TUNNEL EFFECT BETWEEN SUPERCONDUCTORS IN AN ALTERNATING FIELD

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We obtain for the Josephson current an expression which is not based on the assumption that the voltage applied to the contact is small and varies at a slow rate. Various limiting cases are considered.

1. A tunnel current can flow between two superconductors separated by a dielectric barrier even when the voltage on the barrier is zero. If a voltage V(t) is applied to the barrier, then the current becomes alternating. The usual expression for the tunnel Josephson current is[1]

$$j(t) = j_{max}\sin\Big(\int^{t} 2V(t')\,dt'\Big). \qquad (1)$$

This formula is valid only when the voltage and its frequency are small compared with the gap in the superconductor spectrum. Such conditions are frequently not fulfilled during experiments.

We derive below a general expression for the current through the barrier for arbitrary V(t) and for all temperatures. We consider the limiting cases of slow and fast variation of V(t), and also the case of small V with arbitrary time dependence. In all these cases the expression for the current reduces to four single integrals, for which the limiting values are obtained.

2. The voltage on the Josephson element connected in a certain circuit is not specified, and depends itself on the magnitude of the current and on the impedance of the remainder of the circuit. The problem of finding the current in the circuit breaks up into two. First it is necessary to find the current flowing through the Josephson element for a given voltage, i.e., its current-voltage characteristic. Then, knowing this characteristic and the impedance of the remaining circuit, we can find the current in the circuit. We consider below only the first problem, and the results are directly applicable only for a sufficiently narrow contact. In the case of broad contacts, or in the study of the emission of electromagnetic waves, it is necessary to take into account the variation of the voltage and current over the width of the contact.[2-4] To solve this waveguide problem, we can use the results that follow, which give the local connection between the current and the voltage at a given point of the contact.

At finite temperatures and for a large potential difference, both the superconducting and normal currents flow through the contact. In this case the voltage varies not only on the contact, but also inside the superconductors. The permittivity of the dielectric film is exponentially small, and if the voltage drop across it is of the order of the magnitude of the gap, then the variation of the potential inside the superconductor, over distances of the order of the pair dimension, can be neglected.

In the zeroth approximation in the barrier penetrability, each superconductor is situated in a potential that is variable in time but constant in space. The Hamiltonian of the system has in this approximation the form

$$H^0 = H_1 + H_2 + V_1(t)\sum_{\nu_1} a^+_{\nu_1} a_{\nu_1} + V_2(t)\sum_{\nu_2} a^+_{\nu_2} a_{\nu_2},$$

where the indices ν_1 and ν_2 describe the single-electron states of the first and second superconductor, and H_1 and H_2 are respectively the Hamiltonians of these superconductors and do not depend explicitly on the time. The potential V(t) leads to no physical effects, and its influence reduces to the appearance of trivial phase factors in the operators $\tilde{a}(t)$ and the Green's function[5]

$$G(t, t', \nu_{1,2}) = -i\langle T(\tilde{a}_{\nu_{1,2}}(t)\tilde{a}^+_{\nu_{1,2}}(t'))\rangle$$

$$= \exp\Big[-i\int_{t'}^{t} V_{1,2}(t_1)\,dt_1\Big]\int \frac{d\omega}{2\pi} G_{1,2}(\omega)\exp[-i\omega(t-t')],$$

$$F^+(t, t', \nu_{1,2}) = \langle T(\tilde{a}^+_{\nu_{1,2}}(t)\tilde{a}^+_{\nu_{1,2}}(t'))\rangle$$

$$= \exp\Big[i\int^{t} V_{1,2}(t_1)\,dt_1 + i\int^{t'} V_{1,2}(t_1)\,dt_1\Big]$$

$$\times \int \frac{d\omega}{2\pi} F^+_{1,2}(\omega)\exp[-i\omega(t-t')], \qquad (3)$$

where the brackets $\langle\ \rangle$ denote averaging over the

Gibbs distribution with Hamiltonians H_1 and H_2, and $G_{1,2}(\omega)$ and $F^+_{1,2}(\omega)$ are the Green's function without the external field.

The penetration of the electrons through the potential barrier can be analyzed with the aid of a tunnel Hamiltonian, making the total Hamiltonian in the representation of the states ν_1 and ν_2:

$$H = H_0 + \hat{T}, \qquad \hat{T} = \sum_{\nu_1\nu_2} (T_{\nu_1\nu_2} a_{\nu_1}^+ a_{\nu_2} + T^*_{\nu_1\nu_2} a_{\nu_2}^+ a_{\nu_1}). \quad (4)$$

The operator $a^+_{\nu_1} a_{\nu_2}$ transforms the electron from the state ν_2 on one side of the barrier to the state ν_1 on the other side.

It was shown in [6], with the case $V = 0$ as an example, that the results of such an analysis coincide with the solution of the equations for the Green's functions for superconductors separated by a potential barrier.

3. The change in the number of electrons on one side of the barrier is described by the operator

$$\dot{N} = \frac{\partial}{\partial t}\left\{\exp\left[i\int^t H\,dt_1\right]\sum_{\nu_1} a_{\nu_1}^+ a_{\nu_1} \exp\left[-i\int^t H\,dt_1\right]\right\}. \quad (5)$$

In second order in $\hat{T}$ we obtain

$$\dot{N} = \int_{-\infty}^{t} [\hat{T}(t_1)[\hat{N}(t)\hat{T}(t)]]\,dt_1.$$

Averaging this expression over the Gibbs distribution with Hamiltonian $H_1 + H_2$, we obtain for the current through the barrier

$$J(t) = -4\,\mathrm{Re}\sum_{\nu_1\nu_2}|T_{\nu_1\nu_2}|^2 \int_{-\infty}^{t} \{[F_{\nu_1}{}^*(t, t_1)\, F_{\nu_2}{}^{+*}(t, t_1) - F_{\nu_1}{}^+(t, t_1) F_{\nu_2}(t, t_1)]' + [G_{\nu_1}(t_1, t) G_{\nu_2}(t, t_1) - G_{\nu_1}{}^*(t, t_1) G_{\nu_2}{}^*(t_1, t)]\}\,dt_1. \quad (6)$$

The Green's functions depend only on the energies ϵ_ν and have a sharp maximum near the Fermi surface. The transition probability averaged over all the states varies slowly with the energy, and is expressed near the Fermi surface in terms of the resistance of the layer between the normal metals:

$$R^{-1} = 4\pi \sum_{\nu_1\nu_2} |T_{\nu_1\nu_2}|^2 \delta(\varepsilon_{1\Phi} - \varepsilon_{\nu_1})\delta(\varepsilon_{2\Phi} - \varepsilon_{\nu_2}). \quad (7)$$

Taking (7) into account, the general expression for the current through the contact takes the form

$$RJ(t) = \frac{1}{2\pi^3}\,\mathrm{Re}\int_{-\infty}^{0} i\,dt_1 \Big\{ \iint_{-\infty}^{\infty} d\omega_1\, d\omega_2 \exp\Big[-i\int^t V(t')\,dt' - i\int^{t+t_1} V(t')\,dt'\Big] \times \exp[-i(\omega_1+\omega_2)t_1]\,\mathrm{Im}(F_1{}^+(\omega_1)F_2(\omega_2)) - \iint_{-\infty}^{\infty} d\omega_1\, d\omega_2 \exp\Big[i\int_t^{t+t_1} V(t')\,dt'\Big]\cdot\exp[-i(\omega_1-\omega_2)t_1] \times \mathrm{Im}(G_1(\omega_1)G_2(\omega_2))\Big\}. \quad (8)$$

Here

$$V(t) = V_2(t) - V_1(t). \quad (9)$$

The Green's functions integrated over the energies

$$G_{1,2}(\omega) = \int d\varepsilon_{1,2} G(\omega, \nu_{1,2}), \quad F^+_{1,2}(\omega) = \int d\varepsilon_{1,2} F^+(\omega, \nu_{1,2}) \quad (10)$$

have the same form as for an infinite superconductor, and do not depend on the presence of nonmagnetic impurities and on the character of reflection of the electron from the surface.

4. To simplify the general expression for the current, assumptions must be made concerning the character of the time dependence of the potential. In the integral (8), the important frequencies are $\omega \sim T_c$, and consequently $t_1 \sim T_c^{-1}$. Therefore if the frequency of variation of the potential is small compared with the transition temperature in the interval from t to $t + t_1$, the function $V(t')$ can be replaced by its value at the point t. As a result we obtain

$$RJ^0 = I_1 \sin\Big(\int^t 2V(t_1)\,dt_1\Big) + I_2\cos\Big(\int^t 2V(t_1)\,dt_1\Big) + I_3, \quad (11)$$

where

$$I_1 = \frac{\mathrm{Re}}{2\pi^3}\iint_{-\infty}^{\infty} i[\omega_1+\omega_2+V+i\delta]^{-1}\,\mathrm{Im}(F_1{}^+(\omega_1) \times F_2(\omega_2))\,d\omega_1\,d\omega_2,$$

$$I_2 = -\frac{\mathrm{Re}}{2\pi^3}\iint_{-\infty}^{\infty} [\omega_1+\omega_2+V+i\delta]^{-1}\,\mathrm{Im}(F_1{}^+(\omega_1) \times F_2(\omega_2))\,d\omega_1\,d\omega_2,$$

$$I_3 = \frac{\mathrm{Re}}{2\pi^3}\iint_{-\infty}^{\infty} [\omega_1-\omega_2-V+i\delta]^{-1}\,\mathrm{Im}(G_1(\omega_1)G_2(\omega_2))\,d\omega_1\,d\omega_2,$$

$$I_4 = \frac{\mathrm{Re}}{2\pi^3}\iint_{-\infty}^{\infty} i[\omega_1-\omega_2-V+i\delta]^{-1}\,\mathrm{Im}(G_1(\omega_1)G_2(\omega_2))\,d\omega_1\,d\omega_2. \quad (12)$$

The integral I_4 does not enter in (11), but it will be useful in what follows. An expression similar to (11) was obtained by another method in [7].

Sometimes the next term of the adiabatic approximation is important. To calculate it we expand expression (8) for the current up to first order in $\partial V/\partial t$, making the substitution

$$\exp\Big[i\int_{t'}^{t} V(t_1)\,dt_1\Big] \to \exp[iV(t)(t-t')]\Big[1 + \frac{i}{2}\frac{\partial V}{\partial t}(t-t')^2\Big]. \quad (13)$$

The first term of this expression yields formula (11), and the second is equal to

$$RJ^{(1)}=\frac{1}{2}\frac{\partial V}{\partial t}\left[\frac{\partial^2 I_2}{\partial V^2}\sin\left(\int^{t}2V(t_1)\,dt_1\right)\right.$$
$$\left.-\frac{\partial^2 I_1}{\partial V^2}\cos\left(\int^{t}2V(t_1)\,dt_1\right)+\frac{\partial^2 I_4}{\partial V^2}\right]. \qquad (14)$$

It is of interest to consider the case when in addition to the slowly-varying potential there is a small rapidly varying potential. We denote the slowly varying part of the potential by $V_s(t)$, and the rapidly varying part by

$$V_r(t)=\int_{-\infty} V(\omega)\exp[-i\omega t]\frac{d\omega}{2\pi}. \qquad (15)$$

Expanding the right side of (8) in powers of $V(\omega)$, we obtain the increment of the current

$$RJ=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi\omega}\left\{\mathrm{Re}\left(V(\omega)\exp\left[-i\left(\omega t+2\int^{t}V_s(t_1)\,dt_1\right)\right]\right)\right.$$
$$\times[I_2(V_s+\omega)-I_2(V_s)]-\mathrm{Im}\left(V(\omega)\right.$$
$$\left.\times\exp\left[-i\left(\omega t+2\int^{t}V_s(t_1)\,dt_1\right)\right]\right)[I_1(V_s+\omega)-I_1(V_s)]$$
$$-\mathrm{Re}(V(\omega)\exp(-i\omega t))[I_3(V_s-\omega)-I_3(V_s)]$$
$$\left.-\mathrm{Im}(V(\omega)\exp(-i\omega t))[I_4(V_s-\omega)-I_4(V_s)]\right\}. \qquad (16)$$

In the principal terms of (11) and (14) it is necessary here to replace V by V_s only in I_n. Formula (16) is valid not only when $V_r \ll \Delta$, but also for large ω in a broader region $V_r \ll \omega$.

Thus, for both rapid and slow variation of the potential, the current is expressed in terms of the four integrals of (12).

5. The integrals of (12) can be reduced to single integrals. For I_1 and I_4 we obtain in trivial fashion

$$I_1=\frac{1}{2\pi^2}\int_{-\infty}^{\infty}\mathrm{Im}(F_1^+(\omega)F_2(\omega+V))\,d\omega,$$

$$I_4=\frac{1}{2\pi^2}\int_{-\infty}^{\infty}\mathrm{Im}(G_1(\omega)G_2(\omega-V))\,d\omega. \qquad (17)$$

To calculate the integrals I_2 and I_3, we use the dispersion relations[5] between the time-dependent Green's function and the functions G^R and $\mathcal{F}$, which are obtained by analytically continuing the temperature functions to the real axis of the variable $i\omega_n$:

$$G(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1-\mathrm{th}(x/2T)}{x-\omega+i\delta}+\frac{1+\mathrm{th}(x/2T)}{x-\omega-i\delta}\right]\mathrm{Im}\,G^R(x)\,dx,$$

$$iF^+(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1-\mathrm{th}(x/2T)}{x-\omega+i\delta}\right.$$
$$\left.+\frac{1+\mathrm{th}(x/2T)}{x-\omega-i\delta}\right]\mathrm{Im}\,\mathcal{F}^+(-ix)\,dx. \qquad (18^*)$$

Substituting these expressions in (12), we obtain

$$I_2=\frac{1}{2\pi^2}\int_{-\infty}^{\infty}\left[\mathrm{th}\frac{x}{2T}-\mathrm{th}\frac{x+V}{2T}\right]\mathrm{Im}\,\mathcal{F}_1^+(-ix)$$
$$\times\mathrm{Im}\,\mathcal{F}_2^+(i(x+V))\,dx,$$
$$I_3=\frac{-1}{2\pi^2}\int_{-\infty}^{\infty}\left[\mathrm{th}\frac{x-V}{2T}-\mathrm{th}\frac{x}{2T}\right]\mathrm{Im}\,G_1^R(x)$$
$$\times\mathrm{Im}\,G_2^R(x-V)\,dx. \qquad (19)$$

The obtained expressions are valid also for superconductors with paramagnetic impurities.

6. We consider first superconductors without magnetic impurities. In this case

$$G^R(\omega)=-\frac{\pi\omega}{\sqrt{\Delta^2-(\omega+i\delta)^2}},\quad \mathcal{F}(-i\omega)=\frac{\pi\Delta}{\sqrt{\Delta^2-(\omega+i\delta)^2}}; \qquad (20)$$

$$G(\omega)=\mathrm{Re}\,G^R(\omega)+i\,\mathrm{th}\frac{\omega}{2T}\mathrm{Im}\,G^R(\omega),$$
$$iF^+(\omega)=-iF(\omega)=\mathrm{Re}\,\mathcal{F}(-i\omega)$$
$$+i\,\mathrm{th}\,(\omega/2T)\,\mathrm{Im}\,\mathcal{F}(-i\omega). \qquad (21)$$

Substituting these expressions in (17) and (19), we obtain

$$I_1=\frac{\Delta_1\Delta_2}{2}\int_{-\infty}^{\infty}\left[\frac{\theta(\Delta_1-|\omega-V|)\theta(|\omega|-\Delta_2)}{\sqrt{\Delta_1^2-(\omega-V)^2}\sqrt{\omega^2-\Delta_2^2}}\right.$$
$$\left.-\frac{\theta(|\omega|-\Delta_1)\theta(\Delta_2-|\omega+V|)}{\sqrt{\omega^2-\Delta_1^2}\sqrt{\Delta_2^2-(\omega+V)^2}}\right]\mathrm{th}\frac{|\omega|}{2T}\,d\omega$$

$$I_2=-\frac{\Delta_1\Delta_2}{2}\int_{-\infty}^{\infty}\left(\mathrm{th}\frac{\omega}{2T}-\mathrm{th}\frac{\omega+V}{2T}\right)$$
$$\times\frac{\mathrm{sign}\,\omega\cdot\mathrm{sign}(\omega+V)\theta(|\omega|-\Delta_1)\theta(|\omega+V|-\Delta_2)}{\sqrt{\omega^2-\Delta_1^2}\sqrt{(\omega+V)^2-\Delta_2^2}}\,d\omega$$

$$I_3=-\frac{1}{2}\int_{-\infty}^{\infty}\left(\mathrm{th}\frac{\omega-V}{2T}-\mathrm{th}\frac{\omega}{2T}\right)|\omega||\omega-V|$$
$$\times\frac{\theta(|\omega|-\Delta_1)\theta(|\omega-V|-\Delta_2)}{\sqrt{\omega^2-\Delta_1^2}\sqrt{(\omega-V)^2-\Delta_2^2}}\,d\omega$$

*th ≡ tanh.

$$I_4 = \frac{1}{2}\int_{-\infty}^{\infty} |\omega|\,\mathrm{th}\,\frac{\omega}{2T}\Big[\frac{(\omega - V)\theta(|\omega| - \Delta_1)\theta(\Delta_2 - |\omega - V|)}{\sqrt{\omega^2 - \Delta_1^2}\sqrt{\Delta_2^2 - (\omega - V)^2}} + \frac{(\omega + V)\theta(|\omega| - \Delta_2)\theta(\Delta_1 - |\omega + V|)}{\sqrt{\omega^2 - \Delta_2^2}\sqrt{\Delta_1^2 - (\omega + V)^2}}\Big]d\omega. \qquad (22)$$

At zero temperature and for $V < \Delta_1 + \Delta_2$ different from zero, only the first integral, which describes the superconducting Josephson current, differs from zero:

$$I_1 = \begin{cases} \dfrac{2\Delta_1\Delta_2}{\sqrt{(\Delta_1+\Delta_2)^2 - V^2}} K\left(\sqrt{\dfrac{(\Delta_1-\Delta_2)^2 - V^2}{(\Delta_1+\Delta_2)^2 + V^2}}\right), & |V| < |\Delta_1 - \Delta_2| \\ \sqrt{\Delta_1\Delta_2}\,K[(V^2 - (\Delta_1-\Delta_2)^2)^{1/2}(4\Delta_1\Delta_2)^{-1/2}], & |V| > |\Delta_1 - \Delta_2|. \\ \dfrac{2\Delta_1\Delta_2}{\sqrt{V^2 - (\Delta_1-\Delta_2)^2}} K\left(\sqrt{\dfrac{4\Delta_1\Delta_2}{V^2 - (\Delta_1-\Delta_2)^2}}\right), & |V| > \Delta_1 + \Delta_2 \end{cases} \qquad (23)$$

When $V = 0$ we obtain the well known expression of [6]; when V is close to $\Delta_1 + \Delta_2$, the amplitude of the Josephson current becomes logarithmically large. When $V > \Delta_1 + \Delta_2$, the normal current appears abruptly:[7]

$$I_3(\Delta_1 + \Delta_2) = {}^1\!/_2\pi\sqrt{\Delta_1\Delta_2}. \qquad (24)$$

At finite temperatures, the integrals (22) can be calculated only in limiting cases, When $V \gg \Delta$ we have

$$I_1 = \frac{\pi\Delta_1\Delta_2}{V}, \quad I_2 = -\frac{2\Delta_1\Delta_2}{V}\ln\left(\frac{V}{\sqrt{\Delta_1\Delta_2}}\right),$$

$$I_3 = V\left[1 - \frac{\Delta_1^2 + \Delta_2^2}{2V^2}\right], \quad I_4 = -\frac{\pi(\Delta_1\Delta_2)^2}{2V^3}. \qquad (25)$$

For V close to $\Delta_1 + \Delta_2$, the integrals (22) have a singularity. The singular parts of these integrals are

$$I_1 = -I_4 = \frac{\sqrt{\Delta_1\Delta_2}}{4}\left(\mathrm{th}\frac{\Delta_1}{2T} + \mathrm{th}\frac{\Delta_2}{2T}\right) \times \ln\left(\frac{\Delta_1 + \Delta_2}{|V - (\Delta_1 + \Delta_2)|}\right),$$

$$I_3(+) - I_3(-) = -(I_2(+) - I_2(-)) = \frac{\pi\sqrt{\Delta_1\Delta_2}}{4}\left(\mathrm{th}\frac{\Delta_1}{2T} + \mathrm{th}\frac{\Delta_2}{2T}\right). \qquad (26)$$

When $V \ll \Delta$ or T, the integrals of (22) can be calculated only for identical superconductors on both sides of the barrier, i.e., when $\Delta_1 = \Delta_2$. In this case

$$I_1 = I_4 = \frac{\pi\Delta}{2}\mathrm{th}\frac{\Delta}{2T};$$

$$I_2 = I_3 = \frac{\Delta}{4\,\mathrm{ch}^2(\Delta/2T)}\left(\frac{V}{T}\right)\ln\left(\frac{\min(T,\Delta)}{V}\right). \qquad (27)^*$$

*ch ≡ cosh.

Thus, for a potential V that is large or has a high frequency, the main contribution to the current is made by the first term in I_3 from formula (25). Substitution of this term in (11) and (16) leads to the same connection between the current and the voltage as in the normal metal: $RJ = V$. The Josephson current, which has in this case a frequency strongly different from the frequency of the potential, will be determined by integrals I_1 and I_2 of (25). For large V it tends to zero like $V^{-1}\ln V$.

When $V = \Delta_1 + \Delta_2$, the current acquires a singularity connected with the fact that pair breaking by the electric field becomes possible, and one electron can go through the barrier. When $V < \Delta_1 + \Delta_2$, the normal current at low temperatures is exponentially small, and when $V > \Delta_1 + \Delta_2$, a large normal current appears jumpwise. The jump is described by formula (26) and does not become smeared out with change in temperature. A similar jump occurs in the integral I_1. The Josephson alternating current becomes logarithmically large when V is close to $\Delta_1 + \Delta_2$. A similar singularity, as follows from (16), arises in a weak rapidly-alternating field at a frequency $\omega = \Delta_1 + \Delta_2$. In this case the pairs break and a quantum of alternating electromagnetic field is absorbed. If there is a constant potential V and a weak alternating field of frequency ω, then a singularity arises at $V + \omega = \Delta_1 + \Delta_2$. Usually the alternating field is connected with the existence of an alternating Josephson current and has a frequency $\omega = 2V$. Thus, the singularity should exist at $3V = \Delta_1 + \Delta_2$. For small V and ω, the principal term of the current has the usual form (1), and formulas (14) and (16) lead to small corrections. It is interesting to note that the normal current I_3 which is exponentially small at low temperatures, is proportional to $V \ln V$ for small values of V, i.e., the resistance is logarithmically small. Such a singularity is connected with the high density of states at energies close to the gap in the spectrum. It exists only for identical superconductors, for only in this case can the electron have an energy close to the gap in the spectrum before and after the transition.

7. The general formulas (17) and (19) obtained above are applicable also for superconductors with paramagnetic impurities. In this case the Green's functions are more complicated[8] and can be expressed in parametric form:

$$G^R(\omega) = -\frac{\pi u}{\sqrt{1-u^2}}, \quad \mathcal{F}(-i\omega) = \frac{\pi}{\sqrt{1-u^2}},$$

$$\frac{\omega}{\Delta} = u\left(1 - \frac{1}{\tau_s\Delta\sqrt{1-u^2}}\right) \quad (28)$$

We note that the same formulas are valid also in the case of thin contaminated films situated in a magnetic field.[9] The parameter $(\tau_s\Delta)^{-1}$ is replaced by $2\tau\langle\langle(\mathbf{p}\cdot\mathbf{A})^2\rangle\rangle/\Delta$.

Substituting (28) in (21), (17), and (19) we can obtain a general expression for the tunnel current in this case, too. At a magnetic-impurity concentration which is not close to the critical one at which the gap of the spectrum disappears, the results do not differ qualitatively from the impurity-free case.

The most interesting difference in the gapless region is that at low temperatures and for a small potential difference, a noticeable normal current flows in this case. From formula (19) for I_3 it follows that when T or V $\ll \Delta$ this current is proportional to the product of the densities of states in the left and right superconductors, and is equal to

$$I_3 = V[(1-(\tau_s\Delta)_1{}^2)(1-(\tau_s\Delta)_2{}^2)]^{1/2}. \quad (29)$$

The superconducting Josephson current is determined by the integral I_1, which for small T and V is equal to

$$I_1 = \Delta\left\{\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{(\tau_s\Delta)^2} - 1\right)^{1/2} - \frac{2}{3\tau_s\Delta} + \frac{(1-(\tau_s\Delta)^2)^{1/2}}{\tau_s\Delta} - \frac{(1-(\tau_s\Delta)^2)^{3/2}}{3\tau_s\Delta}\right\} - \frac{T^2\pi^2}{3\Delta}\frac{(\tau_s\Delta)^3}{\sqrt{1-(\tau_s\Delta)^2}} \quad (30)$$

The first term of this expansion was obtained earlier by Kulik.[10]

8. The derived formulas (11), (14), and (16) are the boundary conditions for the Josephson effect. They relate the current density at some location on the contact with the potential difference at the same location. The potential varies little over distances of the order of the pair dimension, so that formulas (11), (14), and (16) give the correct local connection between the current and the potential. To determine the total current through the contact, it is necessary to solve the waveguide equation, which can be obtained from the current-conservation law:[3]

$$J(x,y,t) = \left[\frac{c^2}{4\pi(\lambda_1+\lambda_2)}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - \frac{\varepsilon}{4\pi d}\frac{\partial^2}{\partial t^2}\right]\varphi(x,y,t),$$

where $\varphi = \int^t V(x, y, t')dt'$, d is the thickness of the barrier, and the current is connected with the potential by formulas (11), (14), and (16). The solution of this equation is a separate problem, which will not be dealt with here. We therefore do not compare our results with experiment and confine ourselves only to general remarks.

The usual expression for the Josephson current (1) is valid only for a small and slowly-varying potential. If the potential V or its frequency ω are comparable with the gap Δ, then deviations should be observed, and singularities should occur in the value of the current when $V_S + \omega = \Delta_1 + \Delta_2$.[11] When $V \gg \Delta$ or $\omega \gg \Delta$, the principal term in the current coincides with the current in the normal state. But a small superconducting Josephson current exists even in this case.

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