BREMSSTRAHLUNG IN THE COLLISION OF A MUON WITH AN ELECTRON AT REST

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The bremsstrahlung process in the collision of a fast muon with an electron at rest is considered. The photon spectrum is calculated for frequencies much higher than the mass of the electron.

1. The emission of a photon in the collision of a muon with atomic electrons may give an important contribution to the cross section for the bremsstrahlung of muons incident on atoms even when Z is large. We shall regard the muon as relativistic ($E \gg \mu$) and consider photons with $\omega \gg 1$ ($\hbar = c = m_e = 1$). In this case the electrons of the atomic shell can be regarded as free and at rest.

In lowest order perturbation theory the process is described by two pairs of graphs (Fig. 1). Denoting the contributions of graphs I and II and the interference term by $d\sigma_e$, $d\sigma_\mu$, and $d\sigma_{e\mu}$, we write the cross section in the form

$$d\sigma = d\sigma_e + d\sigma_\mu + d\sigma_{e\mu}. \tag{1}$$

2. Consider the contribution of the graphs I, corresponding to the emission of a photon by the electron:

$$d\sigma_e = \frac{\alpha^3}{2\pi^2}\frac{1}{\sqrt{(p_1p_2)^2-\mu^2}}\frac{1}{\Delta_1^4}f_{\alpha\beta}N^{\alpha\beta}\frac{d\mathbf{p}_3}{p_3^0}\cdot\frac{d\mathbf{p}_4}{p_4^0}\frac{d\mathbf{k}}{k^0}\delta(q_1+p_1-p_3-k), \tag{2}$$

where

$$f_{\alpha\beta} = {}^1/_4\,\mathrm{Sp}(\hat{p}_2+\mu)\gamma_\alpha(\hat{p}_4+\mu)\gamma_\beta,$$

$$N_{\alpha\beta} = -{}^1/_8\,\mathrm{Sp}\,O_{\alpha\rho}(\hat{p}_1+1)\bar{O}_{\beta\rho}(\hat{p}_3+1),$$

$$O_{\alpha\rho} = \gamma_\alpha\frac{\hat{p}_1-\hat{k}+1}{(p_1-k)^2-1}\gamma_\rho + \gamma_\rho\frac{\hat{p}_3+\hat{k}+1}{(p_3+k)^2-1}\gamma_\alpha,$$

and $\Delta_1^2 = -q_1^2$; the remaining notation is evident from the figure.

In order to obtain the photon spectrum, we must integrate (2) over all variables except the frequency of the photon. It is more convenient, however, to integrate over all variables and to introduce in the integrand an additional δ-function of $\omega - kp_1$, which fixes the frequency in the rest system of the electron (lab. system). This allows us to do the integrations in manifestly covariant form and to include one by one the contributions from the different parts of the graphs. This approach has been proposed by Baĭer and one of the authors.[1]

We integrate first over the momenta of the electron and the photon. The tensor

$$T_{\alpha\beta} = \int N_{\alpha\beta}\frac{d\mathbf{p}_3}{p_3^0}\frac{d\mathbf{k}}{k^0}\delta(q_1+p_1-p_3-k)\delta(\omega-(kp_1)) \tag{3}$$

is symmetric and depends only on the vectors q_1 and p_1. In general this tensor is determined by four scalar functions. The condition of gauge invariance ($T_{\alpha\beta}q_1^\beta = 0$) reduces the number of independent functions to two, and the tensor $T_{\alpha\beta}$ can be expressed through $T_{\alpha\beta}g^{\alpha\beta}$ and $T_{\alpha\beta}p_1^\alpha p_1^\beta$:

$$T_{\alpha\beta} = T_\rho^\rho F_{1\alpha\beta} + T_{\rho\sigma}p_1^\rho p_1^\sigma F_{2\alpha\beta}, \tag{4}$$

where

$$2F_1^{\alpha\beta} = g^{\alpha\beta} + \frac{q_1^\alpha q_1^\beta}{\Delta_1^2} - \frac{(q_1p_1)^2}{(q_1p_1)^2+\Delta_1^2}\times\left(\frac{\Delta_1^2}{(q_1p_1)^2}p_1^\alpha p_1^\beta + \frac{q_1^\alpha q_1^\beta}{\Delta_1^2} + \frac{q_1^\alpha p_1^\beta + p_1^\alpha q_1^\beta}{(q_1p_1)}\right)$$

$$2F_2^{\alpha\beta} = \frac{-\Delta_1^2}{(q_1p_1)^2+\Delta_1^2}\left[g^{\alpha\beta} + \frac{q_1^\alpha q_1^\beta}{\Delta_1^2} - \frac{3(q_1p_1)^2}{(q_1p_1)^2+\Delta_1^2}\times\left(\frac{\Delta_1^2}{(q_1p_1)^2}p_1^\alpha p_1^\beta + \frac{q_1^\alpha q_1^\beta}{\Delta_1^2} + \frac{q_1^\alpha p_1^\beta + p_1^\alpha q_1^\beta}{(q_1p_1)}\right)\right].$$

Contracting $N_{\alpha\beta}$ with $g^{\alpha\beta}$ and $p_1^\alpha p_1^\beta$ and taking the trace, we obtain

$$-N_\alpha^\alpha = \frac{\varkappa_1}{\omega} + \frac{1}{\omega^2}(1-2\omega)\left(1-\frac{\Delta_1^2}{2}\right) - \frac{1}{\varkappa_1\omega}\left(2-2\omega-\omega^2+\omega\Delta_1^2-\frac{\Delta_1^4}{2}\right) + \frac{1}{\varkappa_1^2}\left(1-\frac{\Delta_1^2}{2}\right),$$

$$N_{\alpha\beta}p_1^\alpha p_1^\beta = \frac{\varkappa_1}{2\omega^2}(2-3\omega-2\omega^2) + \frac{1}{\omega^2}\left[1-4\omega+\omega^2+\omega^3+\frac{\Delta_1^2}{4}(1-6\omega-2\omega^2)\right] + \frac{1}{\varkappa_1\omega}\left[-2+3\omega+\frac{\omega^2}{2}+\frac{\Delta_1^2}{2}(\omega-3)-\frac{\Delta_1^4}{4}\right]$$

$$+\frac{1}{\varkappa_1^2}\left(1+\frac{\Delta_1^2}{4}\right), \tag{5}$$

where

$$\varkappa_1 = (kp_3) = (qp_1) - \Delta_1^2/2.$$

When we integrate (3) with contracted tensors, we take N_α^α and $N_{\alpha\beta}p_1^\alpha p_1^\beta$ outside the integral, since the δ-functions in (3) fix the values of κ_1 and ω. The integrals can then be easily calculated:

$$T_\alpha{}^\alpha = 2\pi N_\alpha{}^\alpha[\Delta_1^2 + (\varkappa_1 + \Delta_1^2/2)^2]^{-1/2},$$

$$T_{\alpha\beta}p_1{}^\alpha p_1{}^\beta = 2\pi N_{\alpha\beta}p_1{}^\alpha p_1{}^\beta[\Delta_1^2 + (\varkappa_1 + \Delta_1^2/2)^2]^{-1/2}.$$

Multiplying the tensors $f_{\alpha\beta}$ and $T_{\alpha\beta}$, we write the cross section in the following form:

$$\frac{d\sigma_e}{d\omega} = \frac{\alpha^3}{2p^2}\int\frac{d\varkappa_1 d\Delta_1^2}{\Delta_1^4}[\Delta_1^2+(\varkappa_1+\Delta_1^2/2)^2]^{-1/2}$$
$$\times\Big\{-N_\alpha{}^\alpha\Big[\Delta_1^2-2\mu^2+\frac{\Delta_1^2}{\Delta_1^2+(\varkappa_1+\Delta_1^2/2)^2}$$
$$\times\left(2E^2-2\left(\varkappa_1+\frac{\Delta_1^2}{2}\right)E-\frac{\Delta_1^2}{2}\right)\Big]$$
$$+N_{\alpha\beta}p_1{}^\alpha p_1{}^\beta\frac{\Delta_1^2}{\Delta_1^2+(\varkappa_1+\Delta_1^2/2)^2}$$
$$\times\Big[\left(2E^2-2\left(\varkappa_1+\frac{\Delta_1^2}{2}\right)E-\frac{\Delta_1^2}{2}\right)\frac{3\Delta_1^2}{\Delta_1^2+(\varkappa_1+\Delta_1^2/2)^2}$$
$$+\Delta_1^2-2\mu^2\Big]\Big\}, \tag{6}$$

where $E = (p_1p_2)$ and $p = \sqrt{(p_1p_2)^2 - \mu^2}$ are the initial energy and momentum of the muon. Here we have made the substitution

$$\frac{d\mathbf{p}_4}{p_4^0} = \frac{1}{2p}d\Delta_1^2 d\varkappa_1 d\varphi$$

and integrated over the azimuthal angle φ.

The region of integration in the plane (Δ_1^2, κ_1) is given by the equation

$$a:\ \varkappa_1^2(2\omega-1)-2\varkappa_1\omega\left(\omega-1-\frac{\Delta_1^2}{2}\right)-\omega^2=0,$$

$$b:\ \varkappa_1^2\mu^2+\varkappa_1\Delta_1^2(\mu^2+E)+\frac{\Delta_1^4}{4}(\mu^2+2E+1)-\Delta_1^2p^2=0$$

and is illustrated in Fig. 2. The limiting values of Δ_1^2 and κ_1 have a different form, depending on the relation between the frequency ω and the characteristic frequency $\omega^* = (E-\mu)/E + p + 1$. If $\omega = \omega^*$, the upper branch of the curve a touches the curve b in the point P. If $\omega > \omega^*$, it lies below. Figure 2 refers to the case $\omega < \omega^*$. If a photon with the maximal frequency $\omega_{max} = (E-\mu)/E - p + 1$ is emitted, the lower branch of the curve a touches the curve b in the point P, and the area of the dashed region vanishes.

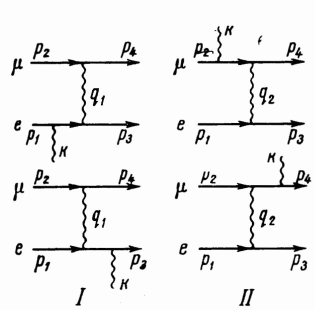

FIG. 1.

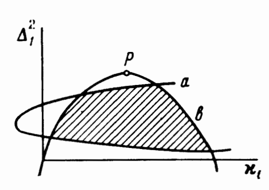

FIG. 2.

In the region of frequencies $\omega \ll \omega^*$ the cross section $d\sigma_e$ can be written in the form of a product of the cross section for the elastic scattering of the muon by the electron and the probability for emission of a classical photon, which is equal to

$$dw(\mathbf{k}) = e^2\left(\frac{p_1{}^\alpha}{(kp_1)}-\frac{p_3{}^\alpha}{(kp_3)}\right)^2\frac{d\mathbf{k}}{2(2\pi)^3k^0}.$$

For relativistic muons $\omega^* \approx 1/2$ independently of E, and the region where the classical approximation is valid encloses a very narrow portion of the spectrum. In the following we shall not consider this region.

3. The remaining double integration in (6) is difficult to carry out, but in the physically most interesting case of relativistic muons and frequencies large compared to the mass of the electron the calculations can be carried to the end.

In the region of frequencies $\omega_{max} - \omega \gg \omega_{max}\mu/E$ we obtain the following result:

$$\frac{E}{\alpha^3}\frac{d\sigma_e}{dx} = 2\left(1-\frac{x}{x_0}\right)\left[\frac{4}{3x^2}+\frac{1}{x}\left(\frac{5}{6x_0}+x_0\right)-\frac{1}{6x_0^2}+\frac{1}{2}\right]$$
$$\times\ln\frac{2E(x_0-x)}{\mu x_0}-\left[\frac{1}{x}\left(\frac{1}{2x_0}+1\right)-\frac{1}{x_0}-\frac{1}{2}\right]\ln\frac{1-x}{1-x_0}$$
$$-\left[\frac{1}{x}\left(\frac{1}{x_0}-x_0\right)+\frac{2}{x_0^2}-\frac{2}{x_0}-\frac{1}{2}-\frac{x}{x_0}\left(\frac{1}{3x_0^2}-1\right)\right]$$
$$\times\ \ln\frac{x_0}{x}-\frac{1}{xx_0}\ln\frac{x_0}{x}\ln\xi-\left(1+\frac{1}{xx_0}\right)$$
$$\times\left(\frac{1}{2}\ln xx_0\ln\frac{x_0}{x}+f(x_0)-f(x)\right)-\left(1+\frac{2}{xx_0}\right)$$
$$\times\left(f\left(\frac{x}{x_0}\right)-f\left(\frac{x_0}{x}\right)+\ln\frac{x_0}{x}\ln 2E\right)-\left(1-\frac{x}{x_0}\right)$$
$$\times\left[\frac{4}{3x^2}+\frac{1}{x}\left(\frac{2}{3x_0}+1+3x_0\right)-\frac{1}{x_0}\left(\frac{2}{3x_0}-\frac{1}{2}\right)\right] \tag{7}$$

Here

$$x=\frac{\omega}{E},\quad \xi=\frac{2E}{\mu^2},\quad x_0=\frac{\xi}{1+\xi}\approx\frac{\omega_{max}}{E},$$

and $f(x)$ is the Spence function

$$f(x) = -\int_0^x\frac{\ln|1-z|}{z}dz.$$

In the limiting case $\omega \ll \omega_{max}$, expression (7) simplifies greatly:

$$\frac{E}{\alpha^3}\frac{d\sigma_e}{dx}=\frac{8}{3x^2}\left(\ln\frac{2E}{\mu}-\frac{1}{2}\right). \tag{8}$$

We see that $d\sigma_e/dx$ decreases rapidly with increasing frequency. This result is easy to understand qualitatively, if we consider the process in the c.m.s. of the electron and the muon. The electron in the c.m.s. will be relativistic since $E \gg \mu$; the muon will be relativistic for $\xi \gg 1$ and nonrelativistic for $\xi \ll 1$. In the c.m.s. the electron radiates mainly in the direction of its motion into a small angle with opening $\sim 1/\tilde{\epsilon}$. The photons with the maximal energy, which are emitted in the direction of the momentum of the electron in the c.m.s., have the energy $\omega^* \approx 1/2$ in the lab. system. Therefore, only those photons make a contribution to the cross section for $\omega \gg 1$ which are emitted into large angles in the c.m.s.

If $\omega \gg 1$, the Weizsäcker-Williams method is not applicable to the calculation of $d\sigma_e$, since large momentum transfers are important ($\Delta^2_{\min} \approx \omega$), and the estimates obtained by this method[2,3] are inaccurate. An analogous result is obtained when the colliding particles have equal mass.[5]

Formula (7) does not apply to the interval of frequencies some hundred Mev at the end of the spectrum; the calculations for these frequencies have been carried out separately.

In the region of frequencies $\omega_{\max} - \omega \ll \omega_{\max}(1 - \omega_{\max}/E)$ the cross section has the form

$$\frac{E}{\alpha^3}\frac{d\sigma_e}{dx}=\frac{1}{x_0}[(x_0-x)\ln\varphi-r], \tag{9}$$

where

$$\varphi=\frac{E}{\mu x_0}(x_0-x+r),\quad r=[(x_0-x)^2-x_0^2\mu^2/E^2]^{1/2}.$$

In the intermediate region

$$\omega_{max}\mu/E \ll \omega_{max}-\omega \ll \omega_{max}(1-\omega_{max}/E)$$

the expressions (7) and (9) go over into each other.

4. Consider now the contribution from the graphs II. After averaging and summing over the polarizations, we can write $d\sigma_\mu$ in the form

$$d\sigma_\mu=\frac{\alpha^3}{4\pi^2}\frac{1}{p\Delta_2^4}\left[a\varkappa_2+b+\frac{c}{\varkappa_2}+\frac{d}{\varkappa_2^2}\right]\frac{d\mathbf{p}_3}{p_3^0}\frac{d\mathbf{p}_4}{p_4^0}\frac{d\mathbf{k}}{k_0}$$

$$\times\,\delta(q_2+p_2-k-p_4). \tag{10}$$

Here

$$a=-\frac{1}{\eta}\left(1-\frac{\Delta_2^2}{2}\right),$$

$$b=\frac{\mu^2}{\eta^2}\left\{-2(E-\omega)^2+\frac{\Delta_2^2}{2}[2(E-\omega)+\mu^2+1]-\frac{\Delta_2^4}{4}\right\}$$

$$-\frac{1}{\eta}\left[\Delta_2^2(E+\mu^2+1)-\frac{\Delta_2^4}{2}\right],$$

$$c=\frac{1}{\eta}\Big\{4E\mu^2(E-\omega)+\Delta_2^2[E^2+(E-\omega)^2-\mu^2(2E-\omega$$
$$+\mu^2+1)]-\frac{\Delta_2^4}{2}(2E-\omega+1)+\frac{\Delta_2^6}{4}\Big\}-\Delta_2^2(E-\omega$$
$$-\mu^2-1)-\eta\left(1-\frac{\Delta_2^2}{2}\right),$$

$$d=-\mu^2\left[2E^2-\frac{\Delta_2^2}{2}(2E+\mu^2+1)+\frac{\Delta_2^4}{2}\right],$$

$$\Delta_2^2=-q_2^2,\quad \eta=kp_2,\quad \varkappa_2=kp_4.$$

Here it is convenient to begin with the integrations over the momenta of the electron and the muon, which are most simply carried out in the corresponding c.m.s. Going over to the variables η, Δ_2^2, we obtain then

$$\frac{d\sigma_\mu}{d\omega}=\frac{\alpha^3}{p^2}\int\frac{d\Delta_2^2 d\eta}{\Delta_2^4}\left[a\frac{S}{R^{3/2}}+b\frac{1}{R^{1/2}}+c\frac{1}{Q^{1/2}}+d\frac{S}{Q^{3/2}}\right], \tag{11}$$

where

$$Q=1/4(\omega+\eta)^2\Delta_2^4-\Delta_2^2(\omega+\eta)(\eta E-\omega\mu^2)+\eta^2p^2,$$
$$R=(E-\omega)^2-\mu^2+2\eta,$$

$$S=\eta[E(E-\omega)+\eta-\mu^2]$$
$$-1/2\Delta_2^2[\eta(E-\omega+1)-\omega(E-\omega+\mu^2)].$$

The subsequent calculations were done for relativistic muons and large frequencies compared with the electron mass, as in Sec. 3. In the region $\omega_{\max} - \omega \gg \omega_{\max}\mu/E$ the cross section has the form

$$\frac{E}{\alpha^3}\frac{d\sigma_\mu}{dx}=\frac{2}{\rho}\Big\{(1-\rho)\Big[\left(1-x+\frac{1}{1-x}-\frac{2}{3}\right)\left(1+\frac{\rho}{2}\right)$$
$$+\frac{5}{3}\rho\left(1+\frac{4\rho}{5}\right)\Big]+\rho\left(1-x+\frac{1}{1-x}+2\rho\right)\ln\rho\Big\}$$
$$\times\ln\frac{2E(x_0-x)}{\mu x}+\frac{1-\rho}{\rho}\Big\{-\left(1-x+\frac{1}{1-x}-\frac{2}{3}\right)$$
$$\times\left(1+\frac{4}{\xi^2(1-x)}\right)-\frac{16}{3}\rho+\frac{8}{3}\rho^2$$
$$-\frac{1}{\xi}\left(\frac{4}{3(1-x)}+4+\frac{x^2}{2(1-x)^2}\right)-3(1-x_0)-\frac{1-x_0}{1-xx_0}\Big\}$$
$$-\Big[2-x+\frac{x}{2(1-x)}+2\rho+\frac{1-x_0}{1-xx_0}$$
$$\times\left(\frac{x}{2(1-x)}-1-\frac{x_0(1-x)}{1-xx_0}\right)\Big]\ln\rho$$

$$+\Big[\frac{2}{x}-\frac{3}{2(1-x)}+\frac{1}{\xi}\left(\frac{8}{x}-\frac{x}{2(1-x)^2}-\frac{1}{2(1-x)^2}\right)$$
$$+\frac{8}{3\xi^2}\left(\frac{2}{x}+\frac{1}{1-x}-\frac{x}{(1-x)^2}\right)+\frac{1}{x}(1-x)(1-x_0)$$

$$+\frac{1-x_0}{1-xx_0}\left(\frac{1}{x}+x_0(1-x)+\frac{1}{2x_0(1-x)}-\frac{x_0(1-x)}{1-xx_0}\right)\Bigg]$$
$$\times\ \ln\frac{1-x}{1-x_0}+\frac{1}{x}\left(x_0-x-\frac{x_0(1-x)}{1-xx_0}\right)\ln\frac{x_0}{x}-\frac{1+2/\xi}{1-x}$$
$$\times\left[\ln(1-x)\ln\frac{1-x}{1-x_0}+\ln(1-x_0)\ln\frac{x_0}{x}+2\ln(1-x)\right.$$
$$\left.\times\ \ln\frac{x_0}{x}+2f(x_0)-2f(x)\right]+\left(\frac{2-x}{x}+\frac{2}{\xi x}\right)\left[\ln(1-x_0)\right.$$
$$\left.\times\ \ln\frac{1-x}{1-x_0}-\ln(1-xx_0)\ln\frac{x_0}{x}-f(x_0)+f(x)\right]$$
$$+\left(\frac{2-x}{x}+\frac{2}{\xi x}-\frac{1+2/\xi}{1-x}\right)\left[f\left(\frac{x_0(1-x)}{1-xx_0}\right)\right.$$
$$\left.-f\left(\frac{x(1-x_0)}{1-xx_0}\right)\right]+2\left(1-x+\frac{1}{1-x}+2\rho\right)$$
$$\times\left(-\ln\rho\ln\frac{x_0-x}{1-x}+\frac{\pi^2}{6}-f(\rho)+f\left(-\frac{1}{\xi}\right)\right.$$
$$\left.-f\left(-\frac{1-x}{x}\right)\right), \qquad (12)$$

where $\rho = x/\xi(1-x)$.

In the limiting case $\xi \gg 1$ and frequencies such that $E-\omega \gg \mu^2$ the cross section $d\sigma_\mu$ goes over into the Bethe-Heitler formula for $Z = 1$:

$$\frac{E}{\alpha^3}\frac{d\sigma_\mu}{dx}=\frac{2}{\rho}\left(1-x+\frac{1}{1-x}-\frac{2}{3}\right)\left(\ln\frac{2E(1-x)}{\mu x}-\frac{1}{2}\right). \qquad (13)$$

In this case the recoil of the electron is unimportant, and the formula looks as if the mass of the electron were infinite. Formula (13) is in agreement with the results of [4,5], where the bremsstrahlung in electron-electron (or electron-positron) collisions was calculated, and disagrees with an earlier work.[6]

At the end of the spectrum, $d\sigma_\mu$ has the form

$$\frac{E}{\alpha^3}\frac{d\sigma_\mu}{dx}=\frac{\xi}{x_0^2}\left(1-x_0+\frac{1}{1-x_0}\right)[(x_0-x)\ln\varphi-r], \qquad (14)$$
$$\omega_{max}-\omega \ll \omega_{max}(1-\omega_{max}/E).$$

In the intermediate region, formulas (12) and (14) coincide.

5. Estimates show that the interference term is small for $\xi \gg 1$ and $\xi \ll 1$. For $\xi \sim 1$ the cross section $d\sigma_{e\mu}$ is in general of the same order of magnitude as $d\sigma_e$ and $d\sigma_\mu$. The cross section $d\sigma_{e\mu}$ has different signs for μ^+ and μ^- mesons and leads only to small corrections for cosmic muons.

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