ & + D_1 - i(v \varepsilon_1 v - u \varepsilon_1 u) \eta - i \eta^+(v \varepsilon_1 u - u \varepsilon_1 v). \end{aligned} \quad (46)$$

We note that in fact $\bar{\varepsilon}$ only occurs in (46) in second order terms in which we no longer need distinguish between $\hat{\varepsilon}$, ε_0 , or $\bar{\varepsilon}$.

Using the explicit form of D_1 , ε_1 , and η we obtain $\hat{\varepsilon}$ after complicated calculations in the form

$$\hat{\varepsilon} = \varepsilon_0 + a_i a_k \left(-\frac{1}{4\varepsilon^3} \frac{\xi}{m} \delta_{ik} + \frac{1}{8\varepsilon^3} v_i v_k \right) - b_{ik} \frac{\Delta}{2\varepsilon^3} v_i v_k. \quad (47)$$

We note that the operators $i\partial/\partial p_i$ no longer occur explicitly. They are merely contained in the operator ε_0 . The presence of such operators separately would have meant that the answer depended on the choice of the origin $\mathbf{r} = 0$ and that should clearly not be the case.

If we now substitute for a_i and b_{ik} their values in terms of the derivatives of Δ and if we change from the derivatives of Δ to these of φ and compare (47) with (30) we find expressions for β_{ik} and γ_{ik} :

$$\begin{aligned} \beta_{ik} = & -\frac{\Delta}{4} \left(\frac{1}{4\varepsilon^3} \frac{\xi}{m} \delta_{ik} + \frac{1}{8\varepsilon^3} v_i v_k \right) = \beta_1 \delta_{ik} + \beta_2 v_i v_k \\ \gamma_{ik} = & -\frac{\Delta^2}{8\varepsilon^3} v_i v_k = \gamma_2 v_i v_k. \end{aligned} \quad (48)$$

Knowing the Schrödinger equation (40) for the wave function Ψ of the elementary excitation we can easily write down the equation for the density matrix. We shall consider the equation for that matrix in a mixed representation, i.e., for the "quantum distribution function" $n(\mathbf{p}, \mathbf{x})$ which is connected with $\Psi(\mathbf{p})$ through the formula

$$n(\mathbf{p}, \mathbf{x}) = \frac{1}{(2\pi)^6} \int e^{i\mathbf{k}\mathbf{x}} \Psi^* \left(\mathbf{p} - \frac{\mathbf{k}}{2} \right) \Psi \left(\mathbf{p} + \frac{\mathbf{k}}{2} \right) d^3k. \quad (49)$$

It is well known (see [10] and [11]) that the kinetic equation for $n(\mathbf{p}, \mathbf{x})$ has the form

$$\frac{\partial}{\partial t} n(\mathbf{p}, \mathbf{x}) = \frac{i}{(2\pi)^6} \int d^3\tau d^3k d^3p' d^3x'.$$

$$\begin{aligned} \exp \{ ik(\mathbf{x}' - \mathbf{x}) - i\tau(\mathbf{p} - \mathbf{p}') \} n(\mathbf{p}', \mathbf{x}') (\varepsilon_{+}' - \varepsilon_{-}'), \\ \varepsilon_{\pm}' = \varepsilon(\mathbf{p}' \pm \mathbf{k}/2, \quad \mathbf{x}' \pm \tau/2). \end{aligned} \quad (50)$$

We have here denoted by ε the classical function corresponding to the operator (47). In other words, we must replace in ε the operator by the classical function $\Delta(\mathbf{x})$.

In accordance with our basic assumptions we must transform this equation assuming that n and ε are slowly varying functions of \mathbf{x} . To obtain the formula which corresponds to the same degree of approximation as in Eq. (30) we must retain in (50) terms of the approximation following the classical one in the derivatives of n and ε . Expanding ε_{\pm} in terms of \mathbf{k} and τ exactly up to terms of sixth order inclusively and performing all integrations, we get

$$\begin{aligned} \frac{\partial n}{\partial t} + \frac{\partial \varepsilon}{\partial p_i} \frac{\partial n}{\partial x_i} - \frac{\partial \varepsilon}{\partial x_i} \frac{\partial n}{\partial p_i} = & -\frac{1}{8 \cdot 3} \left\{ \frac{\partial^3 \varepsilon}{\partial x_i \partial x_k \partial x_m} \frac{\partial^3 n}{\partial p_i \partial p_k \partial p_m} \right. \\ & - 3 \frac{\partial^3 \varepsilon}{\partial x_i \partial x_k \partial p_m} \frac{\partial^3 n}{\partial p_i \partial p_k \partial x_m} + 3 \frac{\partial^3 \varepsilon}{\partial x_i \partial p_k \partial p_m} \frac{\partial^3 n}{\partial p_i \partial x_k \partial x_m} \\ & \left. - \frac{\partial^3 \varepsilon}{\partial p_i \partial p_k \partial p_m} \frac{\partial^3 n}{\partial x_i \partial x_k \partial x_m} \right\}. \end{aligned} \quad (51)$$

The left-hand side of (51) is the usual collisionless classical kinetic equation, the right-hand side gives the quantum correction connected with the spatial inhomogeneity. On the right-hand side we must, of course, understand by ε simply ε_0 .

In the static case when $\partial n/\partial t = 0$, the classical kinetic equation has as solution any function $n_0(\epsilon)$ of the excitation energy ϵ . This is no longer the case when the right-hand side of (51) is taken into account. By direct substitution one can verify that (49) is in the lowest approximation satisfied by the function

$$n = n_0(\epsilon) + n_0^{\text{II}}, \quad (52)$$

where

$$\begin{aligned} n_0^{\text{II}} = & -\frac{1}{8 \cdot 3} \left\{ \frac{d^3 n_0}{d\epsilon^3} \left[\frac{\partial^2 \epsilon}{\partial x_h \partial x_m} \frac{\partial \epsilon}{\partial p_h} \frac{\partial \epsilon}{\partial p_m} \right. \right. \\ & + \left. \frac{\partial^2 \epsilon}{\partial p_h \partial p_m} \frac{\partial \epsilon}{\partial x_h} \frac{\partial \epsilon}{\partial x_m} - 2 \frac{\partial^2 \epsilon}{\partial x_i \partial p_h} \frac{\partial \epsilon}{\partial x_h} \frac{\partial \epsilon}{\partial p_m} \right] \\ & + \left. 3 \frac{d^2 n}{d\epsilon^2} \left[\frac{\partial^2 \epsilon}{\partial p_h \partial p_m} \frac{\partial^2 \epsilon}{\partial x_h \partial x_m} - \frac{\partial^2 \epsilon}{\partial p_h \partial x_m} \frac{\partial^2 \epsilon}{\partial x_h \partial p_m} \right] \right\}. \quad (53) \end{aligned}$$

Equations (30), (48), and (51) obtained here allow us to determine the distribution function n for a given function $\Delta(x)$. To complete the set it is still necessary to have an equation determining $\Delta(x)$ in terms of $n(\mathbf{p}, \mathbf{r})$.

4. EQUATION FOR THE GAP

Now that we have obtained the kinetic equation which is valid for any dependence of the gap on the coordinates we must find with the same approximation the equation expressing Δ in terms of the distribution function. To do this we must add to the functional S terms with higher derivatives of Δ . It is essential that in the framework of the set of Gor'kov equations the functional must be linear in n . The fact is that in the approximation considered there is in the theory of superconductivity no direct interaction between elementary excitations. In the theory there is only an indirect self-consistent interaction which manifests itself in that the excitation spectrum depends on Δ while Δ itself is determined by the excitation distribution function. This absence of an interaction manifests itself in the fact that the Gor'kov equations (1) for a given Δ are linear equations for G and F .

The form of some of the terms in the functional can be established a priori from general considerations similar to those used for deriving Eq. (23). Omitting these considerations we give at once the expression for the complete functional satisfying all requirements:

$$\begin{aligned} S = & S_0 + 2 \int d^3 r dt d^3 p \left\{ \left[\epsilon_0 + \hat{\beta}_{ik} \nabla_i (\varphi + \varphi^*) \nabla_k (\varphi + \varphi^*) \right. \right. \\ & + \left. \hat{\gamma}_{ik} \nabla_i \nabla_k (\varphi + \varphi^*) + \frac{i}{4m} (\mathbf{p} \nabla) \right. \\ & \left. \left. \times (\varphi^* - \varphi) + \frac{i}{4} \frac{\partial \epsilon}{\partial \mu} \frac{\partial (\varphi^* - \varphi)}{\partial t} \right] n \right. \end{aligned}$$

$$\begin{aligned} & + \frac{1}{2} \frac{\partial^2 \epsilon}{\partial \mu^2} \left(\frac{1}{32m} (\nabla \varphi - \nabla \varphi^*)^2 + \frac{i}{4} \frac{\partial}{\partial t} (\varphi - \varphi^*) \right)^2 \\ & + \left(\frac{\partial \hat{\beta}_{ik}}{\partial \mu} \nabla_i (\varphi + \varphi^*) \nabla_k (\varphi + \varphi^*) + \frac{\partial \hat{\gamma}_{ik}}{\partial \mu} \nabla_i \nabla_k (\varphi + \varphi^*) \right) \\ & \times \left(\frac{1}{32m} (\nabla \varphi - \nabla \varphi^*)^2 + \frac{i}{4} \frac{\partial}{\partial t} (\varphi - \varphi^*) \right) \Big] n \\ & + \left[\hat{a} \left[\frac{\partial}{\partial t} (\varphi + \varphi^*) \right. \right. \\ & + \left. \frac{i}{4m} ((\nabla \varphi^*)^2 - (\nabla \varphi)^2) \right] + i \hat{b} \left[\frac{\partial}{\partial t} (\varphi + \varphi^*) \right. \\ & \left. \left. + \frac{i}{4m} ((\nabla \varphi^*)^2 - (\nabla \varphi)^2) \right] \nabla^2 (\varphi - \varphi^*) \right] n_0 \Big\}. \quad (54) \end{aligned}$$

Here $\hat{\beta}$, $\hat{\gamma}$, \hat{a} , and \hat{b} are some as yet unknown differential operators acting upon n as function of \mathbf{p} . We shall, however, integrate by parts. After this the functional will depend only on n , and not on its derivatives. We emphasize that the latter consideration has merely a formal character since after integrating by parts there may occur diverging integrals. Nonetheless, assuming S to depend merely on n makes all calculations more compact. Any divergences appearing can be removed at the end by the inverse integration by parts. In this connection we shall for the present assume that $\hat{\beta}_{ik}$, $\hat{\gamma}_{ik}$, \hat{a} , and \hat{b} are simply functions of \mathbf{p} , and not operators. In the last two terms in (54) we replaced n by the equilibrium distribution function n_0 . The fact is that the deviation of the distribution function from its equilibrium value is clearly proportional to the quantity $\partial \Delta / \partial t$. Taking that deviation into account would therefore exceed the accuracy to which we are working.

Let us now determine the coefficients $\hat{\beta}_{ik}$ and $\hat{\gamma}_{ik}$. In the following it will be convenient to write them in the form

$$\hat{\beta}_{ik} = \beta_{ik} + \mu_{ik} = \beta_{ik} + \mu_1 \delta_{ik} + \mu_2 v_i v_k \quad (55)$$

$$\hat{\gamma}_{ik} = \gamma_{ik} + \nu_{ik} = \gamma_{ik} + \nu_1 \delta_{ik} + \nu_2 v_i v_k,$$

where β_{ik} and γ_{ik} are the coefficients occurring in the energy ϵ of the elementary excitations (see Eq. (48)). It is thus necessary to determine four functions μ_1 , μ_2 , ν_1 , and ν_2 . These coefficients are, however, not independent. They are connected by one condition. The fact is that in equilibrium the variational principle used by us must be the same as the usual condition that the thermodynamic potential is a minimum. This means that if we vary (54) for constant n in the static case and substitute

$$n = n_0(\epsilon) + n_0^{\text{II}}$$

in accordance with (52) the expression obtained

$$\delta S \approx \delta S_0 + 2 \int d^3 r dt d^3 p \{ n_0(\epsilon_0) \delta [4 \mu_{ik} \nabla_i \varphi \nabla_k \varphi + 2 \nu_{ik} \nabla_i \nabla_k \varphi] + n_0(\epsilon) \delta \epsilon + n_0^{\text{II}} \delta \epsilon_0 \} \quad (56)$$

must be put equal to the variation of some functional with respect to Δ . We can check that for this it is necessary that the condition

$$\mu_1 + \frac{\nu^2}{3} \mu_2 - \left(|\Delta|^2 \frac{\partial}{\partial |\Delta|^2} + \frac{1}{2} - \frac{|\Delta|^2}{4} \frac{1}{\xi} \frac{\partial}{\partial \xi} \frac{1}{\xi} \right) \times \left(\nu_1 + \frac{\nu^2}{3} \nu_2 \right) = C\xi, \quad (57)$$

is fulfilled, where C is a constant which is independent of \mathbf{p} .

Equation (57) gives us one condition to determine μ_1 , μ_2 , ν_1 , and ν_2 . The remaining three conditions can be found by comparing our equations with the linearized Eqs. (16) following from the micro-theory. Varying the functional with respect to Δ and equating the corresponding variational derivative to zero we get an equation for Δ^* . Linearizing this equation in accordance with (19) and solving it together with the linearized Eqs. (51) we find a linear equation for Δ_1^* containing the unknown μ_{ik} and ν_{ik} . If we expand that equation in \mathbf{k} for $\omega_0 = 0$ terms arise of the following three forms:

$$n_p k^2, \quad n_p (\mathbf{k}\mathbf{v})^2, \quad \frac{\partial n_p}{\partial \mathbf{p}} \mathbf{k} \frac{k^2}{(\mathbf{k}\mathbf{v})^2}.$$

The same terms also arise after expanding Eq. (16), which follows from the microtheory. Comparing coefficients of these three terms we get three equations which together with (57) enable us to determine μ_1 , μ_2 , ν_1 , and ν_2 .

We shall not give the intermediate expressions for these quantities obtained in this way. We note, however, that they contain terms with ξ^2 and ξ^4 in the denominator which leads to a divergence of the integrals in (54). To give the meaning of the expressions obtained it is therefore necessary to substitute these expressions into (54) and formally to integrate by parts until in the coefficients only integrable singularities of the form $\ln|\xi|$ remain. After this, the variational principle giving the equation for Δ will be determined by the functional (54) where, however, μ and ν will be differential operators, acting upon n_p , of the form*

$$\begin{aligned} \nu_{ik} &= -\frac{\Delta^4}{8\varepsilon^5} + \frac{\Delta^2}{8} \ln|\xi| \frac{\partial^2}{\partial p_i \partial p_k} \frac{1}{\varepsilon}, \\ \mu_{ik} &= -|\Delta|^4 \frac{9\xi^2 + 4\Delta^2}{32\varepsilon^7} + \frac{3\Delta^2}{16} \ln|\xi| \frac{\partial^2}{\partial p_i \partial p_k} \frac{1}{\varepsilon} \\ &+ \frac{|\Delta|^4}{64} \ln|\xi| \frac{1}{\nu^2} \frac{\partial^4}{\partial p_i \partial p_k \partial p_l^2} \frac{1}{\varepsilon}. \end{aligned} \quad (58)$$

Substituting these expressions into (54) and varying with respect to Δ we obtain an equation which in the static case replaces the equations of paper [6] when the distribution function is arbitrary.

We can similarly determine the coefficients a and b . To do this we must compare the linearized equation for Δ which is obtained from (54) and (51) with Eq. (16) expanded up to terms of order ω_0^2 . As a result, we have, for instance, for a :

$$\begin{aligned} \frac{a}{\Delta} &= -n \frac{\xi^2}{16\varepsilon^5} - \Delta^2 \frac{\partial n}{\partial \varepsilon} \frac{\xi^2 + 2\Delta^2}{4\varepsilon^6} + \frac{\Delta^2}{4} \frac{\partial^2 n}{\partial \varepsilon^2} \frac{5\xi^2 + 8\Delta^2}{12\varepsilon^5} \\ &+ \frac{\Delta^2}{12} \frac{1}{4\varepsilon^2} \frac{\partial^3 n}{\partial \varepsilon^3}. \end{aligned} \quad (59)$$

The complete equation for Δ obtained by varying (54) is very complex. We shall write it down only for the case of real Δ :

$$\begin{aligned} L_0 + \frac{2}{\varepsilon} \Delta + 8\Delta \frac{\partial (\hat{\beta}_{ik} n)}{\partial |\Delta|^2} \nabla_i \varphi \nabla_k \varphi - 4 \nabla_i (\hat{\beta}_{ik} n \nabla_k \varphi) \\ + 4\Delta \frac{\partial (\hat{\gamma}_{ik} n)}{\partial |\Delta|^2} \nabla_i \nabla_k \varphi \\ + 2 \nabla_i \nabla_k (\hat{\gamma}_{ik} n) + 8\Delta \frac{\partial (\hat{a} n_0)}{\partial |\Delta|^2} \left(\frac{\partial \varphi}{\partial t} \right)^2 - 4 \frac{\partial}{\partial t} \left(\hat{a} n_0 \frac{\partial \varphi}{\partial t} \right) = 0. \end{aligned} \quad (60)$$

Here L_0 is Eq. (20) of [11] in which all terms are transferred to the left-hand side.

Let us discuss the important problem of the region of applicability of the equations obtained. One can show that in order that the expansions in \mathbf{k} and ω_0 which we have used are valid the following inequalities must hold

$$\omega_0 \ll \Delta(T), \quad kv_F \ll \Delta(T). \quad (61)$$

Near T_C where $\Delta_0 \gg \Delta(T)$ these inequalities are very rigid. In particular, the second inequality corresponds to the case of a pure type II superconductor. The case considered by us is thus the opposite of the one considered by Abrahams and Tsuneto.^[12]

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