THE S-MATRIX IN THIRRING'S FOUR-FERMION MODEL

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The correspondence between the traditional solution of the four-fermion Thirring model and the results of perturbation theory is investigated. The total set of Feynman diagrams for the scattering operator is summed. The result obtained in this manner coincides with the Glaser-Berezin solution. However this result depends in an essential manner on the vanishing of the mass of the particles and on consideration of mass-shell matrix elements only.

THE Thirring model [1] has been discussed in a large number of papers [2-5] which are in many respects contradictory. The standard method of construction of an S-matrix, proposed by Glaser [2] and developed further by Berezin [3], consists in finding an exact solution in the nonphysical space H_ψ in which an inequivalent representation of the canonical commutation relations is realized, but where the dynamical variables have a well defined meaning as operators. The next step involves a formal carry-over (with subtraction of singularities) of the functional dependence of the S-matrix on the field operators into Fock space. In a rigorous approach such an extrapolation is not at all justified. Indeed, the nonexistence in the physical space of a Heisenberg field having a local limit, and the absence of eigenstates of the total Hamiltonian (the latter is not an operator), make it impossible to define the scattering matrix in terms of asymptotic fields, or to make use of the Lippman-Schwinger procedure. This circumstance, in particular, has allowed several authors (e.g., Wightman [6]) to assert that the traditional solution of the model has hardly any relevance to the problem itself.

However, there exists another argument for or against the scattering operater obtained as a result of the mentioned extrapolation procedure, namely the correspondence with "prosaic" perturbation theory, and in our opinion this result is decisive. The present paper is devoted to an investigation of this question. In fact, we carry out below a summation of the totality of Feynman diagrams for the scattering operator, a problem which is usually completely hopeless in realistic theories. By this method a new (in a certain sense) solution is obtained, which is free of the usual objections, and can be compared with the traditional solution.

2. In the standard formulation (which from outward appearances is noncovariant) the Thirring model is reduced to the interaction of two fermion fields ψ_1 and ψ_2 in a two-dimensional space-time, with the interaction Lagrangian

$$\mathcal{L}_{int} = g : \psi_1^+(x)\psi_1(x)\psi_2^+(x)\psi_2(x):, \tag{1}$$

where

$$\psi_i(x) = \frac{1}{(2\pi)^{1/2}}\int d^2k\, e^{ikx}\psi_i(k),$$
$$\psi_1(k) = \delta(k^0 - \mathbf{k})[\theta(-\mathbf{k})a(-\mathbf{k}) + \theta(\mathbf{k})b^+(\mathbf{k})],$$
$$\psi_2(k) = \delta(k^0 + \mathbf{k})[\theta(\mathbf{k})a(-\mathbf{k}) + \theta(-\mathbf{k})b^+(\mathbf{k})], \tag{2}$$

here θ is the usual step function.

The Glaser-Berezin solution [2,3] defines the S-matrix by means of the expression

$$S = \exp i\int d^2x\mathcal{L}_{int}(x). \tag{3}$$

Reducing (3) to the normal form, one obtains the following result for the part of the S-matrix responsible for the scattering, in each even order, N = 2n, of g:

$$T_{2n}(g)\int d\mathbf{p}_1 d\mathbf{p}_2[\theta(\mathbf{p}_1)\theta(-\mathbf{p}_2)(:a^+(\mathbf{p}_1)a(\mathbf{p}_1)a^+(\mathbf{p}_2)a(\mathbf{p}_2): + :b^+(\mathbf{p}_1)b(\mathbf{p}_1)b^+(\mathbf{p}_2)b(\mathbf{p}_2):) + (\theta(\mathbf{p}_1)\theta(-\mathbf{p}_2) + \theta(-\mathbf{p}_1)\theta(\mathbf{p}_2))\cdot :a^+(\mathbf{p}_1)a(\mathbf{p}_1)b^+(\mathbf{p}_2)b(\mathbf{p}_2):], \tag{4}$$

For odd order N = 2n + 1, we obtain

$$T_{2n+1}(g)\int d\mathbf{p}_1 d\mathbf{p}_2[\theta(\mathbf{p}_1)\theta(-\mathbf{p}_2)(:a^+(\mathbf{p}_1)a(\mathbf{p}_1)a^+(\mathbf{p}_2)a(\mathbf{p}_2): + :b^+(\mathbf{p}_1)b(\mathbf{p}_1)b^+(\mathbf{p}_2)b(\mathbf{p}_2):) - (\theta(\mathbf{p}_1)\theta(-\mathbf{p}_2) + \theta(-\mathbf{p}_1)\theta(\mathbf{p}_2))\cdot :a^+(\mathbf{p}_1)a(\mathbf{p}_1)b^+(\mathbf{p}_2)b(\mathbf{p}_2):], \tag{5}$$

where the coefficients $T_i(g)$ depend on g only. It is our purpose to compare Eqs. (4) and (5) with the corresponding expressions obtained from the usual time-ordered representation of the scattering matrix

$$S = T\left(\exp i \int \mathcal{L}_{int}(x)\, d^2x\right). \qquad (6)$$

In the α-representation the contribution of an N-th order scattering diagram to the coefficient function is determined by the expression [7, 8]

$$\frac{(ig)^N}{N!}\frac{(-1)^{l+N-1}}{(16\pi)^{N-1}}\lim_{r\to 0}\int_0^\infty \Pi\left[P\left(\frac{\partial}{\partial r}\right)d\alpha\right] \times D^{-1}(\alpha)\exp i\left[\frac{A(\alpha,p)-2B(\alpha,r\,p)-K(\alpha,r)}{D(\alpha)}\right] \qquad (7)$$

(the product in (7) and all subsequent equations is over all internal lines). In Eq. (7) to each internal line we associate a Feynman parameter α and a 2-vector r. Each external line is described by its 2-momentum, and in agreement with Eq. (2) it is assumed that for lines associated with the field ψ_1 (type I) $p^0 = \mathbf{p}$, whereas for the field ψ_2 (type II) $p^0 = -\mathbf{p}$. The number l is the total number of closed loops in the diagram, each consisting of lines of the same type. The operator P is defined by:

$$P\left(\frac{\partial}{\partial r}\right) = \begin{cases} \dfrac{\partial}{\partial r^0} - \dfrac{\partial}{\partial r} & \text{for lines of type I (corresponding to contractions of the field } \psi_1) \\ \dfrac{\partial}{\partial r^0} + \dfrac{\partial}{\partial r} & \text{for lines of type II (corresponding to contractions of the field } \psi_2).. \end{cases} \qquad (8)$$

In (7) the form D is the sum over all trees in the diagram under consideration of products of the parameters α not belonging to these trees. The tree of a diagram is defined as the maximally weakly connected diagram containing the same vertices as the initial diagram.

The form A is defined as the sum over all 2-trees of the diagram of the products of parameters α not belonging to these 2-trees, and where each of these products is multiplied by the square of the algebraic sum of external momenta entering into one of two disjoint vertices of the 2-tree under consideration. A 2-tree of a diagram is obtained from the tree by removing one arbitrary line from the latter.

The form B is linear in all r_i. The coefficients of the r_i can be found by means of the following rule. One takes all trees of the diagram from which the i-th line can be removed, and for each of these one forms the product of parameters α not belonging to the tree under consideration. This product is multiplied by the algebraic sum of all external momenta entering into one of the two halves of the tree, separated by the removal of the i-th line, the sum being taken with positive sign if the i-th enters into the half of the tree under consideration, or with a minus sign—if the line leaves this part. Then the expressions so obtained are summed over all such trees.

The form K, which is quadratic in r, is defined as follows: The squares of the algebraic sums of r belonging to the lines of a given loop of the diagram are taken. Then these squares are multiplied by products of the parameters not belonging to the tree containing the loop under consideration, and the results so obtained are summed over all loops of the diagram. A tree with a given loop is a configuration which becomes the tree of the diagram if any line of the loop is removed[1)].

3. Making use of Eq. (7) and the definitions given above one can prove several propostions.

First of all, one can prove that all scattering diagrams with external lines of the same type give no contribution if the particles are on the mass shell (proposition 1).

Indeed, considering, for instance, a scattering diagram of order N with external lines of type I, it is clear that there will be two more internal lines of type II than of type I. We first carry out all the differentiations in (7) corresponding to lines of type II. After the first differentiation a factor of the form

$$\sum[u(\alpha)\overset{(1)}{L_1}(p) + v(\alpha)\overset{(2)}{L_1}(r)],$$

will appear in front of the exponential function in the integrand, where $u(\alpha)$ and $v(\alpha)$ are some functions of α, and $L_1^{(i)}(x)$ are linear functions of x, depending on it only in the combination $x^0 - \mathbf{x}$. Since in this case on the mass shell $p^0 = \mathbf{p}$, the factor reduces to $\Sigma v(\alpha)\, L_1^{(1)}(\mathbf{r})$. This factor will not be subject to differentiations under the action of the operators p corresponding to the remaining lines of type II, since $(\partial/\partial r^0 + \partial/\partial \mathbf{r})\, L_1^{(1)}(\mathbf{r}) = 0$. As a result of all such differentiations we obtain a factor in front of the exponential for which the effective power of r will be N. Since the number of operators p corresponding to lines of the type I is $(N-2)$, it is

[1)] The prescription for the construction of the forms B and K has been first formulated by B. M. Stepanov, who has kindly informed us about his results, for which we are sincerely indebted.

clear that in the limit $r = 0$ we obtain zero, which proves our proposition.

Thus, on the mass shell one needs to take into account only diagrams involving external lines of different types.

We introduce the following notation. The 2-momenta p_1 and p_2 denote the incident particles, the 2-momenta p_3 and p_4 belong to the outgoing particles, and $p_1^0 = p_1$; $p_3^0 = p_3$; $p_2^0 = -p_2$; $p_3^0 = -p_4$. It is convenient to introduce the combinations

$$\tilde{s} = (p_1 + p_2) = (p_3 + p_4), \quad \tilde{t} = (p_1 - p_3) = (p_4 - p_2), \quad \tilde{u} = (p_1 - p_4) = (p_3 - p_2).$$

such that $\tilde{s}^2 = s$, $\tilde{t}^2 = t$, and $\tilde{u}^2 = u$ are the usual Mandelstam variables. In diagrams which do not vanish on the mass shell and are of order N, we associate to the internal lines of type I the collection of parameters $(\alpha_1 r_1, \ldots, \alpha_{N-1} r_{N-2})$, and to the internal lines of type II—the collection $(\alpha_N r_N, \ldots, \alpha_{2N-2} r_{2N-2})$.

Introducing into (7) the new integration variables λ and ξ_i, defined by $\alpha_i = \lambda\xi_i$ with $0 \le \lambda < \infty$, $0 \le \xi_i \le 1$, and $\Sigma\xi_i = 1$, then the definition of the forms D, A, B, and K yields in N-th order

$$D(\alpha) = \lambda^{N-1} D(\xi), \quad A(\alpha, p) = \lambda^N A(\xi, p),$$
$$B(\alpha, r, p) = \lambda^{N-1} B(\xi, r, p), \quad K(\alpha, r) = \lambda^{N-2} K(\xi, r). \qquad (9)$$

Substituting Eqs. (9) in (7), we transform the latter to the form

$$\frac{(ig)^N}{N!} \frac{(-1)^{l+N-1}}{(16\pi)^{N-1}} \lim_{r\to 0} \int_0^1 (\Pi\, d\xi) \frac{\delta(1-\Sigma\xi)}{D(\xi)} P_2\left(\frac{\partial}{\partial r_1}\right)$$
$$\times \ldots P_2\left(\frac{\partial}{\partial r_{N-1}}\right) \cdot P_1\left(\frac{\partial}{\partial r_N}\right) \ldots P_1\left(\frac{\partial}{\partial r_{2N-2}}\right) \int_0^\infty d\lambda$$
$$\times \lambda^{N-2} \exp i\left[\lambda \frac{A(\xi, p)}{D(\xi)} - \frac{2B(\xi, r, p)}{D(\xi)} - \frac{1}{\lambda}\frac{K(\xi, r)}{D(\xi)}\right]. \qquad (10)$$

4. From now on we shall discuss only the theory on the mass shell, without separately mentioning it every time.

We first note that for each diagram involving the combination $\tilde{s}$ of the incoming momenta, there exists a mutually crossing complementary diagram involving the combination $\tilde{u}$, i.e., there is a diagram with an oppositely oriented open chain of lines of type II. This refers also to nonplanar diagrams, depending simultaneously on s and u. An exception is formed only by the t-diagrams, i.e., by those for which at least one open chain of lines has zero length (owing to the kinematical condition $t = 0$, such diagrams give a contribution to the scattering amplitude which does not depend on the external momenta).

Relative to mutually crossing complementary diagrams one can formulate the following proposition 2.

The sum of the contributions of two crossing-complementary diagrams to the coefficient function of the scattering operator is const. $\epsilon(s)$ in every even order and a constant in every odd order (here $\epsilon(s)$ is the sign function).

In order to prove this proposition we first consider an arbitrary scattering diagram of order N with an incoming momentum combination $\tilde{s}$. Since the theory admits nonplanar diagrams, for such diagrams the form $A(\xi, p)$ is a linear function of s and u. But in the theory under consideration $t = 0$, and hence $s = -u$. Thus $A(\xi, p) = a(\xi)s$, where $a(\xi)$ is a function of the parameters ξ. Moreover, the quadratic dependence of the form $K(\xi, r)$ on r in the limit $r = 0$ implies that a nonvanishing contribution to the integral (10) comes from those terms which appear as a result of the action of an identical number of operators p_1 and p_2 on $\exp[K(\xi, r)/i\lambda D(\xi)]$. If in one such term the exponential function is subjected to $N - k - 1$ operators $p_1 \times p_2$, there will appear a factor $(\lambda)^{-N+k+1}$. The remaining k operators $p_1 \times p_2$, acting on $\exp[-iB(\xi, p, r)/D(\xi)]$ yield a factor proportional to s^k, since in the diagrams of the type under consideration the coefficient of each r_j in the form B will necessarily contain p_1 and p_2. In addition $p_2(p_2) = p_2(\tilde{s})$, $p_1(p_1) = p_1(\tilde{s})$. Thus the contribution of each such diagram of order N to the coefficient function of the scattering operator is given by the expression

$$\frac{(ig)^N}{N!} \frac{(-1)^{l+N-1}}{(16\pi)^{N-1}} \int_0^1 (\Pi\, d\xi) \frac{\delta(1-\Sigma\xi)}{D(\xi)} \sum_{k=0}^{N-1} Y_k(\xi)$$
$$\times s^k \int_0^\infty d\lambda\, \lambda^{k-1} \exp\left[i\lambda \frac{a(\xi)}{D(\xi)} s\right] e^{-\lambda\varepsilon}. \qquad (11)$$

In Eq. (11) $Y_k(\xi)$ denote functions of the parameters ξ. A characteristic feature of this equation is the fact that the variable s occurs in it only in the combination λ_s. We note that this circumstance is also a consequence of dimensional considerations, and that a nonvanishing mass would liquidate this feature.

In order to derive from (11) an expression for the contribution of a crossing complementary diagram to the coefficient function of the scattering operator it is necessary first to perform in (13) a substitution $s \to u = -s$. Then the mutually crossing complementary diagram is obtained from the original one by changing in it the orientation of one of the chains consisting of lines of the same type and having a beginning and an end. Therefore it follows from Eq. (10) and the definition of the forms $B(\xi, r, p)$ and $K(\xi, r)$ that in addition to the substitution $s \to -s$ the expression (11) will be multiplied by $(-1)^{\sigma}$, where σ is the number of lines in the chain whose orientation has been reversed. It turns out that the parity of the number of internal lines in two chains of lines in the diagram having a beginning and end is the same, and is opposite to the parity of the order of the diagram. These facts are simple consequences of Furry's theorem and of the fact that the total number of internal lines of each type in a scattering diagram of order N equals $N-1$.

Thus the total contribution of two mutually crossing-complementary scattering diagrams of order N to the coefficient function of the scattering operator is given by

$$\frac{(ig)^N}{N!}\frac{(-1)^{l+N-1}}{(16\pi)^{N-1}}\int_0^1 (\Pi\, d\xi)\,\frac{\delta(1-\Sigma\xi)}{D(\xi)}\sum_{k=0}^{N-1} Y_k(\xi)$$
$$\times\int_0^\infty d\lambda\left\{s^k\lambda^{k-1}\exp\left[i\lambda\frac{a(\xi)}{D(\xi)}s-\varepsilon\lambda\right]\right.$$
$$\left.+(-1)^{N-1}(-s)^k\lambda^{k-1}\exp\left[-i\lambda\frac{a(\xi)}{D(\xi)}s-\varepsilon\lambda\right]\right\}. \quad (12)$$

It can be inferred from Eq. (12) that in even order of perturbation theory the sum of matrix elements of interest is expressible in terms of the function

$$f_k^{(+)}(s\,|\,\xi)=\int_0^\infty d\lambda e^{-\varepsilon\lambda}\lambda^{k-1}\{s^k\exp[i\lambda\varphi(\xi)s]$$
$$-(-s)^k\exp[-i\lambda\varphi(\xi)s]\}, \quad (13)$$

whereas in odd orders it can be expressed in terms of the functions

$$f_k^{(-)}(s\,|\,\xi)=\int_0^\infty d\lambda e^{-\varepsilon\lambda}\lambda^{k-1}\{s^k\exp[i\lambda\psi(\xi)s]$$
$$+(-s)^k\exp[-i\lambda\psi(\xi)s]\}. \quad (14)$$

In Eqs. (13) and (14) $\varphi(\xi)$ and $\psi(\xi)$ are functions of the parameter ξ.

It is easy to see that there are no divergences in (13), the variable s occurs only in the combination λ_s, and that the functions $f_k^{(+)}(s\,|\,\xi)$ are odd in s; thus the functions $f_k^{(+)}(s\,|\,\xi)$ depend only on the sign of s:

$$f_k^{(+)}(s\,|\,\xi)=f_k^{(+)}(\xi)\,\varepsilon(s), \quad (15)$$

where

$$f_k^{(+)}(\xi)=\int_0^\infty dx e^{-\varepsilon x}\{x^k\exp[i\varphi(\xi)x]-(-x)^k\exp[-i\varphi(\xi)x]\}.$$

At the same time it follows from (14) that among the functions $f_k^{(-)}(s\,|\,\xi)$ there is one which is defined by a divergent integral, namely the function with $k = 0$. On the other hand it is known that the model under consideration involves no charge renormalization[1-3]. The latter circumstance means that although in individually taken sums of crossing-complementary diagrams divergences may occur, all these divergences compensate each other within the same order[9,10].

This circumstance gives us the possibility to operate formally with divergent integrals. It follows then from Eq. (14) that all the functions $f_k^{(-)}(s\,|\,\xi)$ are even in s, and since s again enters only in the combination λ_s, it follows that

$$f_k^{(-)}(s\,|\,\xi)=f_k^{(-)}(\xi), \quad (16)$$

where

$$f_k^{(-)}(\xi)=\int_0^\infty dx e^{-\varepsilon x}\{x^k\exp[i\psi(\xi)x]+(-x)^k\exp[-i\psi(\xi)x]\}.$$

Substituting these expressions into Eq. (12) we arrive directly at the result stated in propostion 2.

We note that diagrams of the t-type which have no mutually crossing complementary partners are essential only in odd orders, since otherwise such a diagram would contain at least one closed loop consisting of an odd number of lines of the same type. By Furry's theorem all such diagrams cancel out.

If nevertheless it should turn out that in this model there are proper vertex singularities, these should be removed by means of the usual procedure, the R-operation. In this case the model would be determined not only by the interaction Lagrangian, and in order to determine it completely one has to specify the subtraction point. If one chooses the latter in a manner which is natural for the model (all external momenta equal to zero), the regularized expressions (15) would have all the properties of the unregularized ones, and thus proposition 2 remains in force.

Making use of the results of proposition 2 and of the fact that $s = (p_1 + p_2)^2 = -4p_1 \cdot p_2$, it is easy to show that the in even orders of perturbation theory $(N = 2n)$, the on-mass-shell term in the S-matrix responsible for scattering has the form (4), whereas in odd orders $N = 2n + 1$ the corresponding term is given by Eq. (5).

This means that as a result of summing all Feynman diagrams for the coefficient function of the S-matrix corresponding to scattering on the mass shell, we obtain an expression containing the sum of two operator structures, defined by the expressions (4) and (5), in complete agreement with the results of Glaser[2] and Berezin[3].

5. Thus the summation of the perturbation theory series confirms the traditional result[2,3] for the scattering operator and may serve as a justification of the legitimacy of the procedure used by Glaser and Berezin. However, it is clear from the above that the simplicity of the model is essentially due to the vanishing mass of the particles and to the fact that all matrix elements are considered on the mass shell only. Indeed, for $m \neq 0$, or if one goes off the mass shell (even for $m = 0$), the class of nonvanishing diagrams turns out to be extremely large, and the diagrams acquires a nontrivial dependence on the external momenta. Therefore the hope to obtain local expressions for a theory connected with an off-mass-shell S-matrix (even for $m = 0$) is just as illusory as the hope to be able to sum the perturbation theory series in a real four-dimensional theory.

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