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MULTIPLE SCATTERING OF PARTICLES IN A MAGNETIC FIELD WITH RANDOM INHOMOGENEITIES

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Motion of charged particles in a magnetic field having regular or random components is considered. A kinetic equation for the particle distribution in such a field is derived under the assumption that the magnetic-field inhomogeneities move with a certain velocity. It is assumed in this case that the particles are not deflected much by the random field in a distance equal to the correlation length. A solution of the kinetic equation is obtained for some particular cases. Transition to the diffusion approximation is considered and the equation for diffusion of particles in a medium with moving magnetic-field inhomogeneities is derived.

1. FORMULATION OF PROBLEM

IN many problems of plasma theory, astrophysics, cosmic-ray physics, etc. it becomes necessary to consider the passage of charged particles through random magnetic fields. This includes problems in the passage of cosmic rays through the magnetic fields of the solar system, problems in the diagnostics of a turbulent plasma, problems of motion of clouds of ionized gas through magnetic fields of cosmic medium, etc. These problems are usually considered in the diffusion approximation. In many real cases, however, this approximation is not valid, and it becomes necessary to use more accurate equations. Even in the diffusion approximation, if the random magnetic fields are carried by clouds of a moving plasma, the equation for the distribution function of the diffusing particles does not have at all the usual form, a fact disregarded by a number of authors (for example,[1]). We therefore consider in this paper a more general formulation of the problem.

We consider the motion of charged particles in a magnetic field $\mathbf{H}(\mathbf{r}, t)$ that fluctuate in space and in time; the fluctuations may move with a certain velocity. In this case the particle motion is best described by a distribution function satisfying Boltzmann's equation. We average this equation over the random field and derive an equation for the average distribution function $F(\mathbf{r}, \mathbf{p}, t)$. For some simple cases we construct an approximate solution of the obtained equations. We then go over to the diffusion approximation and obtain an equation describing the diffusion of particles in a medium with moving magnetic-field inhomogeneities.

2. DERIVATION OF KINETIC EQUATION

We assume that the magnetic field $\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_1$ has regular and random components $\mathbf{H}_0(\mathbf{r})$ and $\mathbf{H}_1(\mathbf{r}, t)$ respectively, with $\mathbf{H}_0 = \langle \mathbf{H} \rangle$ and $\langle \mathbf{H}_1 \rangle = 0$. The angle brackets denote averaging over the random fields. For a complete description of the random field, we specify a second-rank correlation tensor

$$B_{\alpha\beta}(\mathbf{r}_1 t_1, \mathbf{r}_2 t_2) = \langle H_{1\alpha}(\mathbf{r}_1 t_1) H_{1\beta}(\mathbf{r}_2 t_2) \rangle \qquad (1)$$

and an entire aggregate of similar tensors of higher rank. If the probabilities have a Gaussian distribution, all the higher correlators are expressed in terms of (1). If the field is transported by clouds of magnetized plasma, then the motion of the fields relative to the observer must be taken into account. The velocity that must be ascribed to the field depends on the degree of freezing-in of the field in the plasma. We shall consider the case of a completely frozen-in field dragged by the plasma with a velocity $\mathbf{u}$ that is in general different at different points; $u \ll c$. We assume that the concentration of the particles passing through the field is sufficiently small so that interaction between them can be neglected.

The classical distribution function of the noninteracting particles in a magnetic field $\mathbf{H}$ satisfies the Boltzmann equation

$$\frac{\partial f}{\partial t}+\mathbf{v}\frac{\partial f}{\partial \mathbf{r}}-\mathbf{HD}f=0. \qquad (2)$$

Here

$$\mathbf{D}=\frac{e}{c}\left[\mathbf{v}-\mathbf{u},\frac{\partial}{\partial \mathbf{p}}\right], \qquad (3)$$

$\mathbf{p}$ is the momentum, $\mathbf{v} = c^2\mathbf{p}/\epsilon$ the velocity, and ϵ the energy of the particle.

The function f varies rapidly and follows the variations of the random field. Interest attaches to the distribution function averaged over the random field, $F(\mathbf{r}, \mathbf{p}, t) = \langle f(\mathbf{r}, \mathbf{p}, t)\rangle$. In order to find the equation for F, let us change (2) to a more convenient form.

We expand $f(\mathbf{r}, \mathbf{p}, t)$ in a Fourier series inside a cube of side L (we shall later let $L \to \infty$)

$$f(\mathbf{r},\mathbf{p},t)=\left(\frac{2\pi}{L}\right)^3\sum_{\mathbf{k}} f_{\mathbf{k}}(\mathbf{p},t)\exp[i\mathbf{k}(\mathbf{r}-\mathbf{v}t)].$$

The coefficient $f_{\mathbf{k}}(\mathbf{p}, t)$ satisfies the equation

$$\frac{\partial f_{\mathbf{k}}(\mathbf{p},t)}{\partial t}=\left(\frac{2\pi}{L}\right)^3\sum_{\mathbf{l}} e^{i\mathbf{k}\mathbf{v}t}(\mathbf{DH})_{\mathbf{l}}\, e^{-i(\mathbf{k}-\mathbf{l})\mathbf{v}t} f_{\mathbf{k}-\mathbf{l}}(\mathbf{p},t), \qquad (4)$$

where

$$(\mathbf{DH})_{\mathbf{l}}=(2\pi)^{-3}\int d\mathbf{r}\,(\mathbf{DH})\,e^{-i\mathbf{l}\mathbf{r}}. \qquad (5)$$

Solving Eq. (4) by the iteration method, we obtain

$$f_{\mathbf{k}}(\mathbf{p},t)=\sum_{n=0}^{\infty}\left(\frac{2\pi}{L}\right)^{3n}\sum_{\mathbf{l}_1\ldots\mathbf{l}_n}\int_0^t dt_1\cdots\int_0^{t_{n-1}} dt_n e^{i\mathbf{k}\mathbf{v}t}(\mathbf{DH})_{\mathbf{l}_1}$$
$$\times\, e^{-i(\mathbf{k}-\mathbf{l}_1)\mathbf{v}(t-t_1)}(\mathbf{DH})_{\mathbf{l}_2}\cdots e^{-i(\mathbf{k}-\mathbf{l}_1-\cdots-\mathbf{l}_n)\mathbf{v}t_n} f_{\mathbf{k}-\cdots-\mathbf{l}_n}(\mathbf{p},0), \qquad (6)$$

where $f_q(\mathbf{p}, 0)$ is the value of $f_q(\mathbf{p}, t)$ at $t = 0$.

In order to integrate with respect to time, the right side of (6), let us expand each of the factors $(\mathbf{D}\cdot\mathbf{H})$ in a Fourier time integral

$$(\mathbf{DH})_{\mathbf{l}}=\int(\mathbf{DH}(\omega))_{\mathbf{l}}\,e^{-i\omega t}\,d\omega \qquad (7)$$

and let us substitute (7) in (6). The arguments of the exponentials in (6) will then take the form $-i[(\mathbf{k}-\mathbf{l}_1-\ldots-\mathbf{l}_j)\cdot\mathbf{v}+\omega_1+\ldots+\omega_j](t_j-t_{j+1})$. We multiply both sides of (6) by $\exp(-i\mathbf{k}\cdot\mathbf{v}t)$ and take their Laplace transforms with respect to time. In the left side we obtain

$$f_k(\mathbf{p},s)=\int_0^{\infty} f_k(\mathbf{p},t)\exp(-st-i\mathbf{k}\mathbf{v}t)\,dt.$$

The integrals with respect to time in the right side of (6) will be evaluated by going over to new variables $\tau = t - t_1$, $\tau_1 = t_1 - t_2$, ..., $\tau_{n-1} = t_{n-1} - t_n$, and $\tau_n = t_n$. In terms of these variables, we can integrate independently from zero to infinity. As a result we get

$$f_k(\mathbf{p},s)=\sum_{n=0}^{\infty}\left(\frac{2\pi}{L}\right)^{3n}\sum_{\mathbf{l}_1\cdots\mathbf{l}_n}\int d\omega_1\cdots\int d\omega_n\,[s$$
$$+\,i\mathbf{k}\mathbf{v}]^{-1}(\mathbf{DH}(\omega_1))_{\mathbf{l}_1}\cdot[s+i(\mathbf{k}-\mathbf{l}_1)\mathbf{v}+i\omega_1]^{-1}$$
$$\times(\mathbf{DH}(\omega_2))_{\mathbf{l}_2}\cdots(\mathbf{DH}(\omega_n))_{\mathbf{l}_n}\cdot[s+i(\mathbf{k}-\mathbf{l}_1-\cdots-\mathbf{l}_n)\mathbf{v}$$
$$+\,i\omega_1+\cdots+i\omega_n]^{-1} f_{\mathbf{k}-\mathbf{l}_1-\cdots-\mathbf{l}_n}(\mathbf{p},0). \qquad (8)$$

We now average (8) over all possible values of the random magnetic field. We assume for concreteness that the probability distribution has a Gaussian character. This assumption is not significant in the case when H_1 is small, for only the pair correlator (1) will enter in the equation in this case, regardless of the probability distribution. Let us consider a stationary random field, which can be inhomogeneous in space, and let us choose $B_{\alpha\beta}$ in the form

$$B_{\alpha\beta}(\mathbf{r}_1t_1,\,\mathbf{r}_2t_2)=B_{\alpha\beta}(\mathbf{r},\,\mathbf{x}-\mathbf{u}t), \qquad (9)$$

where $\mathbf{r} = (\mathbf{r}_1 + \mathbf{r}_2)/2$, $\mathbf{x} = \mathbf{r}_1 - \mathbf{r}_2$, and $t = t_1 - t_2$.

The first argument in the right side describes the change in the mean square of the random component of the field in space, while the second describes the weakening of the correlation with the increasing distance, with account taken of the motion of the inhomogeneities. The velocity $\mathbf{u}$ of their motion will be assumed directed along $\mathbf{r}$. It should be noted that the character of the dependence of $B_{\alpha\beta}$ on the first and second arguments is different. Let $B_{\alpha\beta}$ change noticeably when the first argument changes by an amount on the order of L_0, and the second by an amount on the order of L_c, where L_0 characterizes the spatial scale of the change in the square of the random field, and L_c has the meaning of the linear dimension of that region within which the values of the random field are essentially correlated. We assume that these scales are different: $L_0 \gg L_c$. We shall assume that similar inequalities are valid also for the spatial scale L_{H_0} of the variation of the regular field, and for the scale L_u of the variation of $\mathbf{u}$: $L_{H_0} \gg L_c$, and $L_u \gg L_c$.

Maxwell's equation div H = 0 leads to the condition $\partial B_{\alpha\beta}/\partial x_{1\alpha} = 0$, which reduces, owing to the proposed inequality $L_0 \gg L_c$, to the requirement

$$\partial B_{\alpha\beta}(\mathbf{r},\,\mathbf{x})\,/\,\partial x_\alpha=0. \qquad (10)$$

From the definition (1) of the tensor $B_{\alpha\beta}$ it follows that it satisfies also the condition $B_{\alpha\beta}(\mathbf{r}, \mathbf{x}) = B_{\alpha\beta}(\mathbf{r}, -\mathbf{x})$.

For a statistically isotropic random field $B_{\alpha\beta}$ takes the form

$$B_{\alpha\beta}(\mathbf{r},\mathbf{x})=\frac{1}{12}\langle H_1^2\rangle\left(\frac{\partial^2}{\partial x_\alpha\,\partial x_\beta}-\frac{\partial^2}{\partial x_\gamma^2}\delta_{\alpha\beta}\right)\Phi(\mathbf{r},x), \quad (11)$$

where $\langle H_1^2 \rangle$ is a dimensional factor and $\Phi(\mathbf{r}, x)$ is a scalar function. In concrete calculations we shall take Φ in the form

$$\Phi(\mathbf{r}, x) = L_c^2 \varphi(\mathbf{r}) \exp(-x^2 / L_c^2). \tag{12}$$

If the regular field $\mathbf{H}_0$ does not depend on the time, then the quantities $(\mathbf{D} \cdot \mathbf{H}(\omega))_{\mathbf{l}}$, which enter in the right side of (8), take the form

$$(\mathbf{DH}(\omega))_{\mathbf{l}} = (\mathbf{DH}_0)_{\mathbf{l}}\, \delta(\omega) + (\mathbf{DH}_1(\omega))_{\mathbf{l}}. \tag{13}$$

If the probability distribution is Gaussian, for a random field the average of the product $\langle (\mathbf{D} \cdot \mathbf{H}_1(\omega_1)) \ldots (\mathbf{D} \cdot \mathbf{H}_1(\omega_n))_{\mathbf{l}} \rangle$ for even n vanishes, and for odd n it is expressed in the form of a sum of products of all the possible combinations of the paired averages:

$$\langle (\mathbf{DH}_1(\omega_i))_{\mathbf{l}_i} (\mathbf{DH}_1(\omega_j))_{\mathbf{l}} \rangle = (2\pi)^{-3} \delta(\omega_i + \omega_j) \times \int \exp(-i\mathbf{qr})\, D_\alpha D_\beta B_{\alpha 3}(\mathbf{r}, \mathbf{l}, \omega_i)\, d\mathbf{r}, \tag{14}$$

where $\mathbf{q} = \mathbf{l}_i + \mathbf{l}_j$, $\mathbf{l} = (\mathbf{l}_i - \mathbf{l}_j)/2$, and

$$B_{\alpha\beta}(\mathbf{r}, \mathbf{l}, \omega) = B_{\alpha\beta}(\mathbf{r}, \mathbf{l})\, \delta(\omega - \mathbf{ul}) \tag{15}$$

is the Fourier component of the correlator (9) with respect to t and $\mathbf{x}$. In integrating with respect to $\mathbf{r}$ in (14), it must be remembered that D_α and D_β depend on $\mathbf{r}$ via $\mathbf{u}$.

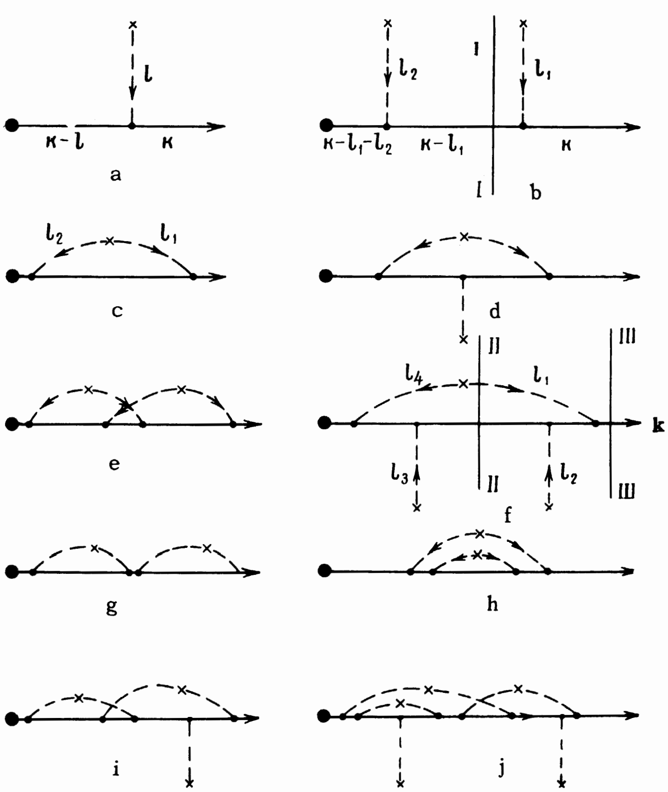

FIG. 1. Certain diagrams which enter in the expansion of the distribution function.

The left side of (8) is transformed after averaging into $F_{\mathbf{k}}(\mathbf{p}, s) = \langle f_{\mathbf{k}}(\mathbf{p}, s) \rangle$, and the quantities $\langle (\mathbf{D} \cdot \mathbf{H})_1 \ldots (\mathbf{D} \cdot \mathbf{H})_n \rangle$ are expressed in terms of (13) and (14). We can then set in correspondence to each term of the averaged equation (8) a diagram describing a definite interaction between the particle and the external field. Examples of such diagrams are shown in Fig. 1. To each diagram we set in correspondence a definite analytic expression in accordance with the following rules:

1. A black circle in a diagram with n vertices corresponds to the quantity $F_{\mathbf{k}-\mathbf{l}_1-\ldots-\mathbf{l}_n}(\mathbf{p}, 0)$.

2. A vertex together with an outgoing dashed line, the other end of which is free, corresponds to the factor $(\mathbf{D} \cdot \mathbf{H}_0)_{\mathbf{l}_j}\, \delta(\omega_j)$, where j is the number of the vertex, counting from right to left.

3. The diagram element containing two vertices and a dashed line joining them, corresponds to the quantity (14).

4. A vertical section line should be drawn between any two neighboring vertices (for example I—I in the diagram 1b, II—II and III—III in the diagram 1f). The section between the vertices j and j + 1 corresponds to a factor $[s + i(\mathbf{k} - \mathbf{l}_1 \ldots - \mathbf{l}_j) \cdot \mathbf{v}$ and $i \sum_r \omega_r]^{-1}$, where $\sum \omega_r$ is the sum of the frequencies corresponding to the dashed lines crossed by the given section lines. All the factors which depend on $\mathbf{v}$ are arranged in the same order as shown in the diagram, reading from right to left.

5. A summation of the type $(2\pi/L)^3 \Sigma_{\mathbf{l}_j} \ldots$ is carried out over all the $\mathbf{l}_j$, and integration is carried out over the ω_j.

Using the diagram concepts and the rules presented above, it is easy to construct the sought kinetic equation. Graphically it takes the form shown in Fig. 2a. The shaded circle with lines emerging from it represent the quantity $F_{\mathbf{k}}(\mathbf{p}, s)$. The left angle plays the role of the collision term of the kinetic equation and corresponds to an aggregate of an infinite number of irreducible diagrams, i.e., diagrams whose parts are connected with one another not only by one solid line. Diagrams of this type include all the diagrams of Fig. 1 with the exception of 1b and 1g.

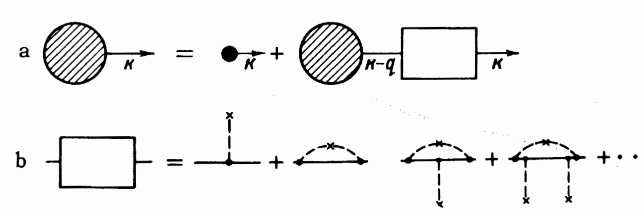

FIG. 2. Graphic representation of Eq. (16).

Let us estimate the contribution of the different diagrams to the kinetic equation. The simplest of them are 1a and 1c. The first describes the interaction of the particle with the average magnetic field, and the second with the inhomogeneities of the field. The ratio of the contributions of diagrams 1d and 1c, as can be readily established with the aid of the preceding formulas, has an order of magnitude L_c/R_{H_0}, where $R_{H_0} = cp/eH_0$ is the Larmor radius in the regular field. Addition of an extra dashed line with one vertex (diagram 1f) introduces an additional factor of the same order. We shall not impose any limitations whatever on the quantity L_c/R_{H_0}, and therefore all the diagrams of type 1d, 1f, etc. must be taken into account. Diagrams of the type 1e and 1h contain, compared with 1c, an extra factor of the order of L_c^2/R_1^2, where $R_1^2 = c^2p^2/e^2\langle H_1^2\rangle$. This factor will be assumed small. Therefore the diagrams 1e and 1h, and the more complicated ones, of the type 1j and 1k, should be discarded. Thus, the collision terms of the kinetic equation is expressed in terms of the diagrams shown in Fig. 2b.

Our assumptions concerning the relative magnitude of the individual diagrams signify that the random magnetic field, within the limits of the single inhomogeneity (linear dimension L_c), deflects the particle by a small angle. At the same time, the deflection of the particle by the regular field H_0 at the same distance may not be small. The assumed relations between L_c, R_{H_0}, and R_1 are satisfied, for example, for a broad energy interval of cosmic rays in magnetic fields of the solar system[2].

On the basis of the assumed choice of diagrams we write out with the aid of Fig. 2 the kinetic equation:

$$(s+i\mathbf{kv})F_{\mathbf{k}}(\mathbf{p},s)=F_{\mathbf{k}}(\mathbf{p},0)+(2\pi/L)^3\sum_{\mathbf{m}}(\mathbf{DH}_0)_{\mathbf{m}}F_{\mathbf{k}-\mathbf{m}}(\mathbf{p},s)$$

$$+(2\pi)^{-6}\sum_{n=0}^{\infty}(2\pi/L)^{3n+6}\sum_{\mathbf{l}_1\mathbf{l}_2\mathbf{m}_1\ldots\mathbf{m}_n}$$
$$\times\int d\mathbf{r}\exp(-i\mathbf{qr})D_\alpha B_{\alpha\beta}(\mathbf{r},\mathbf{l})\cdot[s+i(\mathbf{k}-\mathbf{l}_1)\mathbf{v}$$
$$+i\mathbf{ul}]^{-1}(\mathbf{DH}_0)_{\mathbf{m}_1}[s+i(\mathbf{k}-\mathbf{l}_1-\mathbf{m}_1)\mathbf{v}+i\mathbf{ul}]^{-1}\ldots$$
$$\times(\mathbf{DH}_0)_{\mathbf{m}_n}[s+i(\mathbf{k}-\mathbf{l}_1-\mathbf{m}_1-\ldots-\mathbf{m}_n)\mathbf{v}$$
$$+i\mathbf{ul}]^{-1}D_\beta F_{\mathbf{k}-\mathbf{q}-\mathbf{m}_1-\ldots-\mathbf{m}_n}(\mathbf{p},s). \quad (16)$$

Using (9)—(12), we can readily verify that the essential values of $\mathbf{l}$ and $\mathbf{q}$ in (16) are in order of magnitude equal to L_c^{-1} and L_0^{-1}, respectively, so that $l \gg q$. Therefore we can replace in the denominators of (16) $\mathbf{l}_1\cdot\mathbf{v}$ by $\mathbf{l}\cdot\mathbf{v}$, and neglect the terms $\mathbf{m}_i\cdot\mathbf{v}$ and $\mathbf{k}\cdot\mathbf{v}$ compared with $\mathbf{l}\cdot\mathbf{v}$. Neglect of the $\mathbf{k}\cdot\mathbf{v}$ term signifies that the inhomogeneity in the particle distribution has a scale larger than L_c. Taking next the inverse Fourier transform with respect to the coordinates and the inverse Laplace transform with respect to time, we obtain from (16)

$$\frac{\partial F(\mathbf{r},\mathbf{p},t)}{\partial t}+\mathbf{v}\frac{\partial F(\mathbf{r},\mathbf{p},t)}{\partial \mathbf{r}}=\mathbf{H}_0(\mathbf{r})\mathbf{D}F(\mathbf{r},\mathbf{p},t)$$
$$+\left(\frac{2\pi}{L}\right)^3\sum_{\mathbf{l}}B_{\alpha\beta}(\mathbf{r},\mathbf{l})D_\alpha\int_0^t U_l(\tau)D_\beta F(\mathbf{r},\mathbf{p},t-\tau)\,d\tau, \quad (17)$$

where

$$U_l(\tau)=\exp[i\mathbf{l}(\mathbf{v}-\mathbf{u})\tau+\mathbf{H}_0(\mathbf{r})\mathbf{D}\tau], \quad (18)$$

and $B_{\alpha\beta}(\mathbf{r},\mathbf{l})$ is defined by (15).

In order to carry out the integration with respect to $\mathbf{l}$, it is necessary to factor out the components of the operator (18). Let us consider to this end the equation for the Fourier transform of the particle distribution function in homogeneous static magnetic field H_0 and electric field $-\mathbf{u}\times H_0/c$:

$$\frac{\partial f_{\mathbf{k}}(\mathbf{p},t)}{\partial t}+i\mathbf{kv}f_{\mathbf{k}}(\mathbf{p},t)=\mathbf{H}_0\mathbf{D}f_{\mathbf{k}}(\mathbf{p},t). \quad (19)$$

A solution of this equation, satisfying the initial condition $f_k(\mathbf{p},0)=f_k^{(0)}(\mathbf{p})$, is

$$f_{\mathbf{k}}(\mathbf{p},t)=\exp(-i\mathbf{kv}t+\mathbf{H}_0\mathbf{D}t)f_{\mathbf{k}}^{(0)}(\mathbf{p}). \quad (20)$$

Thus, $U_l^*(t)$ is, apart from the factor $\exp(i\mathbf{l}\cdot\mathbf{u}t)$, the operator for the evolution of the distribution function.

On the other hand, in any regular external field, in the absence of collisions, the distribution function can be written in the form

$$f(\mathbf{r},\mathbf{p},t)=f^{(0)}(\mathbf{r}-\Delta\mathbf{r}(t),\ \mathbf{p}-\Delta\mathbf{p}(t)), \quad (21)$$

where $\Delta\mathbf{r}(t)$ and $\Delta\mathbf{p}(t)$ are the changes of the radius vector and of the momentum of the particle within the time t. Taking the Fourier transform, we obtain (21)

$$f_{\mathbf{k}}(\mathbf{p},t)=\exp[-i\mathbf{k}\Delta\mathbf{r}(t)]\exp\left[-\Delta\mathbf{p}(t)\frac{\partial}{\partial\mathbf{p}}\right]f_{\mathbf{k}}^{(0)}(\mathbf{p}), \quad (22)$$

since $\exp\left(-\Delta\mathbf{p}\frac{\partial}{\partial\mathbf{p}}\right)$ is the shift operator in momentum space. In the particular case when $\mathbf{u}=0$ and there is only the magnetic field $\mathbf{H}_0$, this operator takes the form $\exp(H_0\cdot Dt)$. Comparing (18) with (20) and (22), we can write $U_1(\tau)$ in the form

$$U_l(\tau)=\exp[i\mathbf{l}(\Delta\mathbf{r}-\mathbf{u}\tau)]\exp\left[-\Delta\mathbf{p}\frac{\partial}{\partial\mathbf{p}}\right], \quad (23)$$

where $\Delta\mathbf{r}$ and $\Delta\mathbf{p}$ are the changes in the coordinate and momentum of the particle within the time τ in a field $\mathbf{H}_0(\mathbf{r})$, which can be regarded as homogeneous over distances of the order of L_c.

Integrating with respect to l in (17) with the aid of (23), we get

$$\frac{\partial F(\mathbf{r},\mathbf{p},t)}{\partial t}+\mathbf{v}\frac{\partial F(\mathbf{r},\mathbf{p},t)}{\partial \mathbf{r}}=\mathbf{H}_0\mathbf{D}F(\mathbf{r},\mathbf{p},t)$$
$$+\int_0 D_\alpha B_{\alpha\beta}(\mathbf{r},\Delta\mathbf{r}(\tau)-\mathbf{u}\tau)\exp\left[-\Delta\mathbf{p}\frac{\partial}{\partial\mathbf{p}}\right]$$
$$\times D_\beta F(\mathbf{r},\mathbf{p},t-\tau)\,d\tau. \tag{24}$$

The action of the operator $\exp\left(-\Delta\mathbf{p}\,\frac{\partial}{\partial\mathbf{p}}\right)$ on F reduces to replacement of the argument $\mathbf{p}$ by $\mathbf{p}-\Delta\mathbf{p}(\tau)$.

If the momentum of the particle in the field $\mathbf{H}_0(\mathbf{r})$ changes little over the distances of the order L_c, then we can put $\Delta p = 0$ and $\Delta r = vt$. Assuming that the distribution function F changes little within the time $\tau \sim L_c/v$, and putting $F(\mathbf{r},\mathbf{p},t-\tau)\approx F(\mathbf{r},\mathbf{p},t)$, we obtain for times $t \gg L_c/v$

$$\frac{\partial F}{\partial t}+\mathbf{v}\frac{\partial F}{\partial \mathbf{r}}=\mathbf{H}_0\mathbf{D}F+D_\alpha\bar{B}_{\alpha\beta}(\mathbf{r},\mathbf{v})D_\beta F, \tag{25}$$

where

$$\bar{B}_{\alpha\beta}(\mathbf{r},\mathbf{v})=\int_0^\infty B_{\alpha\beta}(\mathbf{r},\mathbf{v}\tau-\mathbf{u}\tau)\,d\tau. \tag{26}$$

This equation corresponds to allowance for only two diagrams—1a and 1c.

For a statistically isotropic random field we get from (11) and (12)

$$\bar{B}_{\alpha\beta}(\mathbf{r},\mathbf{v})=\frac{\sqrt{\pi}L_c\langle H_1^2\rangle}{12|\mathbf{v}-\mathbf{u}|}\left[\delta_{\alpha\beta}+\frac{(\mathbf{v}-\mathbf{u})_\alpha(\mathbf{v}-\mathbf{u})_\beta}{(\mathbf{v}-\mathbf{u})^2}\right]\varphi(\mathbf{r}). \tag{27}$$

When this quantity is substituted in (25), the second term in the square brackets (27) makes no contribution, and Eq. (25) takes the form

$$\frac{\partial F}{\partial t}+\mathbf{v}\frac{\partial F}{\partial \mathbf{r}}=\mathbf{H}_0\mathbf{D}F+\frac{\sqrt{\pi}L_c\langle H_1^2\rangle}{12}\varphi(\mathbf{r})\mathbf{D}|\mathbf{v}-\mathbf{u}|^{-1}\mathbf{D}F. \tag{28}$$

Finally, scattering in a static magnetic field corresponds to $\mathbf{u} = 0$. The energy ϵ of the particle is conserved here, and D takes the form $ec\mathbf{l}/\epsilon$, where $\mathbf{l} = [\mathbf{p}\times\partial/\partial\mathbf{p}]$ is the operator of angular momentum in momentum space. The distribution function satisfies in this case the equation

$$\frac{\partial F}{\partial t}+\mathbf{v}\frac{\partial F}{\partial \mathbf{r}}=\frac{ec}{\varepsilon}\mathbf{H}_0\mathbf{l}F+\frac{\sqrt{\pi}\,e^2c^2L_c\langle H_1^2\rangle}{12\varepsilon^2 v}\varphi(\mathbf{r})\,l^2F. \tag{29}$$

3. SCATTERING IN A STATIC RANDOM MAGNETIC FIELD

Let us consider the case when: a) the particles are emitted from a stationary point source located at the origin; b) there is no regular magnetic field; c) the velocity of the plasma cloud producing the random magnetic field is small ($u \approx 0$); d) the mean-squared random field depends only on the distance to the source, i.e., the function $\varphi(r)$ has spherical symmetry.

The foregoing assumptions allow us to write Eq. (29) in the form

$$\mathbf{v}\frac{\partial F}{\partial \mathbf{r}}-\frac{\sqrt{\pi}\,L_c v}{12R_1^2}\varphi(r)\,l^2F=Q\delta(\mathbf{r}), \tag{30}$$

where Q is the particle-source strength.

We investigate first that region of space around the source, where the particles have not yet been deflected through a large angle from the initial direction. We denote by θ the angle between $\mathbf{r}$ and $\mathbf{p}$ and introduce the dimensionless length $\rho = r/r_0$, where $r_0 = 12c^2p^2/\sqrt{\pi}\,e^2L_c\langle H_1^2\rangle$. In the case when $\theta \ll 1$, Eq. (30) reduces to

$$\frac{\partial F}{\partial\rho}-\frac{1}{\rho}\theta\frac{\partial F}{\partial\theta}=\varphi(\rho)\,\Delta_\theta F+(Q/4\pi r_0^2\rho^2 v)\,\delta(\rho)\,\delta(\theta). \tag{31}$$

Here $\boldsymbol{\theta}$ is a vector perpendicular to $\mathbf{r}$ and Δ_θ is the two-dimensional Laplace operator acting on the components of $\boldsymbol{\theta}$. The Fourier transform of the distribution function

$$F(\rho,\boldsymbol{\eta})=(2\pi)^{-2}\int F(\rho,\boldsymbol{\theta})\exp(-i\boldsymbol{\eta}\boldsymbol{\theta})\,d\boldsymbol{\theta} \tag{32}$$

satisfies the equation

$$\frac{\partial F(\rho,\boldsymbol{\eta})}{\partial\rho}+\frac{1}{\rho}\frac{\partial}{\partial\boldsymbol{\eta}}\boldsymbol{\eta}F(\rho,\boldsymbol{\eta})+\eta^2\varphi(\rho)F(\rho,\boldsymbol{\eta})$$
$$=\frac{Q\delta(\rho)}{16\pi^3 r_0^2\rho^2 v} \tag{33}$$

The change of variables $(\rho,\eta)\rightarrow(\rho,\zeta=\eta/\rho)$ allows us to get rid of the terms containing the derivatives with respect to η:

$$\left[\frac{\partial}{\partial\rho}+\frac{2}{\rho}+\zeta^2\rho^2\varphi(\rho)\right]F(\rho,\zeta)=\frac{Q}{16\pi^3 r_0^2\rho^2 v}\delta(\rho). \tag{34}$$

Equation (34) can be readily integrated. The result takes the form

$$F(\rho,\zeta)=\frac{Q}{16\pi^3 r_0^2\rho^2}\exp\left\{-\zeta^2\int_0^\rho\varphi(s)s^2\,ds\right\}. \tag{35}$$

Taking the inverse Fourier transform, we obtain the particle distribution function

$$F(\rho,\theta)=Q\left(16\pi^2 r_0^2 v\int_0^\rho\varphi(s)s^2\,ds\right)^{-1}\exp\left\{-\theta^2\rho^2/4\int_0^\rho\varphi(s)s^2\,ds\right\}. \tag{36}$$

The condition for the smallness of the scattering angle reduces to the inequality

$$4\int_0^\rho\varphi(s)s^2\,ds\ll\rho^2, \tag{37}$$

which can be satisfied in all of space if $\varphi(\rho)$ decreases like $1/\rho$ or faster. In the case of a homogeneous field ($\varphi = 1$) the solution is applicable in the region $\rho \ll 1$.

If the particle source is nonstationary

$$S = \frac{Q}{4\pi r^2} f(t)\,\delta(r)\,\delta(\theta), \tag{38}$$

then the distribution function will differ from (36) by the factor $f(t - r_0\rho/v)$, which represents the retardation, and the path covered by the particle will coincide in this approximation with r.

Let us consider now that region of space where the angle between **r** and **p** is not small. For this region the solution can be obtained when $\varphi = 1$, i.e., for a homogeneous field.

Let us expand the distribution function in a Fourier integral in ρ. The Fourier transform $F(k, \vartheta)$ satisfies an equation that follows from (30):

$$ik\cos\vartheta F(k, \vartheta) = l^2 F(k, \vartheta) + Q / 32\pi^4 r_0^2 v, \tag{39}$$

where ϑ is the angle between **k** and **v**. We represent $F(k, \vartheta)$ in the form of a series of Legendre polynomials

$$F(k, \vartheta) = \sum_{l=0}^{\infty} (2l+1) F_l(k) P_l(\cos\vartheta). \tag{40}$$

The expansion coefficients $F_l(k)$ satisfy a difference equation in l, obtained from (39):

$$iklF_{l-1} + l(l+1)(2l+1)F_l + ik(l+1)F_{l+1}$$
$$= \delta_{l0} Q / 32\pi^4 r_0^2 v. \tag{41}$$

This equation can be solved with the aid of continued fractions (see, for example,[3]). We put $32\pi^4 r_0^2 v F_l(k)/Q = \mathcal{F}_l(k)$. From (41) we get $\mathcal{F}_1(k) = -i/k$. With the aid of the identity

$$\mathcal{F}_0(k) = \left(1 - \frac{1}{1 + ik\mathcal{F}_1/\mathcal{F}_0}\right)^{-1} - 1 \tag{42}$$

and the relation

$$ik\frac{\mathcal{F}_l}{\mathcal{F}_{l-1}} = k^2\left[(l+1)(2l+1) + ik\frac{(l+1)\mathcal{F}_{l+1}}{l\mathcal{F}_l}\right]^{-1} \tag{43}$$

we can express $\mathcal{F}_0(k)$ in terms of the continued fraction

$$\left(1 + ik\frac{\mathcal{F}_1}{\mathcal{F}_0}\right)^{-1} = \frac{1}{1+}\,\frac{k^2}{6+}\,\frac{4k^2}{30}$$
$$+ \ldots + \frac{l^2k^2}{l(l+1)(2l+1)} + \ldots \tag{44}$$

The continued fraction (44) has positive elements, and, as can be verified, converges for all real values of k. In this case the exact value of the infinite continued fraction lies between the values of any two neighboring convergents of the fraction, i.e., the finite fraction is obtained from (44) by breaking the sequence at one of the elements. Therefore the infinite sequence (44) can be approximated by means of a certain finite continued fraction. Calculating the convergents of different orders, we can readily establish that for $k^2 \lesssim 100$ is sufficient to confine oneself to the fourth approximation:

$$\mathcal{F}_0(k) \approx \mathcal{F}_0^{(4)}(k) = \frac{504 + 13k^2}{k^2(84 + 0.3k^2)}. \tag{45}$$

With increasing k, the convergence of the continued fraction (44) becomes worse and the approximation (45) becomes invalid.

Thus, we have calculated $\mathcal{F}_0(k)$ and $\mathcal{F}_1(k)$. The other coefficients $\mathcal{F}_l$ ($l = 2, 3, \ldots$) are best determined directly in the coordinate representation. To this end we take an inverse Fourier transform of the series (14). Using the formulas from the theory of Bessel functions, we obtain

$$F(\rho, \theta) = (Q/8\pi^3 r_0^2 v)\sum_{l=0}^{\infty}(2l+1)\mathcal{F}_l(\rho)P_l(\cos\theta) \tag{46}$$

where

$$\mathcal{F}_l(\rho) = i^l \int_0^{\infty} \mathcal{F}_l(k) j_l(k\rho) k^2\, dk, \tag{47}$$

j_l is a spherical Bessel function. With the aid of (47) and (45) we get

$$\mathcal{F}_0(\rho) = \frac{3\pi}{\rho}\left(1 + \frac{809}{126}e^{-\rho\sqrt{280}}\right). \tag{48}$$

The exponential term in the region of applicability of the approximation (45) ($\rho \gtrsim 1$) is a small addition and should be neglected. As a result we obtain

$$\mathcal{F}_0(\rho) = 3\pi / \rho, \quad \mathcal{F}_1(\rho) = \pi / 2\rho^2. \tag{49}$$

The remaining $\mathcal{F}_l(\rho)$ are calculated from relations (41), which take in the coordinate representation the form ($l > 0$):

$$l\left(\frac{d}{d\rho} - \frac{l-1}{\rho}\right)\mathcal{F}_{l-1} + l(l+1)(2l+1)\mathcal{F}_l$$
$$+ (l+1)\left(\frac{d}{d\rho} + \frac{l+2}{\rho}\right)\mathcal{F}_{l+1} = 0. \tag{50}$$

We seek the solution of the system (50), which includes (49), in the form

$$\mathcal{F}_l(\rho) = A_l / (2l+1)\rho^{l+1}, \tag{51}$$

where $A_1 = A_0/2 = 3\pi/2$, as follows from (49). From (50) we obtain the recurrence relation $A_l = A_{l-1}/(l-1)!$, which yields

$$A_l = 3\pi / (l+1)! \tag{52}$$

The series (46) takes the simple form

$$F(\rho, \theta) = \frac{3Q}{8\pi^2 r_0^2 v}\sum_{l=0}^{\infty}\frac{P_l(\cos\theta)}{(l+1)!\,\rho^{l+1}}. \tag{53}$$

This formula describes the angular and spatial distribution of the particles in a homogeneous random magnetic field at distances $\rho \gtrsim 1$ from a point-like isotropic source. For comparison we present the form of the distribution function for the same case in the region $\rho \ll 1$ (see (36))

$$F(\rho, \theta) = \frac{3Q}{16\pi^2 r_0^2 v \rho^3} \exp(-3\theta^2/4\rho). \tag{54}$$

We note that the regions of applicability of (53) and (54) do not overlap.

4. DIFFUSION APPROXIMATION

If the dimensions of the system are sufficiently large and the particles have time to become strongly scattered, so that their direction distributions become close to isotropic, then we can use the diffusion approximation. To obtain the diffusion equation in the presence of a random magnetic field, let us expand the distribution function F in a series in Legendre polynomials of the angles of the vector **p**, and confine ourselves to the first two terms of the expansion

$$F(\mathbf{r}, \mathbf{p}, t) = \frac{1}{4\pi}\left[N(\mathbf{r}, p, t) + \frac{3}{v^2}\mathbf{vJ}(\mathbf{r}, p, t)\right]. \tag{55}$$

We assume that the second term in the right side of (55) is much smaller than the first term. Substituting (55) in (28), we obtain the following system of equations for N and **J**

$$\frac{\partial N}{\partial t} + \operatorname{div}\mathbf{J} = \frac{u^2}{9\varkappa_0}\left[p^2\frac{\partial^2 N}{\partial p^2} + \left(1 + \frac{v^2}{c^2}\right)p\frac{\partial N}{\partial p}\right] + \frac{1}{3\varkappa_0}\mathbf{u}\left(p\frac{\partial \mathbf{J}}{\partial p} + \frac{v^2}{c^2}\mathbf{J}\right) + \frac{1}{R_{H_0}v}[\mathbf{un}_1]\left[p\frac{\partial \mathbf{J}}{\partial p} + \left(1 + \frac{v^2}{c^2}\right)\mathbf{J}\right], \tag{56}$$

$$\mathbf{J} + \frac{\Lambda}{R_{H_0}}[\mathbf{n}_1\mathbf{J}] = -\varkappa_0\frac{\partial N}{\partial \mathbf{r}} - \frac{p}{3}\frac{\partial N}{\partial p}\left(\mathbf{u} + \frac{\Lambda}{R_{H_0}}[\mathbf{n}_1\mathbf{u}]\right) - \frac{\Lambda}{v}\frac{\partial \mathbf{J}}{\partial t}. \tag{57}$$

Here $\mathbf{n}_1 = \mathbf{H}_0/H_0$, $R_{H_0} = cp/eH_0$ is the Larmor radius of the particle in the regular field, and

$$\varkappa_0(\mathbf{r}, p) = \frac{v\Lambda}{3},$$
$$\Lambda = \frac{6c^2p^2}{\sqrt{\pi}\, e^2 L_c \langle H_1^2\rangle \varphi(\mathbf{r})}. \tag{58}$$

The terms of order $(u/v)^3N$ and u^2J/v^3 have been discarded.

The last term in (57) can be discarded if $\Lambda \ll vT$, where T is the time in which **J** experiences a noticeable change. Equation (57) can then be solved with respect to **J**. We direct the coordinate axes along the vectors $\mathbf{n}_1$, $\mathbf{n}_2 = \mathbf{u} \times \mathbf{n}_1$, and $\mathbf{n}_2 = \mathbf{n}_2 \times \mathbf{n}_1$. In terms of these axes we have

$$J_\alpha = -\bar{\varkappa}_{\alpha\beta}\frac{\partial N}{\partial x_\beta} - \frac{p}{3}\frac{\partial N}{\partial p}u_\alpha, \tag{59}$$

where $\bar{\kappa}_{11} = \kappa_0$, $\bar{\kappa}_{22} = \bar{\kappa}_{33} = \kappa_0 R_{H_0}^2 (R_{H_0}^2 + \Lambda^2)$, $\bar{\kappa}_{32} = -\bar{\kappa}_{23} = \kappa_0 \Lambda R_{H_0} (R_{H_0}^2 + \Lambda^2)^{-1}$, and the remaining $\kappa_{\alpha\beta} = 0$. In the absence of a regular magnetic field, (59) goes over into

$$\mathbf{J} = -\varkappa_0\frac{\partial N}{\partial \mathbf{r}} - \frac{p}{3}\frac{\partial N}{\partial p}\mathbf{u}. \tag{60}$$

In this case Λ has the meaning of the transport mean free path, and κ_0 that of a scalar diffusion coefficient.

Substitution of (59) in (56) yields a diffusion equation for the function N(**r**, p, t)—the concentration of particles with specified p:

$$\frac{\partial N}{\partial t} = \nabla_\alpha \varkappa_{\alpha\beta}\nabla_\beta N - \mathbf{u}\nabla N + \frac{1}{3}(\nabla\mathbf{u})p\frac{\partial N}{\partial p}. \tag{61}$$

Equation (61) has been written out in invariant form and is valid for an arbitrary dependence of **u** on **r** ($u \ll c$) and for any orientation of the coordinate axes, if the tensor $\kappa_{\alpha\beta}$ is transformed to the corresponding axes.

Equation (28), from which (61) is derived, is valid if the angle of deflection of the particle over the length L_c is small. However, (61) remains in force also when the random component of the magnetic field is produced by magnetized plasma clouds[1)], each of which can deflect the particle to an arbitrary angle. All that changes here is the definition of Λ: in place of (58) we obtain from the kinetic equation

$$\Lambda^{-1} = \sum_i C_i(\mathbf{r})g_i(p), \tag{62}$$

where $C_i(\mathbf{r})$ is the concentration of magnetic clouds of sort i,

$$g_i(p) = \int (1 - \cos\theta)\, d\sigma_i \tag{63}$$

is the transport cross section of a particle in one cloud, θ is the scattering angle, and $d\sigma_i$ is the differential scattering cross section.

In the spherically-symmetrical case, when $H_0 = 0$, $\mathbf{u} = u\mathbf{r}/r$, and u = const, Eq. (61) takes the form

$$\frac{\partial N}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\varkappa_0\frac{\partial N}{\partial r} - u\frac{\partial N}{\partial r} + \frac{2u}{3r}p\frac{\partial N}{\partial p}. \tag{64}$$

It should be noted that the diffusion equations ob-

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1)The model of magnetized clouds is frequently considered in the theory of motion of cosmic particles (see [4]).

tained in the present work, and particularly (64), differ from the equations derived by Parker[1,5]. The difference is due to the fact that Parker did not allow correctly for the motion of the clouds and is discussed in greater detail in our papers[6,7], where a solution is also obtained for Eq. (64) with a constant diffusion coefficient.

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