PHYSICAL INTERPRETATION OF SOLUTIONS OF THE EINSTEIN EQUATIONS

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A general method based on the correspondence principle is proposed for the physical interpretation of solutions of the Einstein equations. The method enables one to obtain information concerning the solution with an accuracy up to relativistic effects. In particular, one can determine the distribution of the sources of the gravitational field described by the solutions of the Einstein equations for empty space. As an example, the Kerr solution is considered and it is shown that the corresponding gravitational field is generated by a disk-like simple layer of negative masses at the edge of which lies a line of positive masses of infinite linear density. A method for the study of the rotation of local inertial systems and an asymptotic method of investigating local properties of a strong gravitational field are proposed. The latter method is demonstrated on the example of the problem of the motion of particles inside a Schwarzschild sphere and of the passage of the particles through this sphere.

IN the literature there exists a number of exact solutions of the Einstein equations whose physical meaning is not clear or which are even incorrectly interpreted. The reason for this is the absence of a suitable general method for solving such problems. Below we propose such a method based on the correspondence principle[1] which reduces the solution of the initial relativistic problem to the solution of a corresponding problem in the Newtonian potential theory. As an example we consider Kerr's solution.[2]

I. BASIC IDEA OF THE METHOD

Let a line element be given[1)]

$$ds^2 = c^2\,dt^2 - \gamma_{rs}\,d\xi^r\,d\xi^s, \qquad (1.1)$$

the coefficients γ_{rs} of which satisfy the Einstein equations:

$$R_{\mu\nu} - {}^1\!/_2 R g_{\mu\nu} = -\varkappa T_{\mu\nu}, \qquad (1.2)$$

$g_{00} = c^2$, $g_{i0} = 0$, $g_{ik} = -\gamma_{ik}$, $\kappa = 8\pi G/c^2$. It is assumed that the coefficients γ_{ik} and their derivatives have for $c \to \infty$ well defined finite limiting values γ'_{ik}. Then as $c \to \infty$ one obtains from the solution (1.1) the three-dimensional metric form

$$ds'^2 = \gamma_{rs}'\,d\xi^r\,d\xi^s, \qquad (1.3)$$

satisfying the system of equations:[1]

$$R_{00}' = -4\pi G\rho', \quad R_{i0}' = 0, \quad P_{ik}' = 0. \qquad (1.4)$$

Here R'_{00} and R'_{i0} are constructed from γ'_{ik} in the same manner as R_{00} and R_{i0} in (1.2) were constructed from γ_{ik}; P'_{ik} is the Ricci tensor for (1.3); ρ' is the limiting value of the density of the substance generating the given gravitational field. If in addition the relation

$$R'_{0ijk} = 0 \qquad (1.5)$$

is satisfied (R'_{0ijk} are constructed from γ'_{ik} in the same manner in which the components of the curvature tensor R_{0ijk} are constructed from γ_{ik}), then (1.4) represents equations of the gravitational field in Newtonian theory, while (1.3) represents the Newtonian gravitational field corresponding to the relativistic field (1.1) and which is the Newtonian approximation to the latter. But if the relation (1.5) is not satisfied then (1.3) represents the so-called vortex gravitational field of non-Newtonian type corresponding to the given field (1.1) which is a nonrelativistic approximation to the field.[3] In future we shall consider only gravitational fields of the Newtonian type.

The equation $P'_{ik} = 0$ shows that (1.3) is a metric form for flat three-dimensional space. Therefore, there exists a system of rectangular Cartesian coordinates x^i related to the coordinates ξ^i by a certain transformation:

$$x^i = x^i(\xi^k, t). \qquad (1.6)$$

The derivatives $x^i{}_{,0} = \partial x^i/\partial t = v^i(x^k, t)$ are the

[1)]The Latin indices take on the values 1, 2, 3 and the Greek indices take on the values 0, 1, 2, 3.

components of the velocity of motion of the points $\xi^i = \text{const}$ with respect to the Cartesian coordinate system x^i. If the expression

$$\omega_{ik} = {}^1/_2(v^i{}_{,k} - v^k{}_{,i})$$

does not depend on the spatial coordinates x^i, then condition (1.5) is satisfied and we can choose the coordinate system x^i in such a manner that $\omega_{ik} = 0$, and, consequently, v^i can be represented in the form of a gradient

$$v^i = \varphi_{,i}. \tag{1.7}$$

In this coordinate system the Newtonian potential Φ of the gravitational field (1.3) is expressed by the formula[2)]

$$\Phi = \varphi_{,0} + {}^1/_2\varphi_{,s}\varphi_{,s}. \tag{1.8}$$

But if ω_{ik} depends on the coordinates x^i, then condition (1.5) is not satisfied and (1.3) represents a vortex gravitational field.

In order to obtain information on the physical content of a solution of the Einstein equations (1.1) we consider its Newtonian approximation characterized by the potential (1.8). Our information has the same accuracy as the approximation (1.3), i.e., accuracy up to relativistic effects. In particular, if the spatial density of matter is $\rho = 0$, then with the aid of the potential (1.8) one can determine the sources of the gravitational field (1.1).

2. TRANSITION TO THE COORDINATES ξ^i, t

Let a line element be given in the general form:

$$ds^2 = h_{\mu\nu}dy^\mu dy^\nu. \tag{2.1}$$

In order to reduce it to the form (1.1) we introduce a new time coordinate t by means of a transformation

$$t = f(y^\nu), \tag{2.2}$$

where the function f is determined by the differential equation

$$h^{\mu\nu}f_{,\mu}f_{,\nu} = c^{-2}. \tag{2.3}$$

The characteristic system belonging to Eq. (2.3) is the following one:[4]

$$dy^\mu/d\lambda = 2h^{\mu\nu}p_\nu, \quad dp_\mu/d\lambda = -h^{\varkappa\nu}{}_{,\mu}p_\varkappa p_\nu, \tag{2.4}$$

with the integral

$$h^{\mu\nu}p_\mu p_\nu = c^{-2}. \tag{2.5}$$

To the system (2.4) we add the equation $df/d\lambda = p_\nu dy^\nu/d\lambda$, whence in virtue of (2.4), (2.5) and (2.2) it follows that

$$\lambda = {}^1/_2 c^2 t. \tag{2.6}$$

We choose at $t = 0$ the initial set

$$y^\mu = Y^\mu(\xi^1, \xi^2, \xi^3). \tag{2.7}$$

The initial values $p_\mu = P_\mu(\xi^i)$ are determined by the condition

$$P_\mu Y^\mu{}_{,i} = 0 \tag{2.8}$$

and Eq. (2.5). The solution of the system (2.4) with the initial values Y^μ, P_μ is obtained in the form

$$y^\mu = y^\mu(t, \xi^i), \quad p_\mu = p_\mu(t, \xi^i). \tag{2.9}$$

Eliminating from the first four equations in (2.9) the parameters ξ^i, we obtain the relation between the variables t, y^μ, from which follows the transformation (2.2).

In virtue of (2.4), (2.5) and (2.6) the functions (2.9) satisfy the relation

$$\frac{\partial}{\partial t}(p_\nu y^\nu{}_{,i}) = 0.$$

Consequently, the expression $p_\nu y^\nu{}_{,i}$ does not depend on t, and since according to (2.8) at $t = 0$ this expression is equal to zero, we must have

$$p_\nu y^\nu{}_{,i} = 0. \tag{2.10}$$

If we now regard relations (2.9) as formulas for the transition to a new coordinate system t, ξ^i, then in virtue of (2.4), (2.6) and (2.10) the line element (2.1) is transformed to the form (1.1). Thus, we have obtained a general scheme for calculations corresponding to the transition to a coordinate system of the type (1.1). But this scheme is very often not realizable practically even for comparatively simple forms of the line element (2.1). Therefore, the necessity arises to seek for the calculation of the potential Φ other methods which are practically more effective than the method described in Sec. 1.

3. EVALUATION OF THE POTENTIAL

We assume that we have a solution of the Einstein equations in the general form (2.1), supposing only that the lines y^0 are timelike, and the lines y^1 are spacelike. As a first step we introduce the time coordinate t, with the aid of the transformation (2.2) and (2.3). The time coordinate t belongs to one of the coordinate systems of type (1.1), and we shall call it Galilean (in a generalized sense). The line element (2.1) is transformed into the form

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2)A comma preceding a subscript denotes ordinary differentiation; we shall denote a covariant derivative by a semicolon.

$$ds^2 = (c^2 - w^2)\,dt^2 + 2w_s dy^s\,dt - g_{rs} dy^r\,dy^s, \quad (3.1)$$

where

$$w_i = g_{is} w^s, \quad w^2 = w_s w^s, \quad (3.2)$$

while w^i and g_{ik} are determined by the relations

$$w^i = c^2 h^{i\nu} f_{,\nu}, \quad g_{is}(w^k w^s c^{-2} - h^{ks}) = \delta_i^k. \quad (3.3)$$

The same result (3.1) is obtained if we apply to the line element (1.1) the transformation (2.9) for μ = 1, 2, 3. For this transformation we have

$$w^i = y^i_{,0}, \quad \gamma_{ik} = y^r_{,i} y^s_{,k} g_{rs}, \quad (3.4)$$

whence we see that w^i is the velocity of the point ξ^i = const in the space of the points y^i = const evaluated as a derivative with respect to t.

Let $c \to \infty$. If the quantities w^i, g_{ik}, have finite limiting values w'^i, g'_{ik}, then in the limit we obtain a three dimensional space with the metric

$$ds'^2 = g'_{rs}\,dy^r\,dy^s, \quad (3.5)$$

in which there is given the field of the velocities w'^i, which characterizes the motion of the points ξ^i = const of the space (1.3). Since in accordance with (3.4) we have

$$\gamma'_{ik} = y'^r_{,i} y'^s_{,k} g'_{rs}, \quad (3.6)$$

then the space (3.5) is flat, as also is the space (1.3). Here the functions $y'^\mu(t, \xi^i)$ are obtained from the functions (2.9) in the limit $c \to \infty$.

The space (3.5) is generally non-rigid (i.e., the coefficients g'_{ik} depend on t). If this is the case, then our second step is the transition to a certain rigid space by means of a transformation of the form

$$y^i = y^i(t, x^k). \quad (3.7)$$

The metric of this space

$$ds'^2 = g^*_{rs}\,dx^r\,dx^s \quad (3.8)$$

does not depend on the time, t, while the new components v^i of the velocity w'^i will be the following:

$$v_i = y^s_{,i} w'_s, \quad v^i = g^{*is} v_s. \quad (3.9)$$

Evaluating the quantity

$$\omega_{ik} = {}^1\!/_2 (v_{i,k} - v_{k,i}), \quad (3.10)$$

we determine the type of the gravitational field. If $\omega_{ik} = \omega_{ik}(t)$, then the gravitational field is irrotational, and we can choose the coordinate system x^i in such a manner as to have $\omega_{ik} = 0$. We can then determine the function $\varphi(t, x^i)$ from the equation

$$v_i = \varphi_{,i} \quad (3.11)$$

and calculate the potential Φ by means of the formula

$$\Phi = \varphi_{,0} + {}^1\!/_2\, g^{*rs} \varphi_{,r} \varphi_{,s}. \quad (3.12)$$

If the coefficients of the line element (3.1) do not have finite limiting values for $c \to \infty$, then it is necessary to carry out appropriate preliminary transformations, after which they will have finite limiting values.

4. KERR'S SOLUTION

We write Kerr's solution in the form

$$ds^2 = c^2\,d\tau^2 - dr^2 - (r^2 + a^2\cos^2\theta)\,d\theta^2 - (r^2 + a^2)\sin^2\theta\,d\varphi^2$$
$$- 2a\sin^2\theta\,dr\,d\varphi - r r_0 (r^2 + a^2\cos^2\theta)^{-1}$$
$$\times\ (c\,d\tau + dr + a\sin^2\theta\,d\varphi)^2,$$
$$r_0 = 2Gmc^{-2}. \quad (4.1)$$

Equation (4.1) is obtained from the line element considered in [2] after a simple transformation. The Galilean time coordinate t is introduced by means of the transformation

$$t = \tau + f(r),$$
$$r r_0 (cf' - 1)^2 = (r^2 + a^2) c^2 f'^2. \quad (4.2)$$

The line element (4.1) takes on the form

$$ds^2 = c^2(1 - r r_0 \lambda^{-2})\,dt^2 - 2[c^2 f' + c r r_0 \lambda^{-2}(1 - cf')]\,dr\,dt$$
$$-2 c a r r_0 \lambda^{-2}\sin^2\theta\,d\varphi\,dt - (1 - c^2 f'^2)\,dr^2 - \lambda^2 d\theta^2$$
$$-(r^2 + a^2)\sin^2\theta\,d\varphi^2$$
$$- 2a\sin^2\theta\,dr\,d\varphi - r r_0 \lambda^{-2}[(1 - cf')\,dr + a\sin^2\theta\,d\varphi]^2,$$
$$\lambda^2 = r^2 + a^2\cos^2\theta. \quad (4.3)$$

We go over to the limit $c \to \infty$. Since in this case $cr_0 \to 0$ and, therefore, on the basis of (4.2) $cf' \to 0$, we obtain in the limit the three-dimensional Euclidean line element

$$ds'^2 = dr^2 + (r^2 + a^2\cos^2\theta)\,d\theta^2 + (r^2 + a^2)\sin^2\theta\,d\varphi^2$$
$$+ 2a\sin^2\theta\,dr\,d\varphi \quad (4.4)$$

and the velocity field

$$w_1' = -\lim_{c\to\infty} (c^2 f') = -\eta\,(2Gmr)^{1/2}(r^2 + a^2)^{-1/2},$$
$$w_2' = 0, \quad w_3' = 0,$$
$$w'^2 = 2Gmr(r^2 + a^2\cos^2\theta)^{-1}, \quad \eta = \pm 1. \quad (4.5)$$

From this it can be seen that $\omega_{ik} = 0$, so that the gravitational field under consideration is irrotational and according to (3.11) and (3.12), its Newtonian potential is the following one:

$$\Phi = {}^1\!/_2\, w'^2 = Gmr(r^2 + a^2\cos^2\theta)^{-1}. \quad (4.6)$$

The transition to a rectangular cartesian coordinate system is given by the formulas

$$x = (r\cos\varphi + a\sin\varphi)\sin\theta, \quad y = (r\sin\varphi - a\cos\varphi)\sin\theta, \quad z = r\cos\theta. \tag{4.7}$$

Here the potential (4.6) has the form

$$\Phi = (Gm/\sqrt{2})\{\rho^2 - a^2 + [(\rho^2 - a^2)^2 + 4a^2z^2]^{1/2}\}^{1/2}[(\rho^2 - a^2)^2 + 4a^2z^2]^{-1/2}, \quad \rho^2 = x^2 + y^2 + z^2, \tag{4.8}$$

or

$$\Phi = \mathrm{Re}(Gm/\sigma), \tag{4.9}$$

$$\sigma^2 = x^2 + y^2 + (z + ia)^2, \quad \sigma = r + ia\cos\theta. \tag{4.10}$$

The function σ^{-1} and its first derivatives with respect to x, y, z, are continuous over the whole space with the exception of the disk $z = 0$, $x^2 + y^2 \leq a^2$ (or $r = 0$, $\theta = \frac{1}{2}\pi$). On crossing this disk they undergo a discontinuity. Consequently, the sources of the gravitational field lie on the disk. In order to determine the distribution of the sources over the disk, we write

$$\sigma^2 = \rho^2 + (ia)^2 + 2ia\rho\cos\vartheta, \quad z = \rho\cos\vartheta,$$

and expand σ^{-1} in a Legendre series:

$$\sigma^{-1} = \rho^{-1}\sum_{n=0}^{\infty}(-ia/\rho)^n P_n(\cos\vartheta),$$

whence in virtue of (4.9), it follows that

$$\Phi = \frac{Gm}{\rho}\sum_{n=0}^{\infty}(-1)^n\left(\frac{a}{\rho}\right)^{2n} P_{2n}(\cos\vartheta), \quad \rho > a. \tag{4.11}$$

On the other hand we consider the expression

$$U = -\lim_{\varepsilon\to 0}\frac{Gma}{2\pi}\int_0^{2\pi}\left\{\int_0^{a-\varepsilon}(a^2-\rho'^2)^{-3/2}[\rho^2+\rho'^2-2\rho\rho' \times \sin\vartheta\cos(\psi'-\psi)]^{-1/2}\rho' d\rho' - (2a\varepsilon - \varepsilon^2)^{-1/2}[a^2 + \rho^2 - 2a\rho\sin\vartheta\cos(\psi' - \psi)]^{-1/2}\right\}d\psi'. \tag{4.12}$$

Integrating by parts and taking into account the equation

$$\frac{Gm}{\rho} = -\frac{Gm}{2\pi a^2}\int_0^{2\pi}\int_0^{a}\frac{\partial}{\partial\rho'}\{(a^2 - \rho'^2)^{1/2}[\rho^2 + \rho'^2 - 2\rho\rho'\sin\vartheta\cos(\psi' - \psi)]^{-1/2}\}\, d\rho'\, d\psi'.$$

we find that

$$U = -\frac{Gm\rho}{2\pi a}\frac{\partial}{\partial\rho}\int_0^{2\pi}d\psi'\int_0^{a}(a^2 - \rho'^2)^{-1/2} \times [\rho^2 + \rho'^2 - 2\rho\rho'\sin\vartheta\cos(\psi' - \psi)]^{-1/2}\rho'\, d\rho'.$$

We denote

$$\sin\vartheta\cos(\psi' - \psi) = \cos\gamma.$$

Utilizing the addition theorem for Legendre functions we have

$$\int_0^{2\pi} P_n(\cos\gamma)\, d\psi' = 2\pi P_n(\cos\vartheta)P_n(0). \tag{4.13}$$

If in the last expression for U we expand the function

$$(\rho^2 + \rho'^2 - 2\rho\rho'\cos\gamma)^{-1/2}$$

in a Legendre series and then integrate term to term utilizing formula (4.13), we see that U can also be expanded into the series (4.11). From this it follows that $\Phi = U$. But from (4.12) it can be seen that U is the potential of a disk-like simple layer of negative masses of density

$$-(ma/2\pi)(a^2 - \rho'^2)^{-3/2}, \tag{4.14}$$

where ρ' is the distance from the center of the disk, and at the edge of the disk there is situated a line of positive masses of infinite linear density. Such a material system gives rise to the gravitational field described by Kerr's solution. According to (4.12) the total mass of the system is equal to

$$-\lim_{\varepsilon\to 0}\frac{ma}{2\pi}\int_0^{2\pi}\left[\int_0^{a-\varepsilon}(a^2 - \rho'^2)^{-3/2}\rho' d\rho' - (2a\varepsilon - \varepsilon^2)^{-1/2}\right]d\psi' = m.$$

This can also be seen from the asymptotic expression for the potential Φ for $\rho \to \infty$,

$$\Phi \sim Gm/\rho. \tag{4.15}$$

5. ROTATION

From the expression for the Newtonian potential (4.6) or (4.8) it is not possible to conclude whether matter in a given material system is in a state of stationary motion or not. In particular, the question of stationary rotation about the z axis remains indefinite. The order of magnitude of relativistic effects brought about by a rotation of the sources of the field lies beyond the limits of accuracy of the method described above. As is well known, the problem of the rotation of the system is solved by considering the asymptotic behavior of the relativistic solution at infinity.[5] Below we propose another method which enables us to study the effects of the rotation of the sources also in a strong gravitational field.

The coordinate system, t, ξ^i, in which the line element has the form (1.1) is a freely falling one. We consider all possible coordinate systems of this type in the neighborhood of a given world point. Such a set of local (t, ξ)-systems contains all the local inertial systems at a given world point. Although not every one of the local (t, ξ)-systems is inertial in the sense of the special theory of relativity, still it does not rotate with respect to local inertial systems. Consequently, if we shall investigate the rotation of local (t, ξ)-systems with respect to a coordinate system t, y^i in which the line element has the form (3.1), then simultaneously we shall be studying the rotation of local inertial systems with respect to y-space.

The coefficients w_i of the line element (3.1) were the components of the velocity of the point in ξ-space with respect to the y-space. The field of the velocities w_i in y-space contains within itself information concerning the rotation of local (t, ξ)-systems with respect to y-space. The classical quantity $w_{i,k} - w_{k,i}$ is unsuitable for the description of this rotation, since it depends on the choice of the Galilean time coordinate t and, therefore, also on the choice of the local (t, ξ)-system at the given world point, while all the local (t, ξ)-systems given at the same world point must rotate in an identical manner. We shall arrive at the quantity useful for the description of the rotation mentioned above in the following manner.

We consider the skew-symmetric tensors $V_{[\mu,\nu]}$ and $V_{[\lambda}V_{\mu\nu]}$, where V_μ are the components of the four-velocity of a point in y-space written in the coordinate system t, y^i:

$$V_0 = c\mu, \quad V_i = w_i/c\mu, \quad \mu^2 = 1 - w^2/c^2. \quad (5.1)$$

The three-dimensional vector b_i and the skew-symmetric tensor Ω_{ik}, defined by the equations

$$b_i = (2c/\mu)V_{[i,0]} = \mu^{-1}[\partial/\partial t(w_i/\mu) + (w^s/\mu)w_{s;i}], \quad (5.2)$$

$$\Omega_{ik} = -(3/\mu)V_{[0}V_{i,k]} = {}^1/_2[\partial/\partial y^i(w_k/\mu) - \partial/\partial y^k(w_i/\mu)] + {}^1/_2\mu^{-1}c^{-2}(b_i w_k - b_k w_i); \quad (5.3)$$

do not depend on the choice of the Galilean time coordinate t.[6] Instead of the tensor Ω_{ik}, it is useful to introduce the vector Ω^i which is dual to it:

$$\Omega^1 = \Omega_{23}/\sqrt{g}, \quad \Omega^2 = \Omega_{31}/\sqrt{g}, \quad \Omega^3 = \Omega_{12}/\sqrt{g}, \quad (5.4)$$

where g is the determinant constructed from the coefficients g_{ik} of the spatial part of the line element (3.1). In the limit $c \to \infty$ the vectors b_i and Ω^i go over into the following quantities in Newtonian mechanics: b_i will turn out to be equal to the negative of the acceleration of a point in y-space with respect to the set of local inertial systems at a given world point, and the vector Ω^i will turn out to be equal to the negative of the angular velocity of an element of y-space with respect to the set of the local inertial systems. Therefore we are justified in considering the vectors b_i and Ω^i which are independent of the choice of the Galilean time coordinate to be the relativistic equivalents of the classical quantities mentioned above. In particular, the rotation of local (t, ξ)-systems with respect to y-space is characterized by the vector Ω^i which has the meaning of a generalized angular velocity.

In the case of Kerr's solution the line element (3.1) has the specific form (4.3). Here we have introduced the notation $y^1 = r$, $y^2 = \theta$, $y^3 = \varphi$. From (4.2) and (4.3) it follows that

$$w_1 = c[\eta A\mu^2(1+\eta A)^{-1} - 1], w_2 = 0, \ w_3 = ca\sin^2\theta(\mu^2 - 1),$$
$$\mu^2 = 1 - rr_0\lambda^{-2}, A^2 = (r^2 + a^2)/rr_0, \eta = \pm 1, \quad g = \lambda^4\sin^2\theta. \quad (5.5)$$

With the aid of these expressions we evaluate

$$b_1 = -{}^1/_2 c^2 r_0\mu^{-2}\lambda^{-4}(r^2 - a^2\cos^2\theta),$$
$$b_2 = c^2 r r_0 a^2\mu^{-2}\lambda^{-4}\sin\theta\cos\theta, \ b_3 = 0, \quad (5.6)$$

$$\Omega^1 = -cr_0 ra\mu^{-3}\lambda^{-6}\cos\theta(r^2 + a^2 - rr_0),$$
$$\Omega^2 = -{}^1/_2 cr_0 a\mu^{-3}\lambda^{-6}\sin\theta(r^2 - a^2\cos^2\theta),$$
$$\Omega^3 = cr_0 ra\mu^{-3}\lambda^{-6}\cos\theta. \quad (5.7)$$

By going over with the aid of the transformation (4.7) to the x, y, z system of coordinates we obtain the new components

$$\bar{\Omega}^1 = -{}^1/_2 cr_0 a\mu^{-3}\lambda^{-6}\cos\theta[(3r^2 - a^2\cos^2\theta)x - 2r_0 r^2\sin\theta\cos\varphi],$$
$$\bar{\Omega}^2 = -{}^1/_2 cr_0 a\mu^{-3}\lambda^{-6}\cos\theta[(3r^2 - a^2\cos^2\theta)y - 2r_0 r^2\sin\theta\sin\varphi],$$
$$\bar{\Omega}^3 = {}^1/_2 cr_0 ra\mu^{-3}\lambda^{-6}[(3r^2 - a^2\cos^2\theta)\sin^2\theta - 2\lambda^2 + 2r_0 r\cos^2\theta]. \quad (5.8)$$

For large values of r we have

$$\bar{\Omega}^1 \cong -{}^3/_2 cr_0 ar^{-3}\sin\theta\cos\theta\cos\varphi,$$
$$\bar{\Omega}^2 \cong -{}^3/_2 cr_0 ar^{-3}\sin\theta\cos\theta\sin\varphi,$$
$$\bar{\Omega}^3 \cong {}^1/_2 cr_0 ar^{-3}(1 - 3\cos^2\theta). \quad (5.9)$$

Here r, θ, φ can be regarded as the usual polar coordinates, while x, y, z can be regarded as rectangular cartesian coordinates.

As is well known, in a stationary gravitational field far from the sources of the field the angular velocity of rotation of an element of y-space with respect to local inertial systems is given by the formula[7]

$$\mathbf{\Omega} = Gc^{-2}r^{-3}[\mathbf{M} - 3\mathbf{n}(\mathbf{Mn})]. \quad (5.10)$$

Here **M** is the total angular momentum of the gravitating system, **n** is a unit vector in the direction of the radius vector. Setting

$$\mathbf{M} = (0, 0, -acm),$$

the components (5.9) taken with opposite sign can be evaluated also by means of formula (5.10). This shows that for large values of r the vector (5.7) has the meaning of the ordinary angular velocity. At the same time we see how from the asymptotic expression of the vector (5.7) one can determine the total angular momentum.

Taking into account the fact that in a region far from the central disk the effect of the positive masses predominates, one can conclude from the direction of the total angular momentum that for an observer on the positive z axis the disk rotates clockwise. If this is so, then it follows from (5.9) that on the z axis (the axis of symmetry) local inertial systems rotate in the same direction as the positive masses of the central disk, while on the plane $\theta = {}^1\!/_2\pi$ (the equatorial plane) they rotate in the opposite direction. Having this fact in mind, we shall try to determine on the basis of the rotation of local inertial systems near the central disk the general character of motion of matter in the disk.

On the $\theta = {}^1\!/_2\pi$ plane we have in accordance with (5.8)

$$\overline{\Omega}^1 = \overline{\Omega}^2 = 0, \quad \overline{\Omega}^3 = {}^1\!/_2 c r_0 a r^{-3/2} (r - r_0)^{-3/2}.$$

With decreasing r the value of $\overline{\Omega}^3$ increases monotonically and becomes infinite at $r + r_0$. The line $r = r_0$ in the $\theta = {}^1\!/_2\pi$ plane lies on a singular surface determined by the equation $\mu = 0$. In the limiting case of $a = 0$, Kerr's solution goes over into the Schwarzschild solution and the singular surface $\mu = 0$ goes over into the singular Schwarzschild sphere. The rotation of local inertial systems outside the singular surface agrees with the rotation of the ring of positive masses at the edge of the disk.

On the z axis we have

$$\overline{\Omega}^1 = \overline{\Omega}^2 = 0, \quad \Omega^3 = -c r_0 a r (r^2 + a^2)^{-3/2} (r^2 + a^2 - r r_0)^{-1/2},$$

$$r = |z|.$$

We assume that $a > {}^1\!/_2 r_0$, then the singular surface $\mu = 0$ does not intersect the z axis. With decreasing r, the numerical value of $\overline{\Omega}^3$ at first increases up to a certain finite maximum, but then falls to zero value at the center of the disk. This is explained by the fact that the negative masses are rotating in the same direction as the positive masses and the effect of rotation of the former compensates for the effect of the rotation of the latter.

At the point $r = 0$, $\varphi = 0$, $0 < \theta < {}^1\!/_2\pi$ on the disk we have $\overline{\Omega}^1 = \overline{\Omega}^3 = 0$,

$$\overline{\Omega}^2 = -{}^1\!/_2 c r_0 a^{-2} \sin\theta \cos^{-3}\theta.$$

The existence of a non-vanishing $\overline{\Omega}^2$ provides evidence of the existence of radial motion of the masses on the disk.[8] It can be regarded as the limiting case of a perturbed rotational motion in the vertical equatorial plane. Since the negative masses must rotate in the same direction as the local inertial systems in the equatorial plane, i.e., in the direction defined by the vector $\overline{\Omega}^2$, we conclude that matter rises at the edge of the disk, moves radially towards the center and descends there. At the point $r = 0$, $\varphi = 0$, $\pi/2 < \theta < \pi$, situated on the other side of the disk $\overline{\Omega}^2$ is directed along the positive y axis. This means that here also the negative masses move from the edge towards the center of the disk.

6. ASYMPTOTIC METHOD FOR THE LOCAL STUDY OF A STRONG FIELD

In order to investigate the physical properties of the gravitational field near or inside a singular surface, we propose the following method. We place the material system generating the gravitational field at a great distance from the origin of coordinates. In order that the field in the neighborhood of the origin of coordinates should not turn out to be weak we shall at the same time increase in an appropriate manner the characteristic parameters of the system. In this case we shall derive the asymptotic formulas applicable in the neighborhood of the origin. For the sake of simplicity we shall demonstrate the method on the example of the Schwarzschild solution.

We assume that the mass M situated at the point $x = \xi$ on the positive x axis depends on ξ in accordance with $M = m\xi$, where m is a constant. For the motion of the particles along the x axis we have $r = \xi - x$. The gravitational field is described by the Schwarzschild line element without the angular part:

$$ds^2 = (1 - r_0/r)\, c^2\, dt'^2 - 2 r_0^{1/2} r^{-1/2} c\, dt'\, dr - dr^2, \quad r_0 = 2GMc^{-2}$$

or

$$ds^2 = [1 - k^2(1 - x/\xi)^{-1}] c^2\, dt'^2 + 2k(1 - x/\xi)^{-1/2} c\, dt' dx - dx^2,$$

$$k^2 = 2Gmc^{-2}. \tag{6.1}$$

The Schwarzschild sphere intersects the x axis at the point

$$x_0 = (1 - k^2)\xi. \tag{6.2}$$

The interior of the Schwarzschild sphere is situated to the right of the point x_0. For $\xi \to \infty$ there exist three possibilities: 1) $k^2 < 1$, $x_0 \to \infty$, the Schwarzschild sphere recedes to infinity; 2) $k^2 > 1$, $x_0 \to -\infty$, the whole infinite x axis will lie inside the Schwarzschild sphere; 3) $k^2 = 1$, the Schwarzschild sphere will constantly pass through the point $x_0 = 0$.

The transformation

$$x' = {}^2/_3\xi[1-(1-x/\xi)^{3/2}] - kct' \qquad (6.3)$$

brings the line element (6.1) into the form

$$ds^2 = c^2\,dt'^2 - (1-x/\xi)^{-1}\,dx'^2. \qquad (6.4)$$

From this it can be seen that the lines $x' = \text{const}$ are geodesics. The point $x' = \text{const}$ moves towards the center like a freely falling material particle which has zero initial velocity at $x = -\infty$. The coordinate t' is the proper time for the particles mentioned above. After further transformations

$$c\,dt' = c\,dt'' - 2k(1-x/\xi)^{1/2}(1-k^2-x/\xi)^{-1}\,dx, \qquad (6.5)$$

$$dx'' = kc\,dt'' + (1-x/\xi)^{1/2}\,dx, \qquad (6.6)$$

which for simplicity we have written in differential form, we obtain

$$ds^2 = c^2\,dt''^2 - (1-x/\xi)^{-1}\,dx''^2. \qquad (6.7)$$

We see that the points $x'' = \text{const}$ are also freely falling, but they move away from the center and at $x = -\infty$ the velocity of these points is zero. The coordinate t'' is the proper time for these points. The coordinate systems t', x' and t'', x'', moving towards one another are thus constructed in accordance with the same physical principles. The transition from one coordinate system to the other one is derived from (6.3), (6.5), and (6.6) by eliminating x

$$\begin{aligned} c\,dt' &= (1+k^2-x/\xi)(1-k^2-x/\xi)^{-1}\,c\,dt'' \\ &\quad - 2k(1-k^2-x/\xi)^{-1}\,dx'', \\ dx' &= -2k(1-x/\xi)(1-k^2-x/\xi)^{-1}\,c\,dt'' \\ &\quad + (1+k^2-x/\xi)(1-k^2-x/\xi)^{-1}\,dx''. \end{aligned} \qquad (6.8)$$

We now let $\xi \to \infty$. If $k^2 > 1$, then for a sufficiently large value of ξ any fixed segment $-a \le x \le a$ is within the Schwarzschild sphere, and for the points of this segment $x/\xi \to 0$. Consequently, by means of an expansion in powers of the quantity x/ξ one can obtain asymptotic formulas applicable to the investigation of motion along the segment under discussion. According to (6.4) or (6.7) the metric within the limits of the segment is close to the Minkowski metric. The transformation (6.8) assumes the following asymptotic form:

$$\begin{aligned} c\,dt' &= -(k^2+1)(k^2-1)^{-1}[1+O(x/\xi)]\,c\,dt'' \\ &\quad + 2k(k^2-1)^{-1}[1+O(x/\xi)]\,dx'', \\ dx' &= 2k(k^2-1)^{-1}[1+O(x/\xi)]\,c\,dt'' \\ &\quad - (k^2+1)(k^2-1)^{-1}[1+O(x/\xi)]\,dx''. \end{aligned} \qquad (6.9)$$

It is close to a Lorentz transformation in which, however, the positive directions of time and the positive direction along the x' are altered. If along the x axis inside the Schwarzschild sphere motion in both directions is possible then it is impossible to determine which one of the two times t' and t'' has the "true" direction, since there is no obvious reason for a physical nonequivalence of the coordinate systems t', x' and t'', x''. The indefiniteness of the direction of time inside the Schwarzschild sphere disappears if motion here is possible only in one direction (with respect to the coordinate system x). Then one of the two coordinate systems considered above is physically unrealizable inside the sphere.

Taking $k^2 = 1$, we can study the passage of particles through the Schwarzschild sphere. The transformation (6.8) now has a singular character:

$$\begin{aligned} c\,dt' &= (x/\xi)^{-1}[-(2-x/\xi)\,c\,dt'' + 2dx''], \\ dx' &= (x/\xi)^{-1}[2(1-x/\xi)\,c\,dt'' - (2-x/\xi)\,dx'']. \end{aligned} \qquad (6.10)$$

For $x = 0$, i.e., at the instant of passing through the Schwarzschild sphere, each of two observers moving towards one another observes in the case of the other one an infinite dilation of time intervals and an infinite contraction of lengths, just as in the case of motion with relative velocity equal to the velocity of light, while in his own reference system he does not notice anything extraordinary. Here at $x = 0$, the metric is even strictly pseudo-Euclidean, as can be seen from (6.4) and (6.7). Such a contradiction is absent if passage in only one direction is possible through the Schwarzschild sphere.

The motion of particles near a Schwarzschild sphere and inside it has been considered in detail from a different point of view by Zel'dovich and Novikov.[9]

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