WAVE KINETICS IN AN ANISOTROPIC PLASMA

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General relations are derived, which make it possible to construct matrix elements of wave processes of arbitrary order in which free oscillations participate. The Lagrange function of a collisionless plasma is used. The obtained general formulas are further used to calculate the following concrete quantities: the rate of growth of the energy of the Alfven waves resulting from helicon decay, and the relaxation times in a "gas" of Alfven oscillations (3- and 4-wave processes are considered).

1. The investigation of relaxation and kinetic phenomena in a plasma containing superthermal waves and oscillations is the main problem of the theory of collective processes. This problem is solved in most complete form with the aid of the formalism of the kinetic equation for the distribution function of the waves in phase space. The formulation of kinetic equations that take into account, besides the quasilinear effect, also purely nonlinear effects denotes essentially a transition from the dynamic description to the statistical description of the excited oscillations. In a turbulent plasma, as a rule, a tremendous number of waves is excited simultaneously, and therefore such a description is more advantageous and convenient. On the other hand, any concrete transition is realized by averaging over the phases of the oscillations, under the assumption that they are fully uncorrelated (although we cannot point at present to any direct experiment confirming the correctness of this assumption). This is precisely the method used in [1–4] to construct kinetic equations for the waves.

However, the presence of a large number of excited oscillations and the fact that their phases are not correlated makes it possible to simplify the problem to some degree from the very outset, by using the analogy with elementary excitations in condensed media. Indeed, under the conditions noted above, the wave interaction reduces to collisions between waves, and the kinetic equation for the distribution function of the waves in phase space can be written in the same form as the equations for phonons or magnons in solids [5–7]. This is followed by the question of deriving the matrix elements contained in the terms describing the collisions. This problem is simplest to solve with the aid of the complete Lagrange function of a collisionless plasma.

In this paper, which should be regarded as a continuation of [7–9], we derive general relations between the Fourier components of the particle displacement and of the wave field. These relations are very important, since they allow us to construct, using the prescription developed in the paper, the matrix elements of wave processes of any order in which free oscillations participate. It seems to us that the possibility of such a unification of the matrix-element derivation is the advantage of the Lagrange-function method. The matrix elements of each concrete process can be obtained from the general formulas, by taking in them the corresponding limits. It is precisely in this manner that we derive here the matrix elements that determine the probabilities of 3- and 4-wave processes inside a "gas" of Alfven oscillations, and the probability of the decay of a helicon into a helicon and an Alfven wave. By way of examples we obtain estimates of certain weakly-turbulent anisotropic-plasma characteristics that are governed by the foregoing processes.

2. We first separate in the complete Lagrange function of a collisionless plasma the perturbation-induced deviations of the physical quantities from their values in the stationary state; the function is then written in the form [10]

$$\mathscr{L} = \sum_\alpha \int\int d\xi d\mathbf{v} f_\alpha(\xi, \mathbf{v}) \Big\{ \frac{m_\alpha}{2}(\mathbf{v} + D^\alpha \mathbf{r}^\alpha)^2 - e_\alpha \varphi_0(\xi + \mathbf{r}^\alpha) - e_\alpha \varphi'(\xi + \mathbf{r}^\alpha) + \frac{e_\alpha}{c}(\mathbf{v} + D^\alpha \mathbf{r}^\alpha)[\mathbf{A}_0(\xi + \mathbf{r}^\alpha) + \mathbf{A}'(\xi + \mathbf{r}^\alpha)] \Big\} \Big| + \frac{1}{8\pi}\int dv [(\mathbf{E}_0 + \mathbf{E}')^2 - (\mathbf{H}_0 + \mathbf{H}')^2]. \quad (1)$$

Here $\mathbf{r}'$, φ', and $\mathbf{A}'$ are the displacement of the particle and the deviations of the scalar and vector potentials from their equilibrium values φ_0 and $\mathbf{A}_0$ in the initial state; $f_\alpha(\xi, \mathbf{v})$ is the distribution function of the particles of species α; D^α denotes the operator

$$D^\alpha = \frac{\partial}{\partial t} + \mathbf{v}\nabla + \left(\frac{e_\alpha}{m_\alpha}\mathbf{E}_0 + [\mathbf{v}\omega_{H\alpha}]\right)\nabla_v,$$
$$\omega_{H\alpha} = e_\alpha \mathbf{H}_0 / m_\alpha c. \qquad (2)^*$$

Expanding in powers of $\mathbf{r}^\alpha$, φ', and $\mathbf{A}'$, we represent the Lagrangian (1) in the form

$$\mathcal{L} = \sum_n \mathcal{L}_n;$$

$\mathcal{L}_0$ is a functional of stationary quantities and is of no interest; $\mathcal{L}_1$ vanishes identically; by varying the action function $\int \mathcal{L}_2 dt$ with respect to $\mathbf{r}^\alpha$, φ', and $\mathbf{A}'$ we obtain a system of linear equations which is equivalent to the system of Maxwell equations for self-consistent fields and the linearized Vlasov-Boltzmann equation. Consequently, $\mathcal{L}_2$ describes the natural oscillations of the plasma; then the Lagrangians of higher orders in the perturbation amplitude will describe oscillations between the natural oscillations.

3. We are interested in this paper in 3- and 4-wave processes in which transverse oscillations of an anisotropic plasma take place. For the sake of generality we shall not specify concretely the types of waves at the beginning. We choose for the potentials a gauge such that the scalar potential in the wave vanishes; in the case of a homogeneous stationary magnetic field, the Lagrangians of interest to us are of the form

$$\mathcal{L}_3 = \frac{1}{2c}\sum_\alpha e_\alpha \int\int d\xi d\mathbf{v} f_\alpha(\xi, \mathbf{v}) \, (\mathbf{v} r_i^\alpha r_j^\alpha \nabla_i \nabla_j \mathbf{A}'$$
$$+ 2D^\alpha \mathbf{r}^\alpha \cdot r_i^\alpha \nabla_i \mathbf{A}'), \qquad (3)$$

$$\mathcal{L}_4 = \frac{1}{6c}\sum_\alpha e_\alpha \int\int d\xi d\mathbf{v} f_\alpha(\xi, \mathbf{v}) \, (\mathbf{v} r_i^\alpha r_j^\alpha r_l^\alpha \nabla_i \nabla_j \nabla_l \mathbf{A}'$$
$$+ 3D^\alpha \mathbf{r}^\alpha \cdot r_i^\alpha r_j^\alpha \nabla_i \nabla_j \mathbf{A}'). \qquad (4)$$

We shall develop here the mathematical procedure for deriving the matrix element for the case of 3-wave processes only, illustrating by the same token the general scheme.

Following Landau and Rumer [11], we represent the particle displacement $\mathbf{r}^\alpha$ and the potential $\mathbf{A}'$ of the wave field in the form

$$\mathbf{r}^\alpha = \sum_{n=1}^{3} \mathbf{r}_n^\alpha, \quad \mathbf{A}' = \sum_{n=1}^{3} \mathbf{A}_n,$$

where the indices 1, 2, and 3 pertain to waves with wave vectors $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$ and frequencies ω_{k_1}, ω_{k_2}, and ω_{k_3}. We expand $\mathbf{r}_n^\alpha$ and $\mathbf{A}_n$ in Fourier series:

$$\mathbf{r}_n^\alpha = \sum_{k_n} \mathbf{r}_{k_n}^\alpha e^{i k_n \xi}, \quad \mathbf{A}_n = \sum_{k_n} \mathbf{A}_{k_n} e^{i k_n \xi}. \qquad (5)$$

Substituting these expressions in (3) and recognizing that the wave vector is conserved during the interaction, we get

$$\mathcal{L}_3 = \frac{i}{c}\sum e \sum_{k_1 k_2 k_3} \int dv f(v) \{ (\mathbf{k}_3 \mathbf{r}_{k_2})(\mathbf{B}_{k_1}\mathbf{A}_{k_3}) + (\mathbf{k}_2 \mathbf{r}_{k_3})(\mathbf{B}_{k_1}\mathbf{A}_{k_2})$$
$$+ (\mathbf{k}_3 \mathbf{r}_{k_1})(\mathbf{B}_{k_2}\mathbf{A}_{k_3}) + (\mathbf{k}_1 \mathbf{r}_{k_3})(\mathbf{B}_{k_2}\mathbf{A}_{k_1}) + (\mathbf{k}_2 \mathbf{r}_{k_1})(\mathbf{B}_{k_3}\mathbf{A}_{k_2})$$
$$+ (\mathbf{k}_1 \mathbf{r}_{k_2})(\mathbf{B}_{k_3}\mathbf{A}_{k_1}) + i[(\mathbf{r}_{k_1}\mathbf{k}_3)(\mathbf{r}_{k_2}\mathbf{k}_3)(\mathbf{v}\mathbf{A}_{k_3})$$
$$+ (\mathbf{r}_{k_1}\mathbf{k}_2)(\mathbf{r}_{k_3}\mathbf{k}_2)(\mathbf{v}\mathbf{A}_{k_2})$$
$$+ (\mathbf{r}_{k_2}\mathbf{k}_1)(\mathbf{r}_{k_3}\mathbf{k}_1)(\mathbf{v}\mathbf{A}_{k_1})]\}\Delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3), \qquad (6)$$

where

$$\mathbf{B}_k = (D\mathbf{r})_k, \quad \Delta(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}.$$

We have left out the burdensome index for the particle species.

We now express $\mathbf{r}_k$ and $\mathbf{B}_k$ in terms of $\mathbf{A}_k$. To this end we use the equation

$$D^2\mathbf{r} + [\omega_H D\mathbf{r}] = \frac{e}{m}\left(\mathbf{E}' + \frac{1}{c}[\mathbf{v}\mathbf{H}']\right). \qquad (7)$$

This equation is obtained by varying $\int \mathcal{L}_2 dt$ with respect to $\mathbf{r}$. Integrating (7) along the trajectory of undisturbed particle motion, we get[1)]

$$\mathbf{r} = \int_{-\infty}^{t} \mathbf{B}(t')dt', \qquad (8)$$

$$B_x = -\frac{e}{m}\int_{-\infty}^{t} [M_x \cos\omega_H(t'-t) - M_y \sin\omega_H(t'-t)]\,dt', \qquad (9)$$

$$B_y = -\frac{e}{m}\int_{-\infty}^{t} [M_y \cos\omega_H(t'-t) + M_x \sin\omega_H(t'-t)]\,dt', \qquad (10)$$

$$B_z = -\frac{e}{m}\int_{-\infty}^{t} M_z dt', \qquad (11)$$

where we have introduced the notation

$$\mathbf{M} = \frac{1}{c}\frac{\partial \mathbf{A}'}{\partial t} - \frac{1}{c}[\mathbf{v}[\nabla \mathbf{A}']]. \qquad (12)$$

From (9)—(11) it follows that

$$\mathbf{B}_k = \frac{ieA_k}{mc}\int_{-\infty}^{t} \chi \exp\left[i\mathbf{k}\int_{t}^{t'} \mathbf{v}(\tau)d\tau - i\omega_k(t'-t)\right]dt'. \qquad (13)$$

*[vw] ≡ v × w.

1)It is assumed that there is no stationary electric field, and the stationary magnetic field is directed along the z axis.

The equations of the unperturbed particle motion yield

$$v_z(\tau) = v_{z0}, \quad v_x(\tau) = v_{\perp 0}\cos(\beta_0 - \omega_H\tau),$$
$$v_y(\tau) = v_{\perp 0}\sin(\beta_0 - \omega_H\tau);$$

therefore

$$i\mathbf{k}\int_t^{t'} \mathbf{v}(\tau)\,d\tau = i\eta\,[\sin g(t) - \sin g(t')] + ik_z v_{z0}(t'-t),$$

$$g(t) = \beta_0 - \omega_H t - \beta_1, \quad \beta_1 = \operatorname{arctg}(k_y/k_x). \quad (14)$$

We expand the exponential of (13) in a Bessel-function series

$$\exp[-i\eta\sin g(t')] = \sum_{n=-\infty}^{+\infty} J_n(\eta)\exp[-ing(t')]. \quad (15)$$

Taking (14) and (15) into account and using the well known relations

$$J_{n-1}(\eta) = \frac{n}{\eta}J_n(\eta) + J_n'(\eta), \quad J_{n+1}(\eta) = \frac{n}{\eta}J_n(\eta) - J_n'(\eta),$$

we get after rather cumbersome calculations

$$\mathbf{B}_k = \frac{e}{mc\omega_H}\mathbf{P}A_k, \quad \mathbf{B}_{-k} = \frac{e}{mc\omega_H}\mathbf{P}^*A_k^+, \quad (16)$$

where

$$\mathbf{P} = i\sum_{n=-\infty}^{+\infty} e^{i\eta\sin g - ing}\delta_+^{(n)}\,[\mathbf{P}^{(1)}J_n(\eta) + \mathbf{P}^{(2)}J_n'(\eta)],$$

$$\delta_+^{(n)} = -i\int_0^\infty \exp[i(\omega_k - n\omega_H - k_z v_{z0})t]\,dt$$

$$= \mathrm{P}\frac{1}{\omega_k - n\omega_H - k_z v_{z0}} - i\pi\delta(\omega_k - n\omega_H - k_z v_{z0}). \quad (17)$$

Similar calculations yield

$$\mathbf{r}_k = \frac{e}{mc\omega_H}\mathbf{Q}A_k, \quad \mathbf{r}_{-k} = \frac{e}{mc\omega_H}\mathbf{Q}^*A_k^+, \quad (18)$$

where

$$\mathbf{Q} = i\sum_{n=-\infty}^{+\infty} e^{i\eta\sin g - ing}\delta_+^{(n)}\,[\mathbf{Q}^{(1)}J_n(\eta) + \mathbf{Q}^{(2)}J_n'(\eta)]. \quad (18')$$

Expressions for $\mathbf{P}^{(1)}$, $\mathbf{P}^{(2)}$, $\mathbf{Q}^{(1)}$, and $\mathbf{Q}^{(2)}$ are given in the Appendix.

With our gauge for the potentials, the energy of the transverse oscillations in a transparent medium is given by

$$(4\pi c^2)^{-1}\sum_k \omega_k^2\left[\frac{\partial(\omega\varepsilon'_{\alpha\beta})}{\partial\omega} + \varepsilon'_{\alpha\beta}\right]_{\omega=\omega_k} \varkappa_\alpha\varkappa_\beta A_k^+A_k,$$

$$\boldsymbol{\varkappa} = \frac{\mathbf{A}_k}{|A_k|},$$

where $\epsilon'_{\alpha\beta}$ is the hermitian part of the plasma dielectric tensor. We denote the number of transverse waves with wave vector k and frequency ω_k by N_k. We normalize the Fourier components of the vector potential in accord with the equation

$$(4\pi c^2)^{-1}\sum_k \omega_k^2\left[\frac{\partial(\omega\varepsilon'_{\alpha\beta})}{\partial\omega} + \varepsilon'_{\alpha\beta}\right]_{\omega=\omega_k} \varkappa_\alpha\varkappa_\beta A_k^+A_k = \sum N_k\hbar\omega_k \quad (19)$$

and introduce symbols which will prove useful later on

$$A_{k_\nu} = I_\nu a_{k_\nu}, \quad \nu = 1, 2, 3, \quad (20)$$

where

$$I_\nu = \left\{4\pi\hbar c^2/\omega_{k_\nu}\left[\frac{\partial(\omega\varepsilon'_{\alpha\beta})}{\partial\omega} + \varepsilon'_{\alpha\beta}\right]_{\omega=\omega_{k_\nu}} \varkappa_{\nu\alpha}\varkappa_{\nu\beta}\right\}^{1/2}; \quad (21)$$

the normalization condition then becomes

$$\sum_k \hbar\omega_k a_k^+ a_k = \sum_k N_k\hbar\omega_k.$$

It follows hence that the quantities a_k^+ and a_k can be treated respectively as operators for the creation and annihilation of a transverse wave with vector **k** and frequency ω_k. The nonzero matrix elements of these operators are

$$(N_k - 1|a_k|N_k) = \sqrt{N_k}\,e^{-i\omega_k t},$$
$$(N_k + 1|a_k^+|N_k) = \sqrt{N_k + 1}\,e^{i\omega_k t}.$$

With the aid of (16), (18), and (2) we rewrite the Lagrangian (6) in the form

$$\mathcal{L}_3 = \sum_{k_1k_2k_3} \Phi_{k_1k_2k_3}a_{k_1}a_{k_2}a_{k_3}, \quad (22)$$

where

$$\Phi_{k_1k_2k_3} = \frac{i}{c^3}\sum\frac{e^3}{m^2\omega_H^2}I_1I_2I_3\int dvf(v)\,\{(\mathbf{k}_3\mathbf{Q}_2)(\mathbf{P}_1\boldsymbol{\varkappa}_3)$$
$$+ (\mathbf{k}_2\mathbf{Q}_3)(\mathbf{P}_1\boldsymbol{\varkappa}_2) + (\mathbf{k}_3\mathbf{Q}_1)(\mathbf{P}_2\boldsymbol{\varkappa}_3) + (\mathbf{k}_1\mathbf{Q}_3)(\mathbf{P}_2\boldsymbol{\varkappa}_1)$$
$$+ (\mathbf{k}_2\mathbf{Q}_1)(\mathbf{P}_3\boldsymbol{\varkappa}_2) + (\mathbf{k}_1\mathbf{Q}_2)(\mathbf{P}_3\boldsymbol{\varkappa}_1) + i[(\mathbf{Q}_1\mathbf{k}_3)(\mathbf{Q}_2\mathbf{k}_3)(\mathbf{v}\boldsymbol{\varkappa}_3)$$
$$+ (\mathbf{Q}_1\mathbf{k}_2)(\mathbf{Q}_3\mathbf{k}_2)(\mathbf{v}\boldsymbol{\varkappa}_2) + (\mathbf{Q}_2\mathbf{k}_1)(\mathbf{Q}_3\mathbf{k}_1)(\mathbf{v}\boldsymbol{\varkappa}_1)]\}. \quad (23)$$

The summation in the Lagrangian (22) is over wave vectors satisfying the conservation law $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$.

We obtain analogously the general expression for the matrix element in the case of 4-wave processes. It is very complicated and therefore, without writing it out, we proceed directly to consider concrete examples.

4. In an anisotropic plasma it is possible for helicons and Alfven waves to propagate. If we take into account nonlinear effects, then the decay of a helicon into an Alfven wave and a helicon of lower frequency is possible. The corresponding conservation laws are

$$\omega_k^{\mathrm{h}} = \omega_{k-q}^{\mathrm{h}} + \omega_q^{A}. \quad (24)$$

Here ω_k^{h} is the frequency of the helicon with wave vector **k** and ω_q^{A} is the frequency of the Alfven wave with wave vector **q**.

We confine ourselves to the case of a one-dimensional spectrum of excited oscillations propagating along an external magnetic field. Then the dispersion laws are written in the form [12]

$$\omega_k{}^{\mathrm{r}} = \frac{\omega_{He}}{\Omega_{0e}{}^2} c^2 k^2 \quad (\omega_{Hi} < \omega_k{}^{\mathrm{r}} < \omega_{He}); \qquad \omega_q{}^A = c_A q, \tag{25}$$

where c_A is the Alfven velocity and c is the speed of light. Equation (24), as can be readily seen, has a positive solution with respect to the wave number q if $k > k_0$, where $k_0 = \frac{1}{2}(m/M)^{1/2}\Omega_{0e}/c$ (Ω_{0e} = plasma frequency). This condition is satisfied even by the minimum wave number of the helicon $k_{min} = (m/M)\Omega_{0e}/c$. Therefore the helicons from the very beginning of the spectrum can decay into Alfven waves and into helicons with lower frequencies. This process is the Cerenkov radiation of an Alfven wave by the helicon. Let us determine the rate at which this effect increases the energy density of the Alfven oscillations.

We put in the Lagrangian (22) $k_1 = k$, $k_2 = k - q$, and $k_3 = q$, and denote the number of Alfven waves by n_q. We shall use the following expressions for the components of the tensor $\epsilon_{\alpha\beta}$:

$$\varepsilon_{xx} = \varepsilon_{yy} = 1 - \sum \frac{\Omega_0{}^2}{\omega^2 - \omega_H{}^2},$$

$$\varepsilon_{xy} = -\varepsilon_{yx} = -i \sum \frac{\omega_H \Omega_0{}^2}{\omega(\omega^2 - \omega_H{}^2)},$$

$$\varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_{zx} = \varepsilon_{zy} = 0, \qquad \varepsilon_{zz} = 1 - \sum \frac{\Omega_0{}^2}{\omega^2};$$

we then get from (21)

$$I_{1,2} = c\left(\frac{\omega_{He}}{\Omega_{0e}}\right)\left(\frac{2\pi\hbar}{\omega^{\mathrm{h}}_{k_{1,2}}}\right)^{1/2}, \tag{26}$$

$$I_3 = c\left(\frac{\omega_{He}}{\Omega_{0e}}\right)\left(\frac{m}{M}\right)^{1/2}\left(\frac{2\pi\hbar}{\omega_q{}^A}\right)^{1/2}. \tag{27}$$

To find P and Q in the one-dimensional case, we must represent the Bessel function in the form

$$J_n(\eta) = \frac{\eta^n}{2^n n!}\left[1 - \frac{\eta^2}{2(2n+2)} + \ldots\right]$$

and take in (17) and (18′) the limit as $\eta \to 0$. Without presenting this simple mathematical procedure, we write out immediately the resultant approximate expression for the matrix element:

$$\Phi \approx \left(\frac{m}{M}\right)^{1/2} \hbar^{3/2} \left(\frac{\Omega_{0e}}{\omega_{He}}\right) \frac{e}{m} \Omega_{0e} R_D{}^3 (\omega_k{}^{\mathrm{r}} \omega^{\mathrm{r}}_{k-q} \omega_q{}^A)^{-1/2} k^2 (k-q) q. \tag{28}$$

We have left out here the ionic term, since its contribution is smaller than the electronic one by a factor $(m/M)^2$.

The kinetic equation for the distribution function of the Alfven oscillations is set up with the aid of the Lagrangian (22) in accordance with the usual scheme. Retaining in the collision integral of this equation the required terms, we have

$$\frac{\partial n_q}{\partial t} = \frac{2\pi}{\hbar^2} \sum_k |\Phi|^2 N_k N_{k-q} \delta(\omega_k{}^{\mathrm{r}} - \omega^{\mathrm{r}}_{k-q} - \omega_q{}^A). \tag{29}$$

With the aid of the normalization condition (19), we express the functions N_k and n_q in term the spectral field intensity:

$$N_k = \frac{1}{2\pi\hbar\omega_k{}^{\mathrm{r}}}\left(\frac{\Omega_{0e}}{\omega_{He}}\right)^2 |\mathbf{E}_k|^2, \tag{30}$$

$$n_q = \frac{1}{2\pi\hbar\omega_q{}^A}\left(\frac{\Omega_{0e}}{\omega_{He}}\right)^2\left(\frac{M}{m}\right)|\mathbf{E}_q|^2. \tag{31}$$

Substituting these relations in (29) and using the matrix element (28), we get

$$\frac{\partial W}{\partial t} = \frac{1}{8\pi^2}\left(\frac{m}{M}\right)\left(\frac{\Omega_{0e}}{\omega_{He}}\right)^{10}\left(\frac{v_{Te}}{c}\right)^6 \frac{\Omega_0}{nmc^2} \times \sum_k \int |\mathbf{E}_k|^2 |\mathbf{E}_{k-q}|^2 \frac{q^2}{(k-q)^2} \times \frac{\delta(q-q_1)\,dq}{|\partial(\omega_k{}^{\mathrm{r}} - \omega^{\mathrm{r}}_{k-q} - \omega_q{}^A)/\partial q|_{q=q_1}}, \tag{32}$$

where

$$W = \int n_q \hbar\omega_q dq, \qquad q_1 = 2(k - k_0).$$

Assume that we have a packet of helicons of width $k_2 - k_1 = \Delta k_1$, where $k_2 < k_{max}$, $k_1 > k_{min}$, and the energy density in this interval of the wave vectors is constant. Then, integrating (32) with respect to dq, we obtain for the rate of increase of the energy density of the Alfven oscillations the following final order-of-magnitude estimate:

$$\frac{\partial W}{\partial t} \approx 10^{-2}\left(\frac{m}{M}\right)\left(\frac{\Omega_{0e}}{\omega_{He}}\right)^{11}\left(\frac{v_{Te}}{c}\right)^8 \frac{\Omega_{0e}}{(R_D \Delta k)^2} \times \left[\ln\frac{k_2 - 2k_0}{k_1 - 2k_0} + \frac{k_0 \Delta k}{(k_2 - 2k_0)(k_1 - 2k_0)}\right] \frac{|\mathbf{E}|^2}{nmc^2} |\mathbf{E}|^2. \tag{33}$$

5. Let us proceed to estimate certain characteristic quantities describing the "gas" of interacting Alfven oscillations. We consider first processes in which three waves participate - the decay of one wave into two waves and the merging of two waves into one. The corresponding matrix element is obtained from (23) and, provided the magnetic-energy density does not exceed the particle thermal energy density, is of the form

$$\Phi' \approx (2\pi)^2 \hbar^{3/2} \left(\frac{\Omega_{0e}}{\omega_{He}}\right) \frac{e}{m} \left(\frac{m}{M}\right)^{5/2} \Omega_{0e} R_D{}^3 \times (\omega_q{}^A \omega^A_{q-q_1} \omega_{q_1}{}^A)^{-1/2} q^2 (q - q_1) q_1. \tag{34}$$

Let us set up the kinetic equation for n_q and retain in the collision integral of this equation only the terms quadratic in the number of waves; we get

$$\frac{\partial n_q}{\partial t} = \frac{2\pi}{\hbar^2} \sum_{q_1} |\Phi'|^2 [n_{q-q_1} n_{q_1} - n_q n_{q-q_1} - n_q n_{q_1}] \delta(\omega_q^A - \omega_{q-q_1}^A - \omega_{q_1}^A). \quad (35)$$

Substituting here the matrix element (34), we obtain the order of magnitude of the reciprocal relaxation time due to the 3-wave processes:

$$\tau_3^{-1} \approx 10^{-2} \left(\frac{m}{M}\right)^2 \left(\frac{\Omega_{0e}}{\omega_{He}}\right)^4 \left(\frac{v_{Te}}{c_A}\right)^4 (R_D \Delta q)^2 \omega_{q_0} \frac{|\mathbf{E}|^2}{nmc_A^2}. \quad (36)$$

We have again assumed that the energy density of the Alfven oscillations differs from zero and is constant in the wave-vector interval

$$[q_0 - \Delta q/2, \quad q_0 + \Delta q/2].$$

Let us consider now processes in which four Alfven waves take part—decay of one wave into three, merging of three waves into one, and scattering of one wave by another. All these processes are of the same order and we can confine ourselves to one of them in estimates of the characteristic quantities.

The matrix element takes the form

$$\Psi \approx \left(\frac{m}{M}\right)^2 \left(\frac{\Omega_{0e}}{\omega_{He}}\right)^2 \left(\frac{v_{Te}}{c_A}\right)^2 \frac{\hbar^2}{mn} q^2 (qR_D)^2. \quad (37)$$

We substitute this expression in the Lagrangian $\mathcal{L}_4$ and replace a_q by $(n_q)^{1/2}$. Taking (31) into account, we get the following order of magnitude for the energy density of the 4-wave interactions:

$$U_4 \approx 10^{-3} \left(\frac{\Omega_{0e}}{\omega_{He}}\right)^6 \left(\frac{v_{Te}}{c_A}\right)^2 (R_D \Delta q)^2 \frac{|\mathbf{E}|^2}{nmc_A^2} |\mathbf{E}|^2. \quad (38)$$

From the corresponding kinetic equation we can readily obtain for the reciprocal relaxation time due to the 4-wave processes

$$\tau_4^{-1} = \tau_1^{-1} + \tau_2^{-1},$$

where τ_1^{-1} is the contribution from $\mathcal{L}_4$:

$$\tau_1^{-1} \approx 10^{-2} \left(\frac{m}{M}\right)^2 \left(\frac{v_{Te}}{c_A}\right)^4 \left(\frac{\Omega_{0e}}{\omega_{He}}\right)^8 (R_D \Delta q)^4 \omega_{q_0} \left(\frac{|\mathbf{E}|^2}{nmc_A^2}\right)^2 \quad (39)$$

and τ_2^{-1} is the contribution from $\mathcal{L}_3$ in second-order perturbation theory:

$$\tau_2^{-1} \approx 10^2 \left(\frac{m}{M}\right)^2 \left(\frac{v_{Te}}{c_A}\right)^4 \tau_1^{-1}. \quad (40)$$

We can obtain analogously quantities of the type τ_3 and τ_4 for other branches of the oscillations, by starting from the general expressions for the matrix elements. It must be noted that these quantities determine such characteristics of a weakly turbulent plasma as, for example, the time of nonlinear decay of the wave spectrum, the energy transport in the gas of waves, etc. Consequently, they enable us to obtain important information on processes occurring in a weakly turbulent plasma.

I am sincerely grateful to A. A. Vedenov for valuable remarks and interest in the work.

APPENDIX

After cumbersome calculations, the procedure for which is described in detail in Sec. 2., we obtain the following expressions for the components of $\mathbf{P}^{(1)}$, $\mathbf{P}^{(2)}$, $\mathbf{Q}^{(1)}$, and $\mathbf{Q}^{(2)}$:

$$P_x^{(1)} = in\eta^{-1}(\mu_0 \cos g + \mu_1 \sin g)\omega_H + iv_z v_{\perp 0}\omega_H \sin(\beta_0 - \omega_H t),$$
$$P_y^{(1)} = in\eta^{-1}(\mu_0 \sin g - \mu_1 \cos g)\omega_H - iv_z\omega_H v_{\perp 0} \cos(\beta_0 - \omega_H t),$$
$$P_z^{(1)} = i\omega_H \omega_k \varkappa_z + iv_{\perp 0} n\eta^{-1}(\nu_y \cos\beta_1 - \nu_x \sin\beta_1)\omega_H,$$
$$P_x^{(2)} = \omega_H(\mu_1 \cos g - \mu_0 \sin g),$$
$$P_y^{(2)} = \omega_H(\mu_1 \sin g + \mu_0 \cos g),$$
$$P_z^{(2)} = \omega_H v_{\perp 0}(\nu_x \cos\beta_1 + \nu_y \sin\beta_1),$$
$$Q_x^{(1)} = iv_z v_{\perp 0} \cos(g + \beta_1) - i\mu_1 + in\eta^{-1}(\mu_1 \cos g - \mu_0 \sin g - v_z v_{\perp 0} \cos\beta_1),$$
$$Q_y^{(1)} = iv_z v_{\perp 0} \sin(g + \beta_1) + i\mu_1 - in\eta^{-1}(\mu_1 \cos g + \mu_0 \sin g + iv_z v_{\perp 0} \sin\beta_1),$$
$$Q_z^{(1)} = [-\omega_k \varkappa_z - v_{\perp 0} n\eta^{-1}(\nu_y \cos\beta_1 - \nu_x \sin\beta_1)]\omega_H \delta_+^{(n)},$$
$$Q_x^{(2)} = -\mu_0 \cos g - \mu_1 \sin g - \nu_z v_{\perp 0} \sin\beta_1,$$
$$Q_y^{(2)} = -i(\mu_1 \sin g + \mu_0 \cos g - \nu_z v_{\perp 0} \cos\beta_1),$$
$$Q_z^{(2)} = iv_{\perp 0}\omega_H(\nu_x \cos\beta_1 + \nu_y \sin\beta_1)\delta_+^{(n)},$$

where

$$\eta = k_\perp v_{\perp 0}/\omega_H, \quad \boldsymbol{\nu} = [\mathbf{k}\boldsymbol{\varkappa}],$$
$$\mu_0 = \omega_k \varkappa_x - \nu_y v_{z0}, \quad \mu_1 = -\omega_k \varkappa_y - \nu_x v_{z0}.$$

If we put intense formulas $\nu \to 0$ and replace $\omega_k \kappa / c$ by $-\mathbf{k}$, we obtain the matrix elements for the case of potential oscillations of an isotropic plasma.

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