STIMULATED TRANSITIONS IN THE RADIATION OF A RELATIVISTIC ELECTRON IN AN INHOMOGENEOUS MAGNETIC FIELD

A. A. SOKOLOV, V. Ch. ZHUKOVSKIĬ, and Yu. A. KOROVIN

Moscow State University

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Stimulated transitions of relativistic electrons moving in a stationary but inhomogeneous magnetic field are investigated. The regions of variation of the harmonics for which stimulated radiation should exceed absorption is found.

1. Recently, in connection with the more intense development of maser technology, much attention has been paid to stimulated emission. In this note, developing further the papers[1,2] in which similar emission in a constant and homogeneous magnetic field was investigated, and in which, as is well known, no stability was obtained along the field direction, we consider stimulated emission of an electron placed in an axially symmetrical focusing magnetic field

$$H_x = H_y = 0, \qquad H_z = H_0 r^{-q}, \tag{1}$$

where $r = (x^2 + y^2)^{1/2}$, and q is the field fall-off exponent, which in the case of stable motion should lie in the range $0 < q < 1$. The energy spectrum of the electron is given, in the relativistic case (without account of the spin), by the expression

$$E_{lsk} = E_l + \hbar\omega_l(\sqrt{1-q}\, s + \sqrt{q}\, k), \tag{2}$$

where

$$E_l = c\hbar[\gamma^2(2-q)^2 R^{2(1-q)} + k_0^2)^{1/2}, \quad k = m_0 c/\hbar,$$

$$\omega_l = \frac{e_0 H(R) c}{E_l} = \frac{\beta c}{R},$$

$$\gamma = \frac{e_0 H_0}{(2-q)c\hbar}, \qquad R = \left[\frac{l}{\gamma(1-q)}\right]^{1/(2-q)},$$

R is the radius of the equilibrium orbit, ω is the frequency of revolution of the electron, and l, s, and k = 0, 1, 2 ... are respectively the orbital, radial, and axial quantum numbers.

2. Let us consider the transitions l, s, k $\to l'$ $\neq l$, $s' = s$, $k' = k$, stimulated by an external field of frequency $\omega \approx (l - l')\omega_l = \pm\nu\omega_l$, where $\nu > 0$ is the number of the harmonic; in the case of light emission $l - l' = \nu$, while in the case of absorption $l - l' = -\nu$.

We assume that the external electromagnetic wave is linearly polarized and propagates at an angle θ to the direction of the magnetic field. In this case the probability of the stimulated transition is given by the following formula (see, for example[2]):

$$w_{ll'} = \frac{2\pi N(\varkappa) e_0^2 c^4}{\hbar L^3 \omega E_l E_{l'}} S_i g_{l-l'}(\omega), \qquad i = x, z'. \tag{3}$$

Here $N(\kappa)$ is the number of photons with momentum κ in the volume L^3, τ is the average lifetime of the electron in the initial state, and

$$g_{l-l'}(\omega) = \frac{4\tau}{1 + 4\tau^2(\omega_{ll'} - \omega)^2},$$

$$S_x = |\bar{P}_x|^2, \qquad S_{z'} = |\bar{P}_y|^2 \cos^2\theta.$$

The indices x and z′ denote the polarization of the external wave: for the x component the vector of the electric field $\mathscr{E}_x$ lies in the plane of the electron orbit and is directed along the x axis; for the z′ component the vector of the electric field $\mathscr{E}_{z'}$ is perpendicular to $\mathscr{E}_x$ and to the propagation direction of the wave (i.e., it is almost parallel to the z axis).

The matrix elements

$$\bar{P}_{x,y} = \int \psi^+_{l',s,k} e^{-i\varkappa \mathbf{r}} \hat{P}_{x,y} \psi_{l,s,k}\, d^3x$$

are, assuming the amplitudes of the radial and axial oscillations to be small,

$$\bar{P}_{x,y} = \frac{\hbar}{2R}\begin{cases} iX_{l,l\mp\nu}(\theta,\varkappa) \\ Y_{l,l\mp\nu}(\theta,\varkappa) \end{cases}, \tag{4}$$

where $\boldsymbol{\kappa} = \kappa\boldsymbol{\kappa}^0$ is the wave vector of the incident photon, and the quantities

$$X_{l,l\mp\nu} = -2l\frac{2-q}{1-q}J_\nu'(\eta) \pm \frac{\nu(\nu^2-\eta^2-q)}{\eta(1-q)}J_\nu(\eta),$$

$$Y_{l,l\mp\nu} = \pm\frac{2l(2-q)\nu + \eta^2}{\eta(1-q)}J_\nu(\eta) - \frac{\nu^2-q}{1-q}J_\nu'(\eta) \tag{5}$$

are expressed in terms of the Bessel function J_ν and its derivative, which depends on the argument $\eta = \kappa R \sin\theta = \nu\beta\sin\theta$.

The denominator in (3) is equal to

$$\frac{1}{E_l E_{l'}} = \frac{1}{E_l^2}\left(1 \pm \beta^2 \frac{1-q}{2-q}\frac{\nu}{l}\right), \tag{6}$$

and for the factor $g_{l-l'}(\omega)$ we have

$$g_\nu(\omega) = \frac{4\tau}{1+4\tau^2(\omega_{l,l+\nu}-\omega)^2} = \frac{4\tau}{1+\nu^2\xi^2}, \tag{7}$$

$$g_{-\nu}(\omega) = \frac{4\tau}{1+4\tau^2(\omega_{l,l-\nu}-\omega)^2}$$

$$= \frac{4\tau}{1+\nu^2\xi^2} - \frac{16\nu^3\tau^2\xi\beta^2\omega_l(1-q+q/\beta^2)}{(2-q)l(1+\nu^2\xi^2)^2},$$

where $\xi = 2\tau(\omega_{l,l+\nu} - \omega/\nu)$.

We now determine the power radiated by the electron in resonant transitions under the influence of an external electromagnetic wave, at the harmonic ν:

$$W(\nu) = \hbar\omega_{l,l-\nu}w_{l,l-\nu} - \hbar\omega_{l,l+\nu}w_{l,l+\nu}. \tag{8}$$

Using for the formulas (5), (6), and (7) and introducing the intensity of the electric field of the wave $\mathcal{E}$, connected with $N(\kappa)$ by the relation

$$\mathcal{E}^2/4\pi = \hbar\omega N(\varkappa)/L^3,$$

we obtain the following expressions for the radiation power:

1) for the x component

$$W_x(\nu) = -\frac{e_0^2c^2\mathcal{E}^2\tau}{E_l}\frac{4\nu J_\nu'^2(\eta)}{1+\nu^2\xi^2}\left\{\frac{1-\beta^2\sin^2\theta - q/\nu^2}{(1-q)\beta\sin\theta}\frac{J_\nu(\eta)}{J_\nu'(\eta)} - \frac{3\beta^2}{2\nu}\left(1+\frac{q}{3\beta^2(1-q)}\right) + \beta^2\frac{2\xi\omega_l\tau\nu(1+q/\beta^2(1-q))}{1+\nu^2\xi^2}\right\}; \tag{9}$$

2) for the z′ component

$$W_{z'}(\nu) = -\frac{e_0^2c^2\mathcal{E}^2\tau\,\mathrm{ctg}^2\theta}{E_l\beta^2}\frac{4\nu J_\nu^2(\eta)}{1+\nu^2\xi^2} \times\left\{\frac{(1-q/\nu^2)\beta\sin\theta}{1-q}\frac{J_\nu'(\eta)}{J_\nu(\eta)} - \frac{3\beta^2}{2\nu}\left(1+\frac{q}{3\beta^2(1-q)}+\frac{2}{3}\frac{\sin^2\theta}{1-q}\right) + \beta^2\frac{2\xi\omega_l\tau\nu(1+q/\beta^2(1-q))}{1+\nu^2\xi^2}\right\}. \tag{10}$$

From this we get, in the case of a homogeneous field (q = 0), the result of[2], which goes over into the Schneider formula[1] in the weakly-relativistic limit ($\beta^2 \neq 0$ and $\beta^4 = 0$).

In the weakly-relativistic limit we obtain from formulas (9) and (10) the power of the dipole radiation ($\nu = 1$):

$$W = W_x + W_{z'} = -\frac{e_0^2\mathcal{E}^2\tau}{m_0}\frac{1+\cos^2\theta}{1+\xi^2}\left\{\frac{2-3q}{2(1-q)} + \left(\beta^2 + \frac{q}{1-q}\right)\frac{2\xi\Omega\tau}{1+\xi^2}\right\}, \tag{11}$$

where $\Omega = e_0H(R)/m_0c$. It follows from (11) that in an inhomogeneous field ($q \neq 1$), unlike the Schneider formula, stimulated emission is possible ($W > 0$) at the fundamental harmonic in the case of resonance ($\xi = 0$) when $q > 2/3$.

In the ultrarelativistic approximation ($\beta \to 1$), just as in the case of a homogeneous field (see[2]), we have a region of variation of the harmonics ν in which even stimulated emission can prevail over absorption ($W > 0$) in the presence of resonance.

To find this region, let us consider first the case when the external electromagnetic wave has only an x-component and is incident at an angle θ close to $\pi/2$, i.e., the wave vector lies near the plane of the electron orbit. Then

$$\sin\theta = 1 - \alpha^2/2 \quad (\alpha = \pi/2 - \theta), \quad 1-\beta^2 \ll 1, \tag{12}$$

and the following approximations of the Bessel functions hold:

$$J_\nu(\nu\beta\sin\theta) = \frac{(1-\beta^2\sin^2\theta)^{1/2}}{\pi\sqrt{3}}K_{1/3}\left(\frac{\nu}{3}(1-\beta^2\sin^2\theta)^{3/2}\right),$$

$$J_\nu'(\nu\beta\sin\theta) = \frac{1-\beta^2\sin^2\theta}{\pi\sqrt{3}}K_{2/3}\left(\frac{\nu}{3}(1-\beta^2\sin^2\theta)^{3/2}\right). \tag{13}$$

If we assume that the argument of the functions $K_{1/3}$ and $K_{2/3}$ is small,

$$\frac{\nu}{3}(1-\beta^2\sin^2\theta)^{3/2} \ll 1, \tag{14}$$

then it follows from (13) that

$$J_\nu(\eta) = \frac{\Gamma(1/3)}{2^{2/3}3^{1/6}\pi\nu^{1/3}}, \quad J_\nu'(\eta) = \frac{\Gamma(2/3)\cdot 3^{1/6}}{2^{1/3}\pi\nu^{2/3}}, \tag{15}$$

and in the resonance case $\xi = 0$ we obtain for the radiated power the expression

$$W_x(\nu) = \frac{9^{2/3}2^{1/3}[\Gamma(2/3)]^2 e_0^2c^2\mathcal{E}^2\tau}{\pi^2\nu^{4/3}E_l} \cdot \times\left\{\frac{3-2q}{3(1-q)} - \frac{2^{5/3}\pi\nu^{4/3}(1-\beta^2+\alpha^2-q/\nu^2)}{3^{11/6}[\Gamma(2/3)]^2(1-q)}\right\}. \tag{16}$$

We have put here

$$1 - \beta^2 \sin^2 \theta \approx 1 - \beta^2 + \alpha^2.$$

The power W_x will be positive (radiation) in two cases:

1) if $\alpha^2 < 1 - \beta^2$, then $\nu < (1 - \beta^2)^{-3/4} = \sqrt{\nu_{max}}$ (ν_{max} corresponds to the maximum of spontaneous emission);

2) if $\alpha^2 > 1 - \beta^2$, then $\nu < \alpha^{-3/2}$.

When one of these inequalities is violated, the system becomes absorbing, i.e., $W_x < 0$). To increase the interval of harmonics in which amplification of the radiation takes place, it is convenient to choose an angle $\alpha < \sqrt{1 - \beta^2} = m_0c^2/E$.

For the z′ component in the ultrarelativistic case, in the presence of resonance, only absorption will be observed ($W_{z'} < 0$), and this can be used, for example, for electron acceleration.

[1] J. Schneider, Phys. Rev. Lett. **2**, 504 (1959).
[2] A. A. Sokolov and I. M. Ternov, DAN SSSR **166**, 1332 (1966), Soviet Phys. Doklady **11**, 156 (1966).

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220