CONTRIBUTION TO THE NONLINEAR THEORY OF INSTABILITY OF AN ELECTRON BEAM IN A SYSTEM WITH ELECTRODES

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A nonlinear theory of instability of a monoenergetic beam of electrons in a system with electrodes is considered in the paper for the case of small supercriticality. It is shown that when $u_0 > 0$ (u_0 is defined by relation (35)), then the energy of the oscillations excited in the beam jumps suddenly, on going through the boundary of the stability zone, to values comparable with the initial beam energy. At such oscillation amplitudes, the trajectories in the beam can intersect and regions of multistream motion can arise.

1. When an electron beam passes between two electrodes whose potential is maintained constant, an electrostatic instability can arise; this instability, unlike the well known "two-stream" instability [1], is not connected with the relative motions of the particles and therefore occurs even in a system which is not filled with plasma. As shown by Pierce [2], in order for this instability to arise in the absence of a plasma, it is necessary that the current density in the beam exceed a critical value defined by the relation

$$j_{\mathrm{cr}} = \pi m v_0^3 / 4el^2 \qquad (1)$$

(v_0—velocity of the particles on the beam, l—distance between the electrodes). Mikhaĭlovskiĭ [3] investigated similar instabilities when an electron beam moves in a two-electrode system filled with plasma. Experimental investigations of instabilities in a plasma-beam discharge have confirmed that at large current there can occur in the beam, besides the "two-stream" instability whose development leads to the appearance of a "plateau" on the beam-electron velocity distribution function [4,5], also an instability which is similar to that investigated by Pierce. This instability appears even in states with a "plateau" on the electron distribution function, and its development can lead to the blocking of the electron beam and to the occurrence of opposing particle streams [6].

A linear theory of the instability of an electron beam in a two-electrode system[1)] was considered in [2,3]. To determine the variation of the state of the beam due to excitation of the oscillations, it is necessary to consider the dynamics of the instability at large amplitudes, when the linear approximation is not valid.

We consider in this paper a nonlinear theory of the Pierce instability at small supercriticality. The analysis method is the same as in the paper by one of the authors [8], in which the instability of a beam with periodically varying parameters was investigated in a nonlinear approximation.

A characteristic feature of the beam instability in a system with electrodes is that there always exist such initial perturbations of the density and of the velocity in the beam, for which the nonlinear effects cannot stabilize the instability at low amplitudes. In this case, even in the case of small supercriticality, the energy of the oscillations excited as a result of the instability is comparable with the initial beam energy, and crossing of the trajectories and regions of multistream motion can occur in the beam (the hard excitation mode [9-10]).

2. Let a monoenergetic electron beam of specified density and velocity be formed in the plane $x = 0$ (cathode) and absorbed in the plane $x = l$ (anode). The potential of the two electrodes is maintained fixed, and the charge of the electron beam in the stationary state is compensated by the ions.

We consider one-dimensional longitudinal oscillations occurring in the electron beam. Following Pierce [2], we assume that the ions are infinitely heavy and do not take part in the oscillations. Then the system of hydrodynamic equations describing the oscillations is written in the form

1)Stationary nonlinear solutions in an electric beam passing between two electrodes were considered in a recent paper [7].

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{e}{m}E,\ \frac{\partial n}{\partial t}+\frac{\partial}{\partial x}(nv)=0,$$
$$\frac{\partial E}{\partial x}=-4\pi e(n-n_0). \quad (2)$$

Equations (2) were solved simultaneously with the boundary conditions formulated by Pierce, which have the obvious meaning:

$$n(t,0)=n_0,\quad v(t,0)=v_0,\quad \varphi(t,0)=\varphi(t,l)=0. \quad (3)$$

Here n_0 and v_0 are the density and velocity of the beam in the stationary state, and $\varphi=-\int_0^x E dx'$ is the potential of the wave.

According to Kalman[11], (2) it is convenient to go over in the investigation of the nonlinear system from the customary variables t and x to the variables t and ψ, where $\psi(t, x)$ is defined by the equations

$$\frac{\partial\psi}{\partial x}=\frac{n}{n_0},\quad \frac{\partial\psi}{\partial t}=-\frac{nv}{n_0}. \quad (4)$$

$\psi(t, x)$ is the stream function, and is conserved along the particle trajectory

$$\frac{d\psi}{dt}=\frac{\partial\psi}{\partial t}+v\frac{\partial\psi}{\partial x}=0.$$

When (4) is satisfied, the continuity equation is satisfied automatically, and the Poisson equation can be readily integrated and leads to the following relation for the electric field:

$$E=-4\pi e n_0(\psi+v_0t-x)+\mathscr{E}(t). \quad (5)$$

Here $\mathscr{E}(t)$ is the field at x = 0; in the derivation of (5) we have used the fact that, in accord with (3) and (4),

$$\psi|_{x=0}=-\int_0^t\frac{nv}{n_0}dt'=-v_0t.$$

Recognizing that $d\psi/dt=0$, we get

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=\left(\frac{\partial v}{\partial t}\right)_\psi+\frac{\partial v}{\partial\psi}\frac{d\psi}{dt}=\left(\frac{\partial v}{\partial t}\right)_\psi=\left(\frac{\partial^2 x}{\partial t^2}\right)_\psi.$$

Therefore, in terms of the variables t and ψ, the equation of motion is written as

$$\frac{\partial^2x}{\partial t^2}+\omega_0^2x=\omega_0^2(\psi+v_0t)-\frac{e}{m}\mathscr{E}(t)\left(\omega_0^2=\frac{4\pi e^2n_0}{m}\right). \quad (6)$$

From this we get for the displacement $x(t,\psi)$ of the electrons in the beam, in the presence of oscillations,

$$x(t,\psi)=\psi+v_0t+A(\psi)e^{i\omega_0t}+A^*(\psi)e^{-i\omega_0t}+C(t). \quad (7)$$

In free motion of the electron beam $x=\psi+v_0t$, the last three terms of (7) determine the displacement $\Delta X(t,\psi)$, due to the oscillations, of the beam electrons. C(t) is a particular solution of the equation

$$\frac{d^2C}{dt^2}+\omega_0^2C=-\frac{e}{m}\mathscr{E} \quad (8)$$

with conditions $C|_{t=0}=0$ and $dC/dt|_{t=0}=0$.

Having a formula for the displacement of the electrons $x(t,\psi)$, we can obtain the density and velocity of the beam particles in terms of the variables t and ψ:

$$n(t,\psi)=n_0\left(\frac{\partial x}{\partial\psi}\right)^{-1},\quad v(t,\psi)=\frac{\partial x}{\partial t}. \quad (9)$$

The electric field in the beam $E(t,\psi)$ is determined from (5). The connection between the variable ψ and the Euler coordinates x and t is determined from (4), in which we must substitute the values of $n(t,\psi)$ and $v(t,\psi)$ from (9). Thus, the solution of the problem of the oscillations in a beam reduces to the determination of $x(t,\psi)$.

To determine the functions $A(\psi)$, C(t), and $\mathscr{E}(t)$, which enter in the solution, we must use the boundary conditions (3). Inasmuch as for x = 0 we have $\psi=-v_0t$, $\partial x/\partial\psi=1$, and $\partial x/\partial t=v_0$, we obtain, substituting $x(t,\psi)$ from (7), the following equations:

$$C(t)=-A(-v_0t)e^{i\omega_0t}-\text{c.c.}, \quad (10)$$
$$\frac{dA}{dt}(-v_0t)e^{i\omega_0t}+\text{c.c.}=0. \quad (11)$$

Similarly we get from the relation

$$\varphi(t,l)=-\int_0^l E(t,x)dx=0,$$

using formulas (5) and (7) and going over to integration with respect to the variable $\xi=\psi+v_0t$,

$$\mathscr{E}(t)=-4\pi e n_0\Big\{C(t)+\frac{1}{l}\Big[\int_0^{\xi_l}A(\xi-v_0t)d\xi\, e^{i\omega_0t}+\text{c.c.}\Big]$$
$$+\frac{1}{2l}[(A(\xi_l-v_0t)e^{i\omega_0t}+\text{c.c.})^2-(A(-v_0t)e^{i\omega_0t}$$
$$+\text{c.c.})^2]\Big\}, \quad (12)$$

where $\xi_l(t)$ is determined, in accord with (7) from the equation

$$\xi_l-l=(A(-v_0t)-A(\xi_l-v_0t))e^{i\omega_0t}+\text{c.c.} \quad (13)$$

Finally, substituting in (8) the values of C(t) from (10) and $\mathscr{E}(t)$ from (12), we arrive at the following equation:

$$\frac{d^2}{dt^2}[A(-v_0t)e^{i\omega_0t}+\text{c.c.}]+\frac{\omega_0^2}{l}\Big\{\int_0^{\xi_l}A(\xi-v_0t)d\xi\, e^{i\omega_0t}$$
$$+\text{c.c.}+{}^1/_2[(A(\xi_l-v_0t)e^{i\omega_0t}+\text{c.c.})^2$$
$$-(A(-v_0t)e^{i\omega_0t}+\text{c.c.})^2]\Big\}=0. \quad (14)$$

Equations (11) and (14) make up together the sought system for the determination of the complex oscillation amplitude $A(\psi)$. To solve the system we represent $A(\psi)$ in the form

$$A(\psi)=[u(\psi)+iw(\psi)]\exp\left(i\frac{\omega_0}{v_0}\psi\right). \qquad (15)$$

Then (11) and (14) can be rewritten in the form

$$\frac{du}{d\bar{\psi}}-\frac{\omega_0}{v_0}w(\bar{\psi})=0, \qquad (16)$$

$$\frac{d^2u}{d\bar{\psi}^2}+\frac{\omega_0^2}{v_0^2 l}\left\{\int_0^{\xi_l}\left(u(\bar{\psi}+\xi)\cos\frac{\omega_0\xi}{v_0}-w(\bar{\psi}+\xi)\sin\frac{\omega_0\xi}{v_0}\right)d\xi\right.$$
$$+\left[u(\bar{\psi}+\xi_l)\cos\frac{\omega_0\xi_l}{v_0}-w(\bar{\psi}+\xi_l)\sin\frac{\omega_0\xi_l}{v_0}\right]^2$$
$$\left.-u^2(\bar{\psi})\right\}=0. \qquad (17)$$

In these equations, $\bar{\psi}=\psi|_{x=0}=-v_0t$, and ξ_l is related to u and w by

$$\xi_l=l-2u(\psi+\xi_l)\cos\frac{\omega_0\xi_l}{v_0}+2w(\psi+\xi_l)\sin\frac{\omega_0\xi_l}{v_0}+2u(\psi). \qquad (18)$$

Equations (16)—(18) determine the functions $u(\psi)$ and $w(\psi)$ on all the trajectories crossing the cathode, i.e., when $\psi<0$. When $\psi>0$ the functions $u(\psi)$ and $w(\psi)$ are determined from the initial conditions. Assume that at $t=0$ the density and velocity distributions along the beam are

$$n(0,x)=n_0(1+s(x)),\quad v(0,x)=v_0(1+g(x)). \qquad (19)$$

From a comparison of (19) with (3) it follows that $s(0)=0$ and $g(0)=0$. On the other hand, using (7) and (15), and taking into account the fact that $dC/dt|_{t=0}=0$, we obtain for $t=0$:

$$n=n_0\left(\frac{\partial x}{\partial\psi}\right)^{-1}=n_0\left[1+2\frac{d}{d\psi}\left(u(\psi)\cos\frac{\omega_0\psi}{v_0}\right.\right.$$
$$\left.\left.-w(\psi)\sin\frac{\omega_0\psi}{v_0}\right)\right]^{-1},$$
$$v=\frac{\partial x}{\partial t}=v_0-2\omega_0\left(w(\psi)\cos\frac{\omega_0\psi}{v_0}+u(\psi)\sin\frac{\omega_0\psi}{v_0}\right) \qquad (20)$$

The connection between ψ and x at $t=0$ is given by the relation

$$\psi=x+\int_0^x s\,dx'.$$

Comparing (19) and (20), and assuming that the initial amplitudes of the oscillations are sufficiently small, so that we can confine ourselves to first-order terms in $s\ll 1$ and $g\ll 1$, we obtain for $u(\psi)$ and $w(\psi)$ when $\psi>0$:

$$u(\psi)=-\frac{1}{2}\left[\int_0^\psi s\,dx\cos\frac{\omega_0\psi}{v_0}+\frac{v_0}{\omega_0}g(\psi)\sin\frac{\omega_0\psi}{v_0}\right],$$
$$w(\psi)=-\frac{1}{2}\left[\int_0^\psi s\,dx\sin\frac{\omega_0\psi}{v_0}-\frac{v_0}{\omega_0}g(\psi)\cos\frac{\omega_0\psi}{v_0}\right]. \qquad (21)$$

In the derivation of (21) we used the fact that, in accord with (10), we have $u(0)=C(0)=0$. It follows in particular from (21) that $du(0)/d\psi=w(0)=0$.

3. In the case of linearization with respect to u and w, the system (16) and (17) simplifies greatly. In this case we have for u the following integro-differential equations:

$$\frac{d^2u}{d\bar{\psi}^2}+2\frac{\omega_0^2}{v_0^2 l}\int_0^{-\bar{\psi}}u(\xi+\bar{\psi})\cos\frac{\omega_0\xi}{v_0}d\xi$$
$$=\frac{\omega_0^2}{v_0^2 l}\left[\int_{-\bar{\psi}}^{l}\tilde{w}(\xi+\bar{\psi})\sin\frac{\omega_0\xi}{v_0}d\xi-\int_{-\bar{\psi}}^{l}\tilde{u}(\xi+\bar{\psi})\cos\frac{\omega_0\xi}{v_0}d\xi\right]$$
$$\text{for } -l<\bar{\psi}<0, \qquad (22)$$

$$\frac{d^2u}{d\bar{\psi}^2}+2\frac{\omega_0^2}{v_0^2 l}\int_0^{l}u(\xi+\bar{\psi})\cos\frac{\omega_0\xi}{v_0}d\xi-\frac{\omega_0}{v_0 l}u(l+\bar{\psi})\sin\frac{\omega_0 l}{v_0}=0$$
$$\text{for } \bar{\psi}<-l.$$

In these equations we have eliminated $w(\psi)$ with $\psi<0$ with the aid of (16), and separated in the integrals with respect to ξ the region of positive arguments of the functions u and w, in which these functions are known. The corresponding functions, which in accord with (21) are expressed in terms of the initial perturbations of the density and velocity in the beam, will be denoted in this section by $\tilde{u}$ and $\tilde{w}$.

The solution of the equations in (22) can be obtained with the aid of a Laplace transformation. However, when using the Laplace transformation it is more convenient to consider the region of positive values of the variable $\bar{\psi}$. It is consequently advantageous to continue in even fashion the function $u(\bar{\psi})$, defined by (22), into the region of positive values of $\bar{\psi}$ and to consider the equations obtained from (22) by the substitutions $\bar{\psi}\to-\bar{\psi}$ and $u(-\bar{\psi})\to u(\bar{\psi})$. We then multiply these equations additionally by $\exp(-p\bar{\psi})$, integrate the first of them with respect to ψ from 0 to l and the second from l to ∞, and add the resultant relations. After simple transformations we get for the Laplace transform of the function $u(\bar{\psi})$

$$u_p=\int_0^\infty u(\bar{\psi})e^{-p\bar{\psi}}d\bar{\psi},$$

the following formula

$$u_p = N(p)\,/\,D(p), \tag{23}$$

where

$$D(p) = p^2 + \frac{2\omega_0^2}{v_0^2 l}\int_0^l e^{-p\xi}\cos\frac{\omega_0\xi}{v_0}\,d\xi - \frac{\omega_0}{v_0 l}e^{-pl}\sin\frac{\omega_0 l}{v_0}, \tag{24}$$

$$N(p) = \frac{\omega_0^2}{v_0^2 l}\int_0^l e^{-p\psi}\Big[\int_\psi^l \tilde{w}(\xi-\psi)\sin\frac{\omega_0\xi}{v_0}\,d\xi - \int_\psi^l \tilde{u}(\xi-\psi)\cos\frac{\omega_0\xi}{v_0}\,d\xi\Big]\,d\psi. \tag{24'}$$

Knowing u_p, we can, using the inverse Laplace transform, find $u(\bar{\psi})$:

$$u(\bar{\psi}) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{p\bar{\psi}}\frac{N(p)}{D(p)}\,dp, \tag{25}$$

where the integration is carried out along the straight line Re $p = \sigma$, which lies to the right of all the singularities of the integrand function. At large values of $\bar{\psi}$ the significant residue of the integrals with respect to p is at the point $p = p_l$, where p_l is that root of the equation $D(p) = 0$ which has the largest real part. In this case, returning to negative $\bar{\psi}$ with the aid of the condition $u(-\bar{\psi}) = u(\psi)$, we obtain the following asymptotic formula

$$u(\bar{\psi}) \doteq \frac{N(p_l)}{dD/dp_l}e^{-p_l\bar{\psi}}. \tag{26}$$

It follows from (24) that p_l is determined by solving the equation

$$\Theta^2 + \frac{i}{2}\Theta_0\Big\{\frac{\Theta_0-\Theta}{\Theta_0+\Theta}[\exp i(\Theta+\Theta_0)-1] - \frac{\Theta_0+\Theta}{\Theta_0-\Theta}[\exp i(\Theta-\Theta_0)-1]\Big\} = 0. \tag{27}$$

We have used here the notation $ip_l l = \Theta$ and $\omega_0 l/v_0 = \Theta_0$.

The dispersion equation (27) coincides with that obtained in [2]. Inasmuch as $\bar{\psi} = -v_0 t$, the instability in the electron beam arises when Eq. (27) has roots with Re $p_l > 0$. When $\Theta_0 \to 0$, all the roots of (27), with the exception of the real roots $\Theta = \pm\Theta_0$, lie below the real axis, Im $\Theta < 0$, corresponding to Re $p_l < 0$. With increasing Θ_0, these roots approach the real axis and when $\Theta_0 = \pi$ there is a root Θ crossing the real axis. The condition $\Theta_0 = \pi$, as can be readily seen, corresponds to a critical value of the current density in the beam $j = en_0v_0$, defined by relation (1).

In the case of small supercriticality $\Theta_0 = \pi + \delta$, $\delta \ll 1$, we have approximately from (27)

$$\Theta \approx i\pi\delta/4. \tag{28}$$

In this case, too, the expression for the pre-exponential factor in (26) becomes much simpler. From (24) we have $dD(0)/dp = 4/l$. We can calculate in similar fashion $N(0)$, integrating in (24′) by parts and substituting $\tilde{u}$ and $\tilde{w}$ from (21). We thus obtain for $u(\bar{\psi})$ at low supercriticality

$$u(\bar{\psi}) = \frac{1}{8}\int_0^l (s(x)-g(x))\Big(1+\cos\frac{\omega_0 x}{v_0}\Big)dx\cdot\exp\Big(-\frac{\pi}{4}\delta\frac{\bar{\psi}}{l}\Big). \tag{29}$$

Equations (26) and (29) determine the growth of the amplitude of the oscillations at the cathode with time, and inasmuch as the amplitude u of the oscillations is conserved along the trajectories of the particles ψ = const, they determine by the same token the function $u(\psi)$ on all the trajectories crossing the cathode. Knowing $u(\psi)$, we can easily obtain with the aid of (7) the displacement of the electrons in the presence of oscillations. The connection between the functions $A(\psi)$, $C(t)$ and $u(\psi)$, $w(\psi)$ which enter in this relation is given by formulas (15) and (10). Substituting $u(\psi)$ from (26) and $w(\psi)$ from (16), we obtain from (7) the following formula for the beam-electron displacement due to the oscillations:[2)]

$$\Delta x(t,\psi) = \frac{N(p_l)}{dD/dp_l}\Big\{e^{-p_l\psi}\Big[\Big(1+\frac{ip_l v_0}{\omega_0}\Big)\exp\Big(i\omega_0\Big(t+\frac{\psi}{v_0}\Big)\Big) + \Big(1-\frac{ip_l v_0}{\omega_0}\Big)\exp\Big(-i\omega_0\Big(t+\frac{\psi}{v_0}\Big)\Big)\Big] - 2\exp(p_l v_0 t)\Big\}. \tag{30}$$

(If Im $p_l \neq 0$, then (27) has two roots with identical real parts, p_l and p^*_l. In this case it is necessary to add to formulas (26) and (30) terms that differ from those given here by the substitution $p_l \to p^*_l$.) The wavelength of the excited oscillations is $\sim v_0/\omega_0$, and the amplitude buildup time is $\sim 1/\omega_0\delta$. A characteristic feature of the obtained solution, which was noted already by Pierce, is that in a two-electrode system the oscillations are inhomogeneous along the system axis. It follows from (30) that in the presence of instability the amplitude of the oscillations decreases in the direction from the cathode to the anode.

4. In this section we shall consider the instability of a beam at large oscillation amplitudes, when linearization no longer holds. In this case it is sufficient to confine oneself to large t, for

2)To transform in this formula to Euler coordinates x and t in the approximation that is linear in the oscillation amplitude, it is sufficient to use the relation $\psi = x - v_0 t$.

which $\bar{\psi} \ll -l$. Eliminating then $w(\psi)$ from (17) and (18) with the aid of (16), we obtain the following nonlinear system of equations for $u(\bar{\psi})$ and $\xi_l(\bar{\psi})$:

$$\frac{d^2u}{d\bar{\psi}^2} = -\frac{\omega_0^2}{v_0^2 l}\left\{2\int_0^{\xi_l} u(\bar{\psi}+\xi)\cos\frac{\omega_0\xi}{v_0}\,d\xi - \frac{v_0}{\omega_0}u(\bar{\psi}+\xi_l)\sin\frac{\omega_0\xi_l}{v_0} + \left[u(\bar{\psi}) - \frac{\xi_l - l}{2}\right]^2 - u^2(\bar{\psi})\right\}, \tag{31}$$

$$\xi_l = l - 2u(\bar{\psi}+\xi_l)\cos\frac{\omega_0\xi_l}{v_0} + 2\frac{v_0}{\omega_0}\frac{du}{d\bar{\psi}}(\bar{\psi}+\xi_l)\sin\frac{\omega_0\xi_l}{v_0} + 2u(\bar{\psi}). \tag{32}$$

Let us consider the solution of Eqs. (31) and (32) at low supercriticality, when $\omega_0 l/v_0 = \pi + \delta$. In this case the characteristic length of the variation of $u(\bar{\psi})$ is large compared with l and it is possible to expand in these equations $u(\bar{\psi} + \xi)$ with $\xi \ll \xi_l \sim l$ in powers of ξ, and retain the first two terms. At not too large oscillation amplitudes, $u < v_0/\omega_0$, it is possible to retain in the equation for $u(\bar{\psi})$ the higher-order nonlinear terms $\sim u^2$. Eliminating ξ_l from this equation with the aid of (32) and neglecting terms of the order of $\delta^2 u$ $(d^2u/d\bar{\psi}^2$, $\delta(du/d\bar{\psi}))$ and δu^2, we obtain

$$\frac{du}{d\bar{\psi}} + \frac{\delta}{4}\frac{\omega_0}{v_0}u(\bar{\psi}) + \frac{\omega_0^2}{v_0^2}u^2(\bar{\psi}) = 0. \tag{33}$$

For small values of u, when the nonlinear term can be neglected, the amplitude increases exponentially with decreasing $\bar{\psi}$:

$$u(\bar{\psi}) = u_0 \exp\left(-\frac{\pi}{4}\delta\frac{\bar{\psi}}{l}\right), \tag{34}$$

and from a comparison of (34) with the asymptotic formula (29) it follows that

$$u_0 = \frac{1}{8}\int_0^l (s(x) - g(x))\left(1 + \cos\frac{\omega_0 x}{v_0}\right)dx. \tag{35}$$

At large values of $\bar{t}$, when the nonlinear term of (33) becomes significant, the character of the solution depends on the sign of u_0. When $u_0 < 0$, the nonlinear term leads to saturation at small oscillation amplitudes proportional to the supercriticality parameter

$$u_\infty = -\delta v_0 / 4\omega_0. \tag{36}$$

The mean value of the energy density of the electric field in the space between electrodes, corresponding to these amplitudes,

$$\overline{E^2} / 8\pi \approx {}^3/_{64}\delta^2 n_0 m v_0^2,$$

is small compared with the energy density in the beam.

When $u_0 > 0$, the nonlinear term in (33) has the same sign as the linear one, and leads to an increase in $|du/d\bar{\psi}|$ ($du/d\bar{\psi}$ remains less than zero all the time). In this case u increases with increasing t within the limits of applicability of Eq. (33), i.e., up to $u \sim v_0/\omega_0$, which corresponds to $\overline{E^2}/8\pi \sim n_0 m v_0^2$.

Our result admits of the following illustrative interpretation[3)]. The perturbations with $u_0 < 0$ lead, in accord with (35) and (19), to an increase in the mean value of the velocity in the beam and to a decrease in the mean value of the density, i.e., they bring the system closer to the borderline of the stability zone. In this case the nonlinear effects stabilize the instability at sufficiently small amplitudes $\sim\delta$. Perturbations for which $u_0 > 0$, and which increase the deviation of the system from the borderline of the stability zone, lead to excitation of oscillations of large amplitudes.

Since it is natural to assume that the initial perturbations of the density and velocity of the beam always include some for which $u_0 > 0$, it can be assumed that the system under consideration has hard excitation. On passing through the border of the stability zone, oscillations, whose energy is comparable with the initial energy of the beam, should become excited in the beam. At such amplitudes, the displacement of the electrons in the oscillations is comparable with the wavelength $u \sim v_0/\omega_0$, and the oscillations can lead to a crossing of the trajectories in the beam and to the occurrence of regions of multistream motion. A criterion for the crossing is the vanishing of the derivative $dx/d\psi$. At the same time, the distance dx between two trajectories with different values of ψ vanishes. Using (7), (10), and (15) we find that in order for the trajectories to cross the function $u(\psi)$ defined by (31) and (32) should satisfy in the interval $0 < \xi < \xi_l$ the condition

$$\frac{v_0}{2\omega_0} = \sin\frac{\omega_0\xi}{v_0}\left[u(\psi) + \frac{v_0^2}{\omega_0^2}\frac{d^2u}{d\psi^2}\right]. \tag{37}$$

At greater oscillation amplitudes, regions of multistream motion arise in the beam and the analysis of our paper is no longer valid.

We consider also the possible existence of stationary solutions at large amplitudes, when $u \sim v_0/\omega_0$. A stationary solution corresponds to the case when u does not depend on ψ. In this

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[3)]The authors are grateful to B. B. Kadomtsev who pointed this circumstance out.

case the displacement x of the beam electrons, defined by (7), depends only on the quantity $\xi = \psi + v_0 t$:

$$x = \xi + 2u\left(\cos\frac{\omega_0\xi}{v_0} - 1\right). \qquad (38)$$

Then the velocity of the beam electrons $v = v_0\,dx/d\xi$, their density $n = n_0[\,dx/d\xi\,]^{-1}$, and the electric field also depend only on ξ, and in terms of Euler coordinates, only on x.

Putting in (31) and (32) $u = \text{const} = u_\infty$ and $\xi_l = \text{const} = \xi_l^\infty$, we obtain the following system of equations for determining the amplitude u_∞ at which saturation sets in, and for the value of ξ_l^∞:

$$\zeta\sin\eta(1 - \zeta\sin\eta) = 0, \quad \eta = \pi + \delta + 2\zeta(1 - \cos\eta), \qquad (39)$$

where we put $\eta = \omega_0\xi_l^\infty/v_0$ and $\zeta = \omega_0 u_\infty/v_0$.

Besides the root $\zeta = 0$, $\eta = \pi + \delta$, which is unstable in accordance with the linear theory, and the root $\zeta = -\delta/4$, $\eta = \pi$, which was considered earlier (see (36)) and which is significant only for perturbations with $u < 0$, these equations have also roots at $\zeta \sim 1$:

$$\eta^{\rm I} = \pi(2n+1), \quad \zeta^{\rm I} = \pi n/2, \quad n = 1, 2\ldots,$$
$$\eta^{\rm II} = \frac{2(1-\cos\eta^{\rm II})}{\sin\eta^{\rm II}}, \quad \zeta^{\rm II} = \frac{1}{\sin\eta^{\rm II}}. \qquad (40)$$

(The smallest of the roots $\eta^{\rm II}$ is equal to $\eta^{\rm II}_{\min} \approx 3\pi - 0.7$, the corresponding to $\zeta^{\rm II} \approx 1.58$.)

Let us consider the stability of the obtained stationary solutions. Linearizing Eqs. (31) and (32) with respect to deflection from the stationary solution $\Delta\xi_l = \xi_l - \xi_l^\infty$, $\Delta u = u - u_\infty$, and substituting $\Delta\xi_l$, $\Delta u \sim \exp(\kappa\overline{\psi}/\xi_l^\infty)$, we obtain the following equation for the determination of κ:

$$F(\varkappa,\eta,\zeta) = \frac{\pi}{2}\varkappa^2 - \eta^2\left[e^{\varkappa}\left(\eta\frac{\varkappa(1-\cos\eta) - \eta\sin\eta}{\eta^2+\varkappa^2} + \sin\eta\right) + \zeta(1-\cos\eta)\right]. \qquad (41)$$

When $\kappa = 0$ we have $F = \eta^2\zeta(1 - \cos) < 0$, and when $\kappa \to -\infty$ we get $F \approx \pi\kappa^2/2 \to +\infty$. Therefore Eq. (41) has at least one root at $\kappa < 0$. Since $\overline{\psi} = -v_0 t$, the presence of this root denotes instability of the stationary solutions defined by (40). Thus, when $u > 0$, Eqs. (31) and (32) have no stationary solutions corresponding to saturation of the oscillations, and, just as in [8], the amplitude of the oscillations oscillates in time.

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