WAVE RESISTANCE OF SUPERCONDUCTORS IN THE INTERMEDIATE STATE

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We show that in the presence of a direct current the spectrum of the electromagnetic oscillations of the intermediate state of a type I superconductor with unequal numbers of electrons and holes is such that Cerenkov radiation of vibrations of static inhomogeneities is possible. We calculate the resultant wave resistance. One of its characteristic features is its strong dependence on the type of inhomogeneity. If the inhomogeneities are periodic, the plot of the resistance against the current has a large number of maxima. If the metal is sufficiently pure, the wave resistance is appreciably larger than its normal value.

THE volume of a type I superconductor in the intermediate state is divided into a large number of alternating regions, occupied by the normal and the superconducting phase. Such a system has, in contrast to a purely superconducting state, a finite dc resistance. This is connected with the fact that the electrical current flowing through the intermediate state also flows in the normal regions. If the dimensions of these regions are appreciably larger than the electron mean free path, the resistivity of the normal phase will be the same as that of the bulk normal metal. The resistance of the intermediate state is in that case the sum of the resistances of the normal regions.[1]

If the electron mean free path is of the same order of magnitude as the dimensions of the normal regions, or larger than those, one should consider the influence of the scattering of the electrons by the boundaries between the phases. The resistance of the intermediate state can no longer, in general, be expressed in terms of the resistance of the bulk normal metal. An exception is the case when the temperature of the superconductor is small compared with the temperature of the superconducting transition when there is no magnetic field. We have shown[2] that owing to the specific character of the reflection of the electrons at the boundaries dividing the normal and the superconducting phase, the connection between the current and the electrical field in the normal layers is the same as in the bulk normal metal. The resistance of the intermediate state can be expressed in terms of the resistance of the normal metal.

It is of interest to note that the results of[2] remain valid even when the dimensions of the sample are small compared with the electron mean free path. This fact is also closely connected with the character of the reflection of the electronic excitations at the dividing boundary between the phases. It was shown in[3] that all three components of the velocity of the excitation incident onto the boundary differ in sign from the corresponding velocity components of the reflected excitation. Because of this the excitation colliding with the surface of the sample can only penetrate into the normal layer over a distance of the order of the thickness of the latter.[1)] After the first collision with the phase boundaries they reverse the direction of their velocity and move again in the direction of the surface of the sample. However, most excitations do not collide at all with the surface.

Another mechanism will, however, occur when the mean free path of the electrons is sufficiently large; it contributes to the resistance of the intermediate state of superconductors with unequal numbers of electrons and holes. We are speaking here about the fact that weakly damped electromagnetic waves, which are similar to helicons in a normal metal in the intermediate state of such superconductors, can be propagated.[4,5,2] We shall show below that when there is a direct electrical current present the spectrum of these waves is such that they can produce Cerenkov radiation at inhomo-

[1)]We assume that the dimensions of the normal layers are not too large. It is necessary that the Lamor radius in the critical magnetic field be larger for the majority of the electrons than the thickness of the normal layers. Otherwise an appreciable fraction of the electrons colliding with the surface of the sample will move without collisions with the phase boundaries and penetrate into the normal phase over distances of the order of the mean free path.

geneities. As a result, part of the energy of the current changes into the energy of the emitted waves and is subsequently dissipated. This also leads to an additional resistance. By analogy with a well-known effect in hydrodynamics (see[6]) we shall call this "wave" resistance. In contrast to the normal resistance the wave resistance does not vanish when the electron mean free path tends to infinity. It is thus the dominating one for sufficiently pure metals.

Under conditions when the wave resistance is important, the character of the dependence of the electrical field on the current depends strongly on the spectrum of the inhomogeneities, i.e., on their distribution over the different wavelengths. In particular, when there are periodic inhomogeneities, the current-voltage characteristic has a large number of resonance maxima. The occurrence of these maxima is connected with the fact that there is only a discrete set of wavelengths of the oscillations (for a given propagation direction) which can be radiated from static inhomogeneities, while the values of these wavelengths depend on the magnitude of the current. When the current is such that the period of the inhomogeneities coincides with one of the wavelengths of the oscillations, the Cerenkov radiation becomes particularly intense.

1. UNPERTURBED SOLUTION

We shall first calculate the normal resistance. We start from the general equations obtained in[1] for the electrodynamics of the intermediate state, which are valid at temperatures well below the critical one and under conditions when all quantities change little over distances of the order of the period of the structure:

$$\operatorname{div}\mathbf{B}=0,\quad \operatorname{rot}\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t},\quad (\mathbf{H}\nabla)\mathbf{H}=\frac{4\pi}{c}[\mathbf{j}_\perp\mathbf{H}],$$
$$\mathbf{E}\mathbf{H}=0,\quad |\mathbf{H}|=H_{\rm cr},\quad \mathbf{B}=x_n\mathbf{H},\quad j_{\perp\alpha}=\frac{\sigma_{\alpha\beta}}{x_n}E_\beta. \qquad (1)^*$$

Here **B** is the magnetic induction, which is obtained as the result of the averaging of the magnetic field over a volume with dimensions large compared with the period of the layer structure, H the magnetic field, E the average electrical field, H_{cr} the critical magnetic field, x_n the concentration of the normal phase, and $\sigma_{\alpha\beta}$ the two-dimensional conductivity tensor of the bulknormal metal placed in a magnetic field H in the plane perpendicular to H. The indices α and β take on two values in this plane.

*rot ≡ curl, $[\mathbf{j}\,\mathbf{H}] \equiv \mathbf{j}\times\mathbf{H}$.

We consider a plane-parallel plate of thickness a in an external magnetic field $\mathcal{H}$ at right angles to its plane. Let there be a current J flowing through the plate and we shall assume that $J \ll cH_{cr}$. The magnetic field of the current is in that case small compared with H_{cr} and the electrical field causing the current may be considered to be a small perturbation. Taking the z axis along the normal to the plate and recognizing that because of the symmetry of the problem all quantities depend only on the z coordinate, we get from (1)

$$\frac{dH_x}{dz}=\frac{4\pi H_{\rm cr}}{c\mathcal{H}}(\sigma_{yx}E_x+\sigma_{yy}E_y),$$
$$\frac{dH_y}{dz}=-\frac{4\pi H_{\rm cr}}{c\mathcal{H}}(\sigma_{xx}E_x+\sigma_{xy}E_y). \qquad (2)$$

The quantities E_x and E_y are constant, since curl **E** = 0. The electrical current density **j** is defined as $(c/4\pi)$ curl H. If the resulting current **J** is along the x axis, we get thus from (2)

$$\sigma_{yx}E_x+\sigma_{yy}E_y=0,$$
$$\sigma_{xx}E_x+\sigma_{xy}E_y=\frac{\mathcal{H}}{H_{\rm cr}\,a}J, \qquad (3)$$

whence we easily find $E_x(J)$ and $E_y(J)$.

We shall be interested in the case of a superconductor with unequal numbers of holes and electrons under the condition that $\nu \ll \Omega$, where ν is the collision frequency of the electrons and Ω the cyclotron frequency in the critical magnetic field. It is well known[7] that in that case the tensor $\sigma_{\alpha\beta}$ can be written in the form $\sigma_{\alpha\beta} = \sigma^{(0)}_{\alpha\beta} + s_{\alpha\beta}$, where $\sigma^{(0)}_{xy} = -\sigma^{(0)}_{yx} = Nec/H_{cr}$ (N is the difference between the number of electrons and holes per unit volume, and e is the electron charge), $\sigma^{(0)}_{xx} = \sigma^{(0)}_{yy} = 0$, $s_{\alpha\beta}$ is a small extra term which is of the order of magnitude of $(\nu/\Omega)\sigma^{(0)}_{xy}$. Using (3) we find the resistance

$$\rho_\parallel \equiv E_x/J = s_{yy}\mathcal{H}H_{\rm cr}\,/\,(Nec)^2a,$$
$$\rho_\perp \equiv E_y\,/\,J = \mathcal{H}\,/\,Neca. \qquad (4)$$

We note that its dissipative part $\rho_\parallel$ vanishes as $\nu \to 0$.

2. THE OSCILLATION SPECTRUM WHEN A CURRENT IS PRESENT

To find the oscillation spectrum we must linearize Eqs. (1) relative to the unperturbed solution with a current. If we then assume $\sigma_{\alpha\beta} = \sigma^{(0)}_{\alpha\beta}$, i.e., neglect the damping, and use Eqs. (2) for the unperturbed magnetic field, we get the following set of equations.

$$\frac{d\mathbf{H}_{1t}}{dz}+\frac{i}{\mathcal{H}}(\mathbf{k}\mathbf{B}_0)\mathbf{H}_{1t}-B_{1z}\frac{4\pi Ne}{\mathcal{H}^2}\mathbf{E}_{0t}=-\frac{4\pi Ne}{\mathcal{H}}\mathbf{E}_{1t},$$

$$\frac{dB_{1z}}{dz}+i(\mathbf{kB}_1)=0, \quad \mathrm{rot}\,\mathbf{E}_1=\frac{i\omega}{c}\mathbf{B}_1,$$
$$H_{1z}=-\frac{\mathbf{H}_{1t}\mathbf{H}_{0t}}{H_{\mathrm{cr}}}, \quad E_{1z}=-\frac{\mathbf{E}_0\mathbf{H}_1}{H_{\mathrm{cr}}}-\frac{\mathbf{E}_{1t}\mathbf{H}_{0t}}{H_{\mathrm{cr}}}, \tag{5}$$
$$\mathbf{B}_1=x_{n1}\mathbf{H}_0+x_{n0}\mathbf{H}_1,$$

where the quantities corresponding to the unperturbed solution are indicated by the index zero and the small extra terms by the index 1, and where we have assumed that the latter have the form $F(z)\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)$ where $\mathbf{k}$ is a two-dimensional vector lying in the plane of the plate. The components of vectors tangential to the plane of the plate are indicated by an index t.

For the following it is convenient to introduce combinations which are invariant under a rotation around the z axis:

$$P=\mathbf{kH}_1, \quad Q=[\mathbf{kH}_{1t}]_z\equiv k_xH_{1y}-k_yH_{1x},$$
$$R=\mathbf{kE}_{1t}-B_{1z}(\mathbf{kE}_{0t})\,/\,\mathcal{H}. \tag{6}$$

Using (5) we get then

$$\frac{dQ}{dz}+\frac{i}{\mathcal{H}}(\mathbf{kB}_0)Q+\frac{4\pi Ne}{c\mathcal{H}}\tilde{\omega}B_{1z}=0,$$
$$\frac{dP}{dz}+\frac{i}{\mathcal{H}}(\mathbf{kB}_0)P+\frac{4\pi Ne}{\mathcal{H}}R=0,$$
$$\frac{dB_{1z}}{dz}+\frac{i}{\mathcal{H}}(\mathbf{kB}_0)B_{1z}+ix_{n0}\left\{P\left[\frac{(\mathbf{kH}_0)^2}{k^2H_{\mathrm{cr}}^2}+1\right]+Q\frac{(\mathbf{kH}_0)[\mathbf{kH}_{0t}]_z}{k^2H_{\mathrm{cr}}^2}\right\}=0,$$
$$\frac{dR}{dz}+\frac{i}{\mathcal{H}}(\mathbf{kB}_0)R-i\frac{\tilde{\omega}}{c}x_{n0}\left\{\frac{[\mathbf{kH}_{0t}]_z(\mathbf{kH}_0)}{k^2H_{\mathrm{cr}}^2}P+Q\left[1+\frac{[\mathbf{kH}_{0t}]_z^2}{k^2H_{\mathrm{cr}}^2}\right]\right\}=0, \tag{7}$$

where $\tilde{\omega}=\omega-(c/\mathcal{H})\mathbf{k}\times\mathbf{E}_{0tz}$. We note that notwithstanding the fact that we have assumed the electrical field $\mathbf{E}_0$ to be weak it is impossible to neglect the second term in the expression for ω because of the presence of the factor k, which can be large compared to the reciprocal of the thickness of the plate. However, in the expressions in the braces in the last two Eqs. (7) we can substitute for $\mathbf{H}_0$ the magnetic field in the absence of a current. Writing

$$\tilde{Q}=Q\exp\left\{\frac{i}{\mathcal{H}}\int_0^z(\mathbf{kB}_0)\,dz'\right\} \tag{8}$$

and similarly for the other quantities we find

$$\frac{d\tilde{Q}}{dz}+\frac{4\pi Ne\tilde{\omega}}{c\mathcal{H}}\tilde{B}_{1z}=0, \quad \frac{d\tilde{P}}{dz}+\frac{4\pi Ne}{\mathcal{H}}\tilde{R}=0,$$
$$\frac{d\tilde{B}_{1z}}{dz}+i\frac{\mathcal{H}}{H_{\mathrm{cr}}}\tilde{P}=0, \quad \frac{d\tilde{R}}{dz}-i\frac{\tilde{\omega}}{c}\frac{\mathcal{H}}{H_{\mathrm{cr}}}\tilde{Q}=0. \tag{9}$$

The set of equations written down here has the following four linearly independent solutions:

$$\tilde{Q}=-A_ne^{i\Lambda\lambda_nz}, \quad \tilde{P}=-\frac{i\gamma}{\lambda_n^2}A_ne^{i\Lambda\lambda_nz},$$
$$\tilde{B}_{1z}=\frac{i\gamma\mathcal{H}}{\Lambda H_{\mathrm{cr}}}\lambda_nA_ne^{i\Lambda\lambda_nz}, \quad \tilde{R}=\frac{\gamma\Lambda\mathcal{H}}{4\pi Ne\lambda_n}A_ne^{i\Lambda\lambda_nz}, \tag{10}$$

where n = 1, 2, 3, 4;

$$\lambda_1=-\lambda_2=1, \quad \lambda_3=-\lambda_4=i;$$
$$\Lambda=(4\pi|Ne\tilde{\omega}|/cH_{\mathrm{cr}})^{1/2}; \quad \gamma=Ne\tilde{\omega}/|Ne\tilde{\omega}|.$$

The A_n are arbitrary constants. To determine them we must write down the boundary conditions at the surfaces of the plate, i.e., at $z=\pm a/2$. These conditions consist in requiring that the normal components of the induction B and the tangential components of the field H are continuous.

Outside the plate we can assume that the magnetic field satisfies the equations

$$\mathrm{div}\,\mathbf{H}^{(e)}=0, \quad \mathrm{curl}\,\mathbf{H}^{(e)}=0, \tag{11}$$

since the parameter ω/ck is vanishingly small in the range of frequencies of interest to us. We find thus for $z>a/2$ and $z<-a/2$

$$[\mathbf{kH}_{1t}^{(e)}]_z=0, \quad H_{1z}^{(e)}=\pm ik^{-1}(\mathbf{kH}_1^{(e)}),$$

where the upper sign refers to the region $z>a/2$, and the lower to the region $z<-a/2$. The continuity conditions can now be written down for $z=\pm a/2$:

$$Q=0, \quad kB_{1z}\mp iP=0. \tag{12}$$

Writing now the sum of the solutions (10) we get a set of equations for the constants A_n:

$$A_1e^{i\varphi}+A_2e^{-i\varphi}+A_3e^{-\varphi}+A_4e^{\varphi}=0,$$
$$A_1e^{-i\varphi}+A_2e^{i\varphi}+A_3e^{\varphi}+A_4e^{-\varphi}=0,$$
$$(1-\theta)A_1e^{i\varphi}-(1+\theta)A_2e^{-i\varphi}+(i+\theta)A_3e^{-\varphi}-(i-\theta)\times A_4e^{\varphi}=0,$$
$$(1+\theta)A_1e^{-i\varphi}-(1-\theta)A_2e^{i\varphi}+(i-\theta)A_3e^{\varphi}-(i+\theta)\times A_4e^{-\varphi}=0, \tag{13}$$

where $\varphi=\Lambda a/2$ and $\theta=-i\Lambda H_{\mathrm{cr}}/k\mathcal{H}$. From the condition that its determinant must vanish we find

$$\sin 2\varphi\,\mathrm{sh}\,2\varphi=(\mathcal{H}k/H_{\mathrm{cr}}\Lambda)(\mathrm{sh}\,2\varphi\cos 2\varphi-\mathrm{ch}\,2\varphi\sin 2\varphi)+(\mathcal{H}k/H_{\mathrm{cr}}\Lambda)^2(\mathrm{sh}^2\varphi\cos^2\varphi-\mathrm{ch}^2\varphi\sin^2\varphi). \tag{14}$$

If $\varphi(k)$ is the positive root of Eq. (14), then the corresponding oscillation frequency is equal to

$$\omega=\frac{cH_{\mathrm{cr}}}{\pi|Ne|a^2}\varphi^2+\frac{c}{\mathcal{H}}[\mathbf{kE}_0]_z. \tag{15}$$

Introducing the "velocity" $\mathbf{v}=\mathbf{j}/Ne$, where $\mathbf{j}$ is the electrical current density, we can rewrite the last formula in the form

$$\omega = \frac{cH_{cr}}{\pi |Ne| a^2} \varphi^2 + \mathbf{vk}, \quad (16)$$

which is formally the same as the formula for a Galilean transformation, since the first term in (16) is the oscillation frequency when there is no current.

If $ka \ll 1$, the solution of (14) is $\varphi = \pi m/2$ ($m = 0, 1, \ldots$). It is clear from (16) that then in the case considered by us, when $v \ll cH_{cr}/Nea$, the oscillation frequencies are practically unchanged when the current is turned on, and are equal to

$$\omega_m = \frac{\pi}{4} \frac{cH_{cr}}{|Ne| a^2} m^2 \quad (m = 1, 2, \ldots). \quad (17)$$

An exception is the case $m = 0$. The corresponding frequency occurs only when there is a current present.

We note that each of the branches (17) of the spectrum of the electromagnetic oscillations in the plate crosses the curves depicting the spectra of the sound and flexural oscillations. Near the intersections there is a strong interaction between the different types of oscillations. We arrive thus to the conclusion that there must exist coupled electromagnetic-sound and electromagnetic-flexural oscillations.

If $ka \gg 1$ we can write instead of (14) the simpler equation

$$\mathrm{th}^2 \varphi = \mathrm{tg}^2 \varphi. \quad (18)$$

In contrast to the long wavelengths, in this case the oscillation frequencies depend strongly on the current. Moreover, for several **k** each of the frequencies tends to zero. This means that oscillations with these **k** can be emitted by static inhomogeneities of the plate. The maximum wavelength of the emitted oscillations is, as follows from (16) of the order of magnitude of $\lambda \sim aJ/cH_{cr} \ll a$. On the other hand, the condition $\lambda \gg d$ must be satisfied, where d is the period of the structure of the intermediate state, since otherwise the macroscopic equations used by us are inapplicable. The magnitude of the current must thus satisfy the inequality $J \gg cH_{cr}(d/a)$.

3. WAVE RESISTANCE

In this section we evaluate the resistance connected with the above-mentioned Cerenkov radiation of electromagnetic waves. We assume here that the inhomogeneity at which this emission takes place is created by the roughness of the sides of the plate and we shall initially assume that the inhomogeneity is sinusoidal, i.e., that the equations giving the boundaries of the plate are:

$$z = \pm a/2 + \zeta_\pm, \quad \zeta_\pm = \zeta_\pm^{(0)} e^{i\mathbf{kr}}, \quad (19)$$

where **k** is a two-dimensional vector in the xy plane, and $\zeta_\pm$ are the amplitudes of the displacements and satisfy the condition $\zeta \ll a$. Such a perturbation causes a periodic change in the structure of the intermediate state. It is clear from the results of the preceding section that this change will be especially large when the magnitude of the current J through the plate is connected with the wave vector of the perturbation **k** by the relation

$$J = J_m \equiv \frac{cH_{cr}}{\pi a |k_x|} \varphi_m^2, \quad (20)$$

where φ_m is the m-th positive root of Eq. (18), k_x the component of **k** along the direction of the current. In order that the condition $cH_{cr} \gg J_m \gg cH_{cr} d/a$ be satisfied we must assume that $1/d \gg k \gg 1/a$.

To find the field distribution in the plate we must solve the set of equations

$$\frac{d\tilde{Q}}{dz} + \frac{4\pi Ne}{\mathscr{H}} \frac{\tilde{\omega}}{c} \tilde{B}_{1z} = -\frac{4\pi H_{cr}}{c\mathscr{H}} s_{\alpha\beta} n_\alpha n_\beta \tilde{R} - \frac{4\pi H_{cr}}{c^2 \mathscr{H}} \tilde{\omega} [\mathbf{n}, \hat{s}\mathbf{n}]_z \tilde{B}_{1z},$$

$$\frac{d\tilde{P}}{dz} + \frac{4\pi Ne}{\mathscr{H}} \tilde{R} = \frac{4\pi H_{cr}}{c\mathscr{H}} [\mathbf{n}, \hat{s}\mathbf{n}]_z \tilde{R} - \frac{4\pi H_{cr}}{c^2 \mathscr{H}} \tilde{\omega} (s_{\alpha\beta} n_\alpha n_\beta - s_{\alpha\alpha}) \tilde{B}_{1z},$$

$$\frac{d\tilde{B}_{1z}}{dz} + i \frac{\mathscr{H}}{H_{cr}} \tilde{P} = 0, \quad \frac{d\tilde{R}}{dz} - i \frac{\tilde{\omega}\mathscr{H}}{cH_{cr}} \tilde{Q} = 0, \quad (21)$$

where

$$\mathbf{n} = \mathbf{k}/k, \quad [\mathbf{n}, \hat{s}\mathbf{n}]_z \equiv n_x s_{y\alpha} n_\alpha - n_y s_{x\alpha} n_\alpha,$$
$$\tilde{\omega} = -c\mathscr{H}^{-1} [\mathbf{kE}_{0t}]_z.$$

This set differs from Eqs. (9) by the presence of small terms connected with the dissipative part of the conductivity $s_{\alpha\beta}$. The necessity to take these into account is connected with the fact that otherwise the amplitude of the oscillations will become infinite when the conditions (20) of exact resonance are satisfied. Since the perturbation (19) is a static one, the frequency ω is put equal to zero in Eqs. (21). The general solution of the set (21) is the sum of the following four linearly independent solutions:

$$\tilde{Q} = -A_n e^{i\Lambda\lambda_n z} \left(1 - \frac{H_{cr}}{Nec} \frac{\Lambda z}{4\lambda_n} \gamma s_{\alpha\alpha}\right),$$

$$\tilde{P} = -i\gamma\lambda_n^2 A_n e^{i\Lambda\lambda_n z} \left\{1 - \frac{H_{cr}}{Nec} \left([\mathbf{n}, \hat{s}\mathbf{n}]_z + \frac{i\gamma}{\lambda_n^2} \left(s_{\alpha\beta} n_\alpha n_\beta - \frac{1}{2} s_{\alpha\alpha}\right) + \frac{\Lambda z}{4\lambda_n} \gamma s_{\alpha\alpha}\right)\right\},$$

$$\tilde{B}_{1z} = i \frac{\gamma\mathscr{H}}{\Lambda H_{\text{к}}} \lambda_n A_n e^{i\Lambda\lambda_n z} \left\{1 - \frac{H_{cr}}{Nec} \left([\mathbf{n}, \hat{s}\mathbf{n}]_z + \frac{i\gamma}{\lambda_n^2} \left(s_{\alpha\beta} n_\alpha n_\beta - \frac{1}{4} s_{\alpha\alpha}\right) + \frac{\Lambda z}{4\lambda_n} \gamma s_{\alpha\alpha}\right)\right\}, \quad (22)$$

where the notation is the same as in (10). We have not written out the expression for $\widetilde{R}$, as it is not needed in the following.

To determine the arbitrary constants A_n in the present case we have instead of the homogeneous conditions (12) at $z = \pm a/2$ (see the Appendix)

$$Q = 0, \quad B_{1z} = \mp \frac{x_s}{x_n} \mathcal{H} k \zeta_{\pm}. \tag{23}$$

Here $x_s = 1 - x_n$ is the concentration of the superconducting phase. If we now substitute the sum of the solutions (22) into (23) we get a set of four equations for the four unknown constants A_n. Using the fact that the $s_{\alpha\beta}$ are small we can easily solve it. We shall, however, not write down the rather cumbersome solution in the general case, since the immediate vicinity of the resonances (20) is of most interest.

If the current J is close to one of the J_m, then $\varphi \equiv \Lambda a/2$ is close to φ_m. If $\tan\varphi \approx \tanh\varphi$, we have then

$$A_{1,2} = \pm A_0^{(1)} \operatorname{ch}\varphi + \frac{4}{\Delta^{(1)}} \Big\{ \pm \frac{i\tau}{\cos\varphi} (\tau - \operatorname{tg}\varphi)(F_+ + F_-)$$
$$\pm \frac{2H_\kappa}{Nec} e^{-i\varphi} (\tau - i)(F_+ + F_-)$$
$$\times \Big[\frac{i\gamma}{4} \varphi s_{\alpha\alpha} \tau (1 + \tau^2) + [\mathbf{n}, \hat{s}\mathbf{n}]_z \tau^2\Big]$$
$$+ (F_+ - F_-) \frac{(\tau - \operatorname{tg}\varphi)}{\cos\varphi} + \frac{2H_{cr}}{Nec} e^{-i\varphi} (\tau - i)(F_+ - F_-)$$
$$\times \Big[\frac{\gamma}{4}\varphi s_{\alpha\alpha} + \gamma \Big(s_{\alpha\beta} n_\alpha n_\beta - \frac{1}{4} s_{\alpha\alpha}\Big)\tau\Big]\Big\},$$

$$A_{3,4} = \pm i A_0^{(1)} \cos\varphi + \frac{4}{\Delta^{(1)}} \Big\{ \pm \frac{\tau(\tau - \operatorname{tg}\varphi)}{\operatorname{ch}\varphi} (F_+ + F_-)$$
$$\pm \frac{2H_{cr}}{Nec} e^{-\varphi} (1 + \tau)$$
$$\times (F_+ + F_-) \Big[i\gamma\varphi \frac{s_{\alpha\alpha}}{4} \tau (\tau^2 - 1) + [\mathbf{n}, \hat{s}\mathbf{n}]_z \tau^2 \Big]$$
$$- \frac{\tau - \operatorname{tg}\varphi}{\operatorname{ch}\varphi} (F_+ - F_-)$$
$$+ \frac{2H_{cr}}{Nec} e^{-\varphi} (1 + \tau)(F_+ - F_-)$$
$$\times \Big[i\gamma\varphi \frac{s_{\alpha\alpha}}{4} + i\gamma \Big(s_{\alpha\beta} n_\alpha n_\beta - \frac{1}{4} s_{\alpha\alpha}\Big)\tau \Big]\Big\}, \tag{24}$$

where

$$A_0^{(1)} = \frac{8i\tau^2}{\Delta^{(1)} \operatorname{ch}\varphi \cos\varphi} (F_+ + F_-), \quad \tau \equiv \operatorname{th}\varphi,$$

$$F_+ = \frac{ix_s}{x_n} \gamma k \zeta_+ \Lambda H_{cr} \exp\Big\{\frac{i}{\mathcal{H}} \int_0^{a/2} (\mathbf{k}\mathbf{B}_0)\, dz\Big\},$$

$$F_- = -\frac{ix_s}{x_n} \gamma k \zeta_- \Lambda H_{cr} \exp\Big\{-\frac{i}{\mathcal{H}} \int_{-a/2}^{0} (\mathbf{k}\mathbf{B}_0)\, dz\Big\},$$

$$\Delta^{(1)} = 32 i\tau \Big\{ (\tau - \operatorname{tg}\varphi) - \frac{i\gamma}{2} \frac{H_{cr}}{Nec} [(\varphi - \tau) s_{\alpha\alpha} + 4\tau s_{\alpha\beta} n_\alpha n_\beta] \Big\}.$$

If, however, $\tan\varphi \approx -\tanh\varphi$, we have

$$A_{1,2} = A_0^{(2)} \operatorname{sh}\varphi + \frac{4}{\Delta^{(2)}} \Big\{ \pm \frac{2H_{cr}}{Nec} e^{-i\varphi} (\tau + i)(F_+ + F_-) \Big[i\gamma\tau^3 \varphi \frac{s_{\alpha\alpha}}{4}$$
$$+ i\gamma \Big(s_{\alpha\beta} n_\alpha n_\beta - \frac{1}{4} s_{\alpha\alpha}\Big)\tau^2 \Big] \mp \frac{i\tau^2(\tau + \operatorname{tg}\varphi)}{\sin\varphi} (F_+ + F_-)$$
$$+ \frac{2H_{cr}}{Nec} e^{-i\varphi} (\tau + i)(F_+ - F_-) \Big[\gamma\varphi \frac{s_{\alpha\alpha}}{4} (-1 - \tau^2) + i\tau [\mathbf{n}, \hat{s}\mathbf{n}]_z\Big]$$
$$+ \frac{\tau(\tau + \operatorname{tg}\varphi)}{\sin\varphi} (F_+ - F_-) \Big\},$$

$$A_{3,4} = A_0^{(2)} \sin\varphi + \frac{4}{\Delta^{(2)}} \Big\{ \pm \frac{2H_{cr}}{Nec} e^{-\varphi} (1 + \tau)(F_+ + F_-) \Big[i\gamma\varphi \frac{s_{\alpha\alpha}}{4} \tau^3$$
$$+ i\gamma \Big(s_{\alpha\beta} n_\alpha n_\beta - \frac{1}{4} s_{\alpha\alpha}\Big)\tau^2 \Big] \pm \frac{\tau^2(\tau + \operatorname{tg}\varphi)}{\operatorname{sh}\varphi} (F_+ + F_-)$$
$$+ \frac{2H_\kappa}{Nec} e^{-\varphi} (1 + \tau)(F_+ - F_-) \Big[i\gamma\varphi \frac{s_{\alpha\alpha}}{4} (1 - \tau^2) + [\mathbf{n}, \hat{s}\mathbf{n}]_z \tau \Big]$$
$$- \frac{\tau(\tau + \operatorname{tg}\varphi)}{\operatorname{sh}\varphi} (F_+ - F_-) \Big\}, \tag{25}$$

where

$$A_0^{(2)} = -\frac{8\tau^2}{\Delta^{(2)} \operatorname{sh}\varphi \sin\varphi} (F_+ - F_-),$$

$$\Delta^{(2)} = 32 i\tau \Big\{ (\tau + \operatorname{tg}\varphi) + \frac{i\gamma\tau}{2} \frac{H_{cr}}{Nec} [s_{\alpha\alpha}(\varphi\tau - 1) + 4 s_{\alpha\beta} n_\alpha n_\beta] \Big\}.$$

Equations (22), (24), and (25) determine the field inside the plate in the approximation which is linear in the field. The corresponding electrical field strength $\mathbf{E}_1$ oscillates along the length of the plate and therefore does not contribute to the resistance. To evaluate the resistance we must thus find the electrical field $\mathbf{E}_2$ which is quadratic in the wave amplitude.

This can be done if we include in Eq. (1) terms which are quadratic in the deviation from the unperturbed solution. Let the linear and quadratic terms in these equations be written on the left and right sides of the equation. Solving the equation by successive approximations, we drop in first approximation the terms on the right-hand side. All quantities in the solution found in this way ($\mathbf{E}_1$, $\mathbf{H}_1$, $\mathbf{B}_1$, ...) depend on x and y through the factor $e^{i\mathbf{k}\cdot\mathbf{r}}$. Proceeding to the second approximation we must write the deviation from the unperturbed solution in the form $\mathbf{E}_1 + \mathbf{E}_2$, $\mathbf{H}_1 + \mathbf{H}_2$, ..., and on the right-hand side we must retain only terms with $\mathbf{E}_1$, $\mathbf{H}_1$, and so on. Since the terms with $\mathbf{E}_1$, $\mathbf{H}_1$, ... on the left-hand side cancel one another, we obtain for $\mathbf{E}_2$, $\mathbf{H}_2$, ... a set of inhomogeneous linear equations where on the right-hand side we have known functions which are quadratic in $\mathbf{E}_1$, $\mathbf{H}_1$, ... These functions are sums of terms of which some are proportional to $\exp(\pm 2i\mathbf{k}\cdot\mathbf{r})$ while others do not depend at all on x and y. The solution will thus also have this property. Since the oscillating terms in $\mathbf{E}_2$ do not contribute to the resistance it is sufficient for us to know the constant parts of the quantities $\mathbf{E}_2$, $\mathbf{H}_2$, ... To determine those we must retain on the right-hand side only those combinations of the quadratic quantities which are independent of x and y.

Taking all this into account we get from the third of Eqs. (1), where in the present case we can neglect the terms with $s_{\alpha\beta}$,

$$\mathcal{H}\frac{d\mathbf{H}_{2t}}{dz} + 4\pi Ne\mathbf{E}_{2t} = -\frac{1}{2}\operatorname{Re}(\mathbf{B}_1^*\nabla)\mathbf{H}_{1t}. \quad (26)$$

When writing down (26) we took into account the first of Eqs. (1) according to which B_{2z} = const, and const = 0, since B_z must be equal to the external field. We note also that by virtue of the second of Eqs. (1) $\mathbf{E}_{2t}$ is a constant vector.

We now integrate Eq. (26) over z from $-a/2$ to $a/2$. Since the total current through the plate is given, we have

$$\int_{-a/2}^{a/2} (\partial\mathbf{H}_{2t}/\partial z)\,dz = 0$$

and hence

$$\mathbf{E}_{2t} = -\frac{1}{8\pi Nea}\operatorname{Re}\int_{-a/2}^{a/2}(\mathbf{B}_1^*\nabla)\mathbf{H}_{1t}\,dz$$
$$= -\frac{1}{8\pi Nea}\operatorname{Re}\int_{-a/2}^{a/2}\left\{\tilde{B}_{1z}^*\frac{d\tilde{\mathbf{H}}_{1t}}{dz} + i\frac{\mathcal{H}}{H_{cr}}(\tilde{\mathbf{H}}_{1t}^*\mathbf{k})\tilde{\mathbf{H}}_{1t}\right\}dz. \quad (27)$$

The integrand in (27) can conveniently be expressed in terms of the quantities $\tilde{P}$ and $\tilde{Q}$, since we have already found their explicit form. We have

$$E_{2x} = -\frac{1}{8\pi Nea}\int_{-a/2}^{a/2} dz\left\{\frac{k_x}{k^2}\operatorname{Re}\left(\tilde{B}_{1z}^*\frac{d\tilde{P}}{dz}\right) - \frac{k_y}{k^2}\operatorname{Re}\left(\tilde{B}_{1z}^*\frac{d\tilde{Q}}{dz}\right) - \frac{\mathcal{H}}{H_{cr}}\frac{k_y}{k^2}\operatorname{Im}(\tilde{P}\tilde{Q}^*)\right\},$$

$$E_{2y} = -\frac{1}{8\pi Nea}\int_{-a/2}^{a/2} dz\left\{\frac{k_x}{k^2}\operatorname{Re}\left(\tilde{B}_{1z}^*\frac{d\tilde{Q}}{dz}\right) + \frac{k_y}{k^2}\operatorname{Re}\left(\tilde{B}_{1z}^*\frac{d\tilde{P}}{dz}\right) + \frac{\mathcal{H}}{H_{cr}}\frac{k_x}{k^2}\operatorname{Im}(\tilde{P}\tilde{Q}^*)\right\}. \quad (28)$$

Substituting (22) into (28) and using (24) and (25) we get after a simple integration

$$\mathbf{E}_{2t} = -\mathbf{k}\frac{\gamma\tau^3\mathcal{H}}{\pi\Lambda(Nek)^2ca}\left|\frac{8(F_+ - F_-)}{\Delta^{(2)}}\right|^2 \times [s_{\alpha\alpha}(\varphi\tau - 1) + 4s_{\alpha\beta}n_\alpha n_\beta] \quad (29a)$$

if $\tan\varphi \approx -\tanh\varphi$ and

$$\mathbf{E}_{2t} = -\mathbf{k}\frac{\gamma\tau^4\mathcal{H}}{\pi\Lambda(Nek)^2ca}\left|\frac{8(F_+ + F_-)}{\Delta^{(1)}}\right|^2 \times [s_{\alpha\alpha}(\varphi - \tau) + 4\tau s_{\alpha\beta}n_\alpha n_\beta] \quad (29b)$$

if $\tan\varphi \approx \tanh\varphi$.

Since expressions (29) are appreciably different from zero only near resonances, we can for the wave part of the resistance which is defined as the ratio of the electrical field $\mathbf{E}_2$ to the magnitude of the current J write after some simple transformations

$$\rho_{\parallel}^{(w)} = \frac{x_s^2}{x_n}\frac{H_{cr}^2}{8\pi a^2}\left|\frac{k_x}{Ne}\right|\left\{|\zeta_+ - \zeta_-|^2{\sum_m}'\frac{\Gamma_m^{(1)}}{(J-J_m)^2 + (\Gamma_m^{(1)}/2)^2} + |\zeta_+ + \zeta_-|^2{\sum_m}''\frac{\Gamma_m^{(2)}}{(J-J_m)^2 + (\Gamma_m^{(2)}/2)^2}\right\},$$
$$\rho_{\perp}^{(w)} = \frac{k_y}{k_x}\rho_{\parallel}^{(w)}, \quad (30)$$

where

$$\Gamma_m^{(1)} = \frac{H_{cr}}{|Ne|c}\frac{J_m}{\operatorname{th}^2\varphi_m}\left[s_{\alpha\alpha}\left(1 - \frac{\operatorname{th}\varphi_m}{\varphi_m}\right) + \frac{4\operatorname{th}\varphi_m}{\varphi_m}s_{\alpha\beta}n_\alpha n_\beta\right],$$

$$\Gamma_m^{(2)} = \frac{H_{cr}}{|Ne|c}J_m\operatorname{th}\varphi_m\left[s_{\alpha\alpha}\left(\operatorname{th}\varphi_m - \frac{1}{\varphi_m}\right) + \frac{4}{\varphi_m}s_{\alpha\beta}n_\alpha n_\beta\right];$$

Σ' and Σ'' indicate sums over those values of m for which, respectively $\tan\varphi_m = \tanh\varphi_m$ and $\tan\varphi_m + \tanh\varphi_m = 0$.

We see that there are a large number of resonance maxima on the $\rho_{\parallel}^{(w)}(J)$-curve and that the ratio of their width to the distance between neighboring maxima is of the order of magnitude of the ratio of $s_{\alpha\beta}$ to Nec/H_{cr}. We note also that the sign of the Hall part of the wave resistance $\rho_{\perp}^{(w)}$ depends on the direction of the wave vector k.

Comparing (30) and (4) we see easily that the ratio of the wave resistance in the maxima to the normal resistance is of the order of magnitude of $(\zeta/a)^2(ka)^2(Nec/s_{\alpha\alpha}H_{cr})^2$. Because of the large magnitude of the last two factors this ratio may be large compared to unity even when $\zeta \ll a$.

All formulae written down in the foregoing refer to the case of a periodic perturbation of the form (19). When an arbitrary perturbation is present we can expand the displacements $\zeta_{\pm}(x, y)$ in a Fourier series:

$$\zeta_{\pm} = \frac{1}{\sqrt{S}}\sum_{\mathbf{k}}\zeta_{\pm}(\mathbf{k})e^{i\mathbf{kr}},$$

where S is the area of the plate and where the summation is over the values of the wave vector in the plane of the plate. Instead of (30) we then get

$$\rho_{\|}^{(w)} = \frac{x_s^2}{x_n}\frac{c^2 H_K^4}{4\pi^3 J^3 |Ne| a^4}\Big\{\sum_m{}' \varphi_m^4 \int_{-\infty}^{+\infty}\frac{dk_y}{2\pi}|\zeta_+^{(m)}+\zeta_-^{(m)}|^2 + \sum_m{}'' \varphi_m^4 \int_{-\infty}^{+\infty}\frac{dk_y}{2\pi}|\zeta_+^{(m)}-\zeta_-^{(m)}|^2\Big\}, \qquad (31)$$

where $\zeta_\pm^{(m)}$ are the values of the functions $\zeta_\pm(k_x, k_y)$ for $k_x = (cH_{cr}/\pi aJ)\varphi_m^2$. We have assumed here that the sums and the integrals in (31) converge. Otherwise they must be cut off for $k \sim 1/d$. It is clear from (31) that the form of the function $\rho(J)$ depends strongly on the distribution of the perturbations over different wavelengths.

We note finally that for a plate in an inclined field the resonance values of the current are determined by the relation

$$J_m = \frac{c(H_{cr}^2 - \mathscr{H}_t^2)}{\pi |k_x| a H_{cr}}\varphi_m^2, \qquad (32)$$

where $\mathscr{H}_t$ is the component of the external field which is tangential to the plate. If $\mathscr{H}_t = 0$, Eq. (32) goes over into (20), as should be the case.

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APPENDIX

To find the boundary conditions when the perturbation (19) is present we must start from the usual requirement that the normal component of the induction and the tangential components of the magnetic field are continuous at $z = \pm a/2 + \zeta_\pm$:

$$H_z^{(e)} - ik\zeta_\pm \mathbf{H}_{0t}^{(e)} = B_z - ik\zeta_\pm \mathbf{B}_{0t},$$
$$\mathbf{H}_t^{(e)} + ik\zeta_\pm H_{0z}^{(e)} = \mathbf{H}_t + ik\zeta_\pm H_{0z}. \qquad (A.1)$$

Bearing in mind that in the approximation linear in $\zeta_\pm$ we can write

$$\mathbf{H}_t\Big(\pm\frac{a}{2}+\zeta_\pm\Big) = \mathbf{H}_{0t}\Big(\pm\frac{a}{2}\Big) + \zeta_\pm\frac{d\mathbf{H}_{0t}}{dz} + \mathbf{H}_{1t}\Big(\pm\frac{a}{2}\Big)$$

and similarly for the other quantities we write condition (A1) in the form

$$H_{1z}^{(e)} - B_{1z} = ik\zeta_\pm(\mathbf{H}_{0t}^{(e)} - \mathbf{B}_{0t}),$$
$$\mathbf{H}_{1t}^{(e)} - \mathbf{H}_{1t} = \frac{d\mathbf{H}_{0t}}{dz}\zeta_\pm + ik\zeta_\pm(H_{0z} - H_{0z}^{(e)}). \qquad (A.2)$$

We have already noted that in the case considered by us when $J \ll cH_{cr}$ the change in the magnetic field under the influence of the current is small. We can therefore substitute in the right-hand side of Eqs. (A2) substitute instead of H_0 and B_0 their values when there is no current. We get then

$$B_{1z} = H_{1z}^{(e)}, \quad \mathbf{H}_{1t} = \mathbf{H}_{1t}^{(e)} + ik\zeta_\pm(H_{0z} - H_{0z}^{(e)}). \qquad (A.3)$$

The field outside the plate, and also at $\zeta_\pm = 0$ satisfies the relations

$$[\mathbf{k}\mathbf{H}_{1t}^{(e)}]_z = 0, \quad H_{1z}^{(e)} = \pm ik^{-1}(\mathbf{k}\mathbf{H}_1^{(e)}).$$

Substituting here (A3) and bearing in mind that $H_{0z} - H_{0z}^{(e)} = H_{cr} - x_n H_{cr} = (x_s/x_n)H_{cr}$ (x_s is the concentration of the superconducting phase) and also that in the case of interest to us ($ka \gg 1$) the inequality $P \ll kB_{iz}$ holds (as can easily be seen from (22)) we get the boundary conditions (23) given in the text.

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