QUASILINEAR RELAXATIONS DYNAMICS OF A COLLISIONLESS PLASMA

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The paper is devoted to the study of dynamics of the quasilinear relaxation of unstable states of collisionless plasma. A self-similar solution of the quasilinear equation is found for the case when a small group of electrons has at the initial instant of time a high velocity in comparison with the other electrons (i.e., a weak beam is present in the plasma). The distribution function has the shape of a step with a steep front, moving towards lower velocities. The process is shown to exhibit such a behavior for a wide class of initial distribution functions, and an equation is derived for the velocity of the steep front of the wave. The stationary distribution function and the Langmuir-oscillation spectrum are determined as functions of the coordinates in the case of stationary injection of a beam in a half-space occupied by a plasma.

1. IN the first papers on the quasilinear theory of plasma oscillations[1,2] it was shown that one-dimensional plasma oscillations that grow in time modify the initial unstable distribution function in such a way that the growth of the oscillation amplitude in time ceases. As a result, the initially unstable distribution function becomes stable, and noise at a level much higher than thermal is produced in appropriate regions of wave-number space. If there are practically no collisions between particles, this state is stationary. All the parameters of the particle-plus-wave system in the final state can be determined by using the conservation laws that follow from the quasilinear equations. The process producing the transition to the stationary state was never considered. We investigate below the laws governing the time variation of the distribution function during the course of the quasilinear relaxation process.

We start with a system of quasilinear equations for one dimensional Langmuir oscillations. The initial electron distribution function is chosen to be that shown in Fig. 1. The second small peak of the distribution function signifies that a beam of low-velocity electrons moves through the plasma with velocity v_0. The beam velocity is much larger than the thermal velocity scatter of the plasma electrons c_e. It is known that such a distribution function is unstable. According to linear theory, the instability has a different character in two limiting cases.[3] When the beam is dense and almost monoenergetic, oscillations grow in the plasma, and frequency and increment are determined by the parameter of the entire system. On the other hand, if the velocity scatter in the beam c_{1e} is not too small, and the beam density n_1 is not too large compared with the plasma density n_0, namely,

$$c_{1e}/v_0 \gtrsim (n_1/n_0)^{1/3},$$

then the beam excites Langmuir oscillations with an increment which is determined by introducing the distribution function of the beam in the vicinity of the point $v = v_0$. To describe the relaxation of the unstable state to the stable ones in the second limiting case we can use the equations of the quasilinear theory

$$\frac{\partial F}{\partial t} = \frac{\omega_0^2}{2n_0 m}\frac{\partial}{\partial v}\left(\int \varepsilon_k \delta(\omega - kv)\,dk\frac{dF}{\partial v}\right), \qquad (1)$$

$$\frac{\partial \varepsilon_k}{\partial t} = \pi\frac{\omega_0^3}{k^2}\varepsilon_k\int k\frac{\partial F}{\partial v}\delta(\omega - kv)dv, \qquad \omega = \omega_0 = \sqrt{\frac{4\pi n_0 e^2}{m}}, \qquad (2)$$

$$\int F\,dv = n_1/n_0, \qquad (3)$$

where F is the distribution function of the beam-electron velocities, normalized in accordance with relation (3), ϵ_k is the spectral density of the energy of the Langmuir oscillations (noise), and ω and k are the frequency and the wave number.[4] For the problem considered below, the thermal correction to the frequency of the Langmuir oscillations $\omega = \omega_0$ is immaterial. By virtue of the uniqueness of the connection between k and e, which follows from the relation $\omega_0 = kv$, we can choose for the function ϵ_k the velocity v as the independent variable in lieu of k. Then the spectral density in k-space will be connected with the spectral density $\epsilon(v)$ in the following fashion: $\epsilon_k = \epsilon(v)\omega_0/v^2$.

If we now introduce convenient dimensionless variables and functions

$$f=\pi\frac{n_0}{n_1}v_0F,\quad w=\frac{\varepsilon_k\omega_0}{2n_1mv^3},\quad \tau=\omega_0\frac{n_1}{n_0}t,\quad V=\frac{v}{v_0},$$

then (1), (2), and (3) take the following form

$$\frac{\partial f}{\partial\tau}=\frac{\partial}{\partial V}\left(wV^2\frac{\partial f}{\partial V}\right),\qquad(4)$$

$$\frac{\partial w}{\partial\tau}=wV^2\frac{\partial f}{\partial V},\qquad(5)$$

$$\frac{1}{\pi}\int f\,dV=1.\qquad(6)$$

From (4) and (5) follows an important simple relation between the distribution function and the noise:

$$f-f_0=\frac{\partial}{\partial V}(w-w_0),\qquad(7)$$

where f_0 and w_0 are the distribution function and the value of w at the initial instant of time. To obtain this relation, we must substitute $mV^2\partial f/\partial V$ from (5) in (4) and integrate with respect to τ.

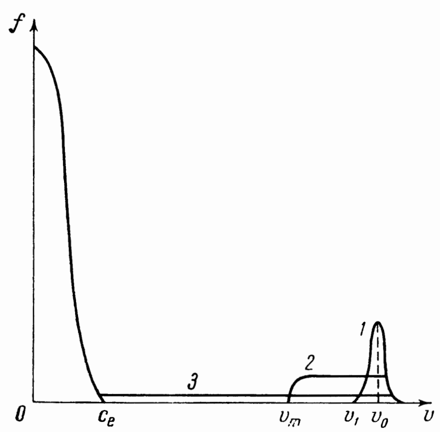

FIG. 1

With the aid of (7) we can eliminate f from (5) and obtain an equation containing only $w(V,\tau)$:

$$\frac{\partial w}{\partial\tau}=V^2w\frac{\partial^2(w-w_0)}{\partial V^2}+Vw\frac{\partial f_0}{\partial V}.\qquad(8)$$

In the region where $\partial f_0/\partial V=0$, Eq. (8) remotely resembles the equation describing the propagation of heat in space, when the thermal-conductivity coefficient is a power-function of the amount of heat.[1)] As is well known, in this case the heat propagates in the form of a wave with a steep front, and regions exist where the heat has not yet arrived and the temperature has the initial value.[5] Using a special form of the boundary and initial conditions, it is possible to obtain self-similar solutions for the equation describing such a wave.[5]

2. Let us attempt, as in the theory of thermal waves, to find a self-similar solution for our problem. We consider those instants of time, when the distribution function of the beam has expanded towards V, to $V=V_m$, such that

$$c_{1e}/v_0\ll 1-V_m\ll 1.\qquad(9)$$

The initial stage of the process of quasilinear relaxation, which leads to a smoothing of the distribution function of the beam, depends, of course, on the initial form of f_0. One can hope, however, that in the succeeding phase of the process, when the left-side inequality of (9) is satisfied, and the detailed form of the initial distribution function of the beam electrons does not influence the solution, just as the law governing the propagation of the thermal wave at a sufficiently large distance from an explosion is determined only by the amount of heat released in the explosion.

When the right-side inequality of (9) is satisfied, we can replace in (8) V^2 by 1, and $w_0(V)$ by $w_0(1)$. After such a simplification, Eq. (8) does not depend explicitly on the velocity, and this greatly facilitates the search for self-similarity.

In the region $V_m < V < V_1$ (V_1 is the point where f_0 is negligibly small) Eqs. (7) and (8) do not depend explicitly on f_0. It then follows from (7) that

$$w(1)-w_0(1)=-\int_{V_1}^{V_m}f\,dV=\pi=-\int_{V_2}^{V_1}f_0\,dV.\qquad(10)$$

The last equality is approximate and is valid accurate to terms of the order of $(1-V_1)(1-V_m)$, and is based on the assumption that the process of quasilinear relaxation leads to a smoothing of the distribution function.

In order to verify that the problem actually has a self-similar solution satisfying the boundary condition (10), we introduce in (7) and (8) a new variable and a new function

$$\xi=\frac{1-V}{\sqrt{\pi\tau}}\qquad\varphi=\frac{w-w_0(V)}{\pi}.\qquad(11)$$

Then they take the form

$$f=-\sqrt{\pi/\tau}\,\varphi',\qquad(12)$$

$$-{}^1\!/_2\,\xi\varphi'=(\varphi+\varphi_0)\varphi'',\qquad(13)$$

where the prime denotes differentiation with respect to ξ, and $\varphi_0=w_0(1)/\pi$.

Since f is positive by definition, only those solutions of (13) for which $\varphi'(\xi)\le 0$ for all ξ have a

[1)] Recently the authors have learned that similar considerations were advanced by E. P. Velikhov.

physical meaning, in accord with (12).

The boundary condition (10), following such a substitution, is $\varphi(0) = 1$, and the boundary condition for $\varphi = \infty$ takes the form $\varphi(\infty) = 0$, since the noise ahead of the front of the propagating wave retains its initial value, and the distribution function is equal to zero.

Thus, to find the distribution function at any instant of time it is necessary to find the solution of (13) with the indicated boundary conditions, such that $\varphi'(\xi) \le 0$. These requirements are sufficient to make the solution unique and possessing a horizontal asymptote $\varphi = 0$.[6] Unfortunately, it cannot be written in terms of quadratures and, generally speaking, the equation must be solved numerically. Figure 2 shows the functions φ and $|\varphi'|/2$ obtained as a result of a numerical solution of (13) for $\varphi_0 \approx 1/130$.

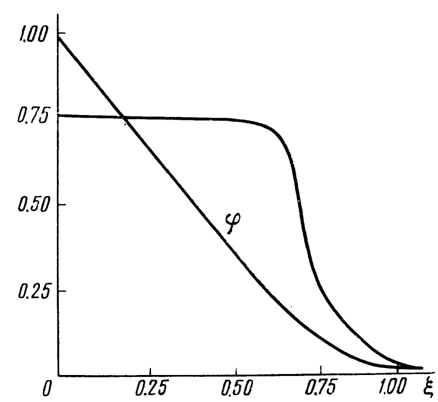

FIG. 2

In the case when $\ln(\varphi/\varphi_0) \gg 1$, we can write an approximate analytic solution of (13). We shall verify directly that the solution will be the following function:

$$\varphi(\xi) = \frac{1}{2}\xi_0(\xi_0 - \xi)\ln\frac{\varphi}{\varphi_0} + \frac{1}{2}(\xi_0 - \xi) - \frac{1}{4}(\xi_0 - \xi)^2. \quad (14)$$

Indeed, differentiating $\varphi(\xi)$, we obtain

$$\varphi'(\xi) = -\frac{1}{2}\xi_0 \ln\frac{\varphi}{\varphi_0} + \frac{1}{2}\xi_0(\xi_0 - \xi)\frac{\varphi'}{\varphi} - \frac{1}{2} + \frac{1}{2}(\xi_0 - \xi).$$

Further, dividing φ' by the function φ defined by (14), we obtain, accurate to small quantities $\sim[\ln(\varphi/\varphi_0)]^{-1}$, that $\varphi'/\varphi = 1/(\xi - \xi_0)$. Therefore

$$\varphi' \approx -\frac{1}{2}\xi_0 \ln\frac{\varphi}{\varphi_0} - \frac{1}{2}(1 + \xi).$$

We obtain φ'' in similar fashion:

$$\varphi'' \approx \frac{1}{2}\frac{\xi}{\xi_0 - \xi}.$$

The terms containing $\ln(\varphi/\varphi_0)$ in the expressions for φ and φ' are, by assumption, the principal ones, but we have retained also small terms, since they make a contribution to the second derivative of φ. In verifying solutions (14), it is possible to substitute into (13) $\varphi(\xi)$ and $\varphi'(\xi)$ without the small terms.

Finally, since the logarithm $\ln(\varphi/\varphi_0)$ is a slowly varying function when the ratio φ/φ_0 is large, we have with "logarithmic accuracy"

$$\varphi = \frac{1}{2}\xi_0(\xi_0 - \xi)\ln\frac{1}{\varphi_0}. \quad (15)$$

The constant ξ_0 determines with "logarithmic accuracy" the "coordinate" of the wave front. Its magnitude can be determined from the boundary condition $\varphi(0) = 1$. It is equal to

$$\xi_0 \approx \sqrt{2}/\sqrt{\ln(1/\varphi_0)}. \quad (16)$$

Since (15) is valid when $\ln(\varphi/\varphi_0) \gg 1$, it does not describe the leading part of the front, where $\varphi \approx \varphi_0$. The form of the leading part of the front can be approximately determined by putting $\xi = \xi_0$ in the left side of (13). The solution of such a simplified equation will be valid when $\xi - \xi_0 \ll \xi_0$. Integrating the obtained equation with respect to ξ, we get

$$\varphi' = -\frac{1}{2}\xi_0 \ln\left(1 + \frac{\varphi}{\varphi_0}\right). \quad (17)$$

Hence

$$\xi - \xi_0 = -\frac{2}{\xi_0}\int_{\varphi_0}^{\varphi}\frac{d\varphi}{\ln(1 + \varphi/\varphi_0)}$$

$$= \frac{2}{\xi_0}\varphi_0\left[\mathrm{li}\left(1 + \frac{\varphi}{\varphi_0}\right) - \mathrm{li}(2)\right], \quad (18)$$

where li is the integral logarithm. We assume here that $\varphi = \varphi_0 \ne 0$ when $\xi = \xi_0$. Expanding $\ln(1 + \varphi/\varphi_0)$ in terms of the small quantity $\varphi/\varphi_0 \ll 1$, we obtain for the leading part of the front the following formula:

$$\varphi = \varphi_0 \exp\left\{-\frac{\xi_0}{2\varphi_0}(\xi - \xi_0)\right\}.$$

From this expression we see that φ approaches zero exponentially, and the characteristic length is the very small quantity $2\varphi_0/\xi_0$.

Let us find also the time after which quasilinear relaxation of the beam occurs on the interval ΔV. It is approximately equal to the time during which the wave front traverses a distance Δv in velocity space. We have from (16)

$$\tau = \pi^{-1}(v_0^{-1}\xi_0^{-1}\Delta v)^2,$$

i.e.,

$$t = \frac{1}{\gamma}\frac{1}{\xi_0^2} = \frac{1}{\gamma}\frac{\ln \varphi_0^{-1}}{2}, \quad \gamma = \pi\omega_0 \frac{n_1}{n_0}\left(\frac{v_0}{\Delta v}\right)^2.$$

Actually, for beams with a relatively large particle density, the ratio $1/\varphi_0$, which is equal to the ratio of the energy of the final and initial noise, is a very large quantity. Thus, if initially the noise was at the thermal level, then $1/\varphi_0 \sim N_D n_1/n_0$, where N_D is the number of particles in the Debye sphere. In other words, $\ln \varphi_0^{-1}$ is practically equal to the Coulomb logarithm. The latter, on the other hand, under conditions which are typical of the experiment, is approximately equal to 20.

The results can be intuitively explained in the following simple manner. After a time $1/\gamma$, the front of the wave traverses a path in velocity space which is approximately equal to $\Delta v/\ln \varphi_0^{-1}$. Behind the wave front, where the noise has almost the final value, a plateau is formed after the same time $1/\gamma$. If the noise level ahead of the front were equal to zero, then the "wave" would stand still and its front would be infinitely steep. Actually, the slope of the front is finite, but is large because the wave moves rather slowly in the scale of the time $1/\gamma$.

3. We shall now show that the picture of the quasilinear relaxation which we have described does not change appreciably, and all the conclusions remain valid, for a broader class of initial distribution functions. To this end we return to the initial system of equations (4)–(7). Now, however, the initial distribution function will be the one shown in Fig. 3, i.e., a beam moving with average velocity V = 0 is present in the plasma, as before, but now $f_0 = 0$ in the entire velocity interval.

We can expect the solution of Eqs. (4)–(7) to describe a wave whose front moves in V-space with a certain velocity $\dot{u}(\tau)$. This means that the quantities f and w depend on V and τ as follows:

$$f(V, \tau) = f(V - u(\tau), \tau) \equiv f(\eta, \tau);$$
$$w(V, \tau) = w(V - u(\tau), \tau) \equiv w(\eta, \tau).$$

The width of the wave front η_1 is small compared with the distance $1 - u$ traversed by the wave. Over the distance η_1, the distribution function f increases rapidly from an initial value f_0 to a certain value $f(V^0, \tau)$ (see Fig. 3), which will be determined later, after which it remains practically unchanged. As the wave moves on, $f(V^0, \tau)$ and the waveform vary, but these changes occur on the time scale of τ. Therefore, on the wave front we have

$$\left|\frac{\partial f}{\partial \tau}\right| \ll \left|\dot{u}\frac{\partial f}{\partial \eta}\right|, \quad \left|\frac{\partial w}{\partial \tau}\right| \ll \left|\frac{\partial w}{\partial \eta}\dot{u}\right|$$

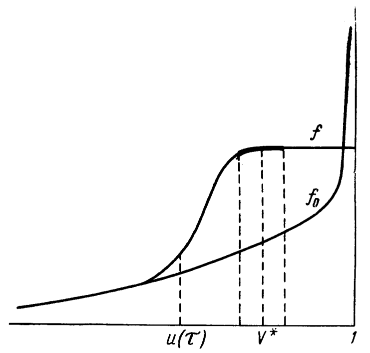

FIG. 3

With equal accuracy, we can simplify and rewrite (5) in the form

$$-\frac{\partial w}{\partial \eta}\dot{u} = wu^2\frac{\partial f}{\partial \eta}, \tag{19}$$

and then, integrating once, we get

$$f(\eta) - f_0(u) = -\frac{\dot{u}}{u^2}\ln\frac{w(\eta)}{w(u)}. \tag{20}$$

The quantity $w(u, \tau)$ is determined by the noise level at the point u at the instant of arrival of the wave front at this point. It must be emphasized that if the linear increment at this point differs from zero, then w(u) must be calculated with the formula of the linear theory, since by definition w(u) is taken at a point where the distribution function w is equal to the initial value $f_0(u)$, and the quasilinear effects are not yet significant.

Equation (7) in terms of the variables η is written as follows:

$$f(\eta) - f_0(\eta) = \frac{\partial(w - w_0(\eta))}{\partial \eta}. \tag{21}$$

In the region of the front we can replace $f_0(\eta)$ in this formula, at the accuracy assumed, by $f_0(u)$, since $f_0(\eta) - f_0(u) = \eta\,\partial f_0/\partial V$, and we can neglect $\partial w_0/\partial \eta$ compared with $\partial w/\partial \eta$, after which, substituting the expression (20) for $f(\eta)$ into (21), we obtain

$$\frac{\partial w}{\partial \eta} = -\frac{\dot{u}}{u^2}\ln\frac{w(\eta)}{w(u)}. \tag{22}$$

The integral of (22) is

$$\eta = \frac{u^2}{\dot{u}}w(u)\left[\mathrm{li}\left(\frac{w(\eta)}{w(u)}\right) - \mathrm{li}(2)\right] \tag{23}$$

In that region of values of η, where $\ln(w(\eta)/w(u)) \gg 1$, the solution of (22) can be written approximately, with "logarithmic accuracy," as follows:

$$w = -\frac{\dot{u}}{u^2}\eta\ln\left(-\eta\frac{\dot{u}}{u^2}\ln\frac{w(V^*)}{w(u)}\right). \tag{24}$$

(V^* is defined in Fig. 3).

The law governing the variation of the distribution function on the front of the wave has, with the same accuracy, the form

$$f = f_0 - \frac{\dot{u}}{u^2} \ln\left(-\eta \frac{\dot{u}}{u^2} \ln \frac{w(V^*)}{w(u)}\right), \tag{25}$$

i.e., the quantity $f - f_0$ is in the form of a step. Expressions (23), (24), and (25) justify the assumptions made in the derivation of (22), namely that the wavefront form changes weakly, only to the extent that u changes.

Such an approximation is not valid on the boundary of the wave in the vicinity of the point $\eta = 0$ ($V = u$). The law governing the manner in which w tends to $w(u)$ can be readily established by expanding $\ln(w/w(u))$ under the assumption that $(w - w(u))/w(u) \ll 1$. Then, linearizing (22), we obtain

$$w - w(u) = w(u) \exp\left\{-\eta \frac{\dot{u}}{u^2 w(u)}\right\}.$$

Thus, w tends to $w(u)$ very rapidly, since $w(u)$ is by definition a small quantity. For this reason we can actually neglect the term $\partial w_0(\eta)/\partial\eta$ in (22).

We now determine the solution on the right of the juncture point $V = V^*$ (see Fig. 3). Here we can assume in first approximation that the electron velocity distribution function is a plateau, the height of which depends on the time, i.e., $f = c(\tau)$. To determine the next approximation, we use (4). Substituting in it $f = c(\tau) + \delta f$, we obtain

$$\delta f = \dot{c} \int_V^1 \frac{1 - V'}{V'^2 w(V')} dV'. \tag{26}$$

If we replace in the integrand V by u and $w(V')$ by $w(V^*)$, where V^* is the point at which the two solutions are joined together, then δf will only increase. Estimating δf in this manner, we can find the region of values of V in which the inequality $|\delta f| \ll c(\tau)$ is satisfied:

$$c \gg \delta f \sim |\dot{c}| \frac{(1-u)^2}{2u^2 w(V^*)}. \tag{27}$$

The distribution function in the region of the plateau can be related with the coordinate of the wave front u. Since the wave front is assumed narrow compared with $1 - u$, the particle-number conservation law can be rewritten in the form

$$(1 - u) c = \int_u^1 f_0 \, dV. \tag{28}$$

It follows therefore that

$$\dot{c}/c \sim \dot{u}/(1 - u). \tag{29}$$

Expressing in (27) c in terms of u and substituting $w(V^*)$ from formula (24), we obtain

$$(1 - u) \Big/ \ln \frac{w(V^*)}{w(u)} \ll \eta. \tag{30}$$

with "logarithmic accuracy" we can replace $\ln[w(V^*)/w(u)]$ by $\ln[w(1)/w(u)]$ in the estimate (30). Thus, $f = c(\tau)$ for all values of $\eta = V - u$ satisfying the condition (30).

Since $\ln[w(1)/w(u)] \gg 1$, there exists a region of values of η for which condition (30) and the condition of the narrowness of the front $\eta \ll 1 - u$ can be simultaneously satisfied (see Fig. 3). Let us join together the distribution function at some point $V = V^*$ of this region. From (21) and (28) it follows that ($w(V^*) \gg w(u)$!):

$$f(V^*) = f_0(V^*) + \frac{\partial w}{\partial \eta} = \frac{1}{1-u} \int_u^1 f_0 \, dV.$$

This equation can be simplified by putting $f_0(V^*) \approx f_0(u)$ and $w(V^*) \approx w(1)$. The relative error in this case will not be larger than $(V^* - u)/u \ll 1$. Then the equation for determining the law of wave motion assumes the form:

$$-\frac{\dot{u}}{u^2} \ln \frac{w(1)}{w(u)} + f_0(u) = \frac{1}{1-u} \int_u^1 f_0 \, dV. \tag{31}$$

We solve it, for example, for the problem already considered above, when $f_0 = \pi\delta(V - 1)$. In this case (31) takes the form

$$-\frac{\dot{u}}{u^2} \ln \frac{w(1)}{w(u)} = \frac{\pi}{1-u}.$$

Integrating the obtained equation with respect to time with logarithmic accuracy, we get

$$\frac{1}{u} + \ln u - 1 = \pi\tau \Big/ \ln \frac{w(1)}{w(u)}.$$

The time is reckoned here in such a manner that $u = 1$ when $\tau = 0$. In order to compare this result with the solution of the self-similar problem, we must consider those instants of time when u is close to unity. Expanding the expression on the left in terms of the small quantity $1 - u \ll u$, we obtain the already known law of motion (16):

$$1 - u / \sqrt{\tau} = \text{const.}$$

If $u \ll 1$, then

$$\tau u = \text{const.}$$

It must be emphasized that the results obtained in Sec. 3 are valid only if at each instant of time $\ln[w(1)/w(u)] \gg 1$, where $w(u) = w_0^0 \exp[\gamma_0(u)\tau]$. The quantities $w(1)$ and w_0^0 are determined here by the noise levels in the final and initial states, and $\gamma_0(u)$ is the increment calculated from the initial distribution function. This is certainly not the case if the initial distribution function is such that the increment γ_0 is the same in the entire instability region. In this case the relaxation occurs within a time $1/\gamma_0$ after the noise grows to a level close to the final one.

If we exclude from consideration this particular case, then the noise will grow first of all in the vicinity of the point V, where the linear increment $\gamma_0(V)$ is maximal. Thus, a situation arises in which the noise is larger in some region of V than in the other regions. This, as we have seen earlier, leads to an increase in the slope of the distribution function and to propagation of the wave in velocity space.

We note the following circumstance, which greatly limits the region of applicability of the theory developed above. At not very small ratios n_1/n_0 the derivatives $\partial f/\partial V$ of the distribution function are very large at the front, and consequently the increment defined by formula (2) is large and can become formally larger than the frequency ω_0. Actually expression (2) for the increment and the quasilinear equation (1) are not valid for sufficiently steep distribution functions.

Thus, a more accurate equation for the increment is

$$\frac{2\gamma}{\mathrm{Re}\,\omega}=\gamma\frac{\omega_0^2}{k}\int\frac{\partial F/\partial v}{(\mathrm{Re}\,\omega-kv)^2+\gamma^2}dv \tag{32}$$

(for weak beams $\mathrm{Re}\,\omega \approx \omega_0$).

If the distribution function varies weakly over a distance γ/k about the vicinity of the point $v = \mathrm{Re}\,\omega/k$, then the derivative $\partial F/\partial v$ can be taken outside the integral sign, and the remaining integral is equal to $\pi/|k|\gamma$.

Consequently, in this case we can use the expression for γ in formula (2), when the value of the increment at the point $k = \omega_0/v$ is related with the value of the function $\partial F/\partial v$ at the point v. Such an approximation of the integral in the right side of (32) is valid if

$$\frac{\partial^2 F}{\partial v^2}\frac{\gamma}{k}\ll\frac{\partial F}{\partial v}.$$

Substituting into this inequality the expression for γ used in formula (2), we obtain

$$V^3\frac{n_1}{n_0}\frac{\partial^2 f}{\partial V^2}\ll 1 \quad\text{or}\quad \frac{n_1}{n_0}\frac{\partial^2 f}{\partial V^2}\ll 1,\quad\text{since}\quad V\approx 1. \tag{33}$$

Here $V = v/v_0$ and $f = \pi v_0 F n_0/n_1$.

This (local) connection is lost if in the vicinity of the point $v_m = \mathrm{Re}\,\omega/k$ the function $\partial F/\partial v$ changes appreciably over the length γ/k. In this case all the particles with velocities $|v - v_m| \lesssim \gamma/k$ contribute to the integral in (32).

If the distribution function is the step shown in Fig. 1, then the increment is equal to

$$\gamma=\left(\frac{n_1}{n_0}\frac{v_m}{2(v_0-v_m)}\right)^{1/2}\omega_0. \tag{34}$$

This expression is valid if the width of the transition region of the step δv is smaller than γ/k, i.e., if

$$\frac{\delta v}{v_0}\ll\left(\frac{n_1}{n_0}\frac{v_m}{2(v_0-v_m)}\right)^{1/2}.$$

The instability of Langmuir oscillations with such an increment is frequently called hydrodynamic. It was investigated in [7], and the quasilinear stage of relaxation of a narrow beam with $c_{1e}/v_0 \lesssim (n_1/n_0)^{1/3}$ was considered in [8]. We have immediately excluded from consideration the initial stage of relaxation of the beam by the inequality $c_{1e}/v_0 \lesssim (n_1/n_0)^{1/3}$.

However, owing to the effect observed by us, namely the increase in the slope, the "hydrodynamic stage" of the buildup of Langmuir oscillations by the beam will be significant at all stages of the relaxation process. A consideration of this stage is the subject of a separate investigation.

The relaxation process is described by Eqs. (1)–(3) from the beginning to the end only for very weak beams, when the "local connection" is valid even in the region of the maximum derivative $\partial F/\partial v$.

Substituting in (33) the concrete expression for the distribution function (12), we obtain the condition for the applicability of our theory, which relates the beam-particle energy with the initial energy density of the noise:

$$\varphi_0\left(\ln\frac{1}{\varphi_0}\right)^3\gtrsim\left[\frac{n_1}{n_0}(1-u)^{-3}\right]^{1/2}. \tag{35}$$

Since $n_1/n_0(1 - u)^3 \ll 1$ from the very beginning, the right side of the inequality (35) is a small quantity (when $1/(1 - u) \sim 1$ it is simply equal to $(n_1/n_0)^{1/2}$.

The quantitative picture of the quasilinear relaxation process apparently does not change much in the case of dense beams. Indeed, as explained above, the presence of a gently-sloping part behind the front is connected with the slow motion of the front. On the other hand, the velocity of the front is determined by the rate of noise growth. The maximum growth rate can be estimated in the case of dense beams by choosing for the increment the maximum value (34), corresponding to a distribution function in the form of a step. In this case, unlike the "local" case, the noise with wave vector k begins to grow when the front of the wave moving with velocity v_m reaches the point $v \approx \omega/k + \gamma/k$ and ceases to grow when the wave is displaced through a distance $\sim 2\gamma/k$. The noise energy increases over this length by the following factor:

$$\exp\left\{2\gamma\frac{dv}{\dot{v}_m}\right\} \approx \exp\left\{4\frac{\gamma^2}{k\dot{v}_m}\right\} \approx \exp\left\{\frac{2}{\dot{u}(1-u)}\right\}.$$

If we assume, on the other hand, that the noise has grown by a factor $w(1)/w(u)$ and use the expression just obtained, then the rate of motion of the wave front in velocity space $\dot{u}$ turns out to be equal to that obtained earlier.

4. Addendum (August 9, 1966). Experimental observation of the temporal process of relaxation of an electron beam is made difficult by the fact that the relaxation time is quite short. As a rule, it is smaller than the time necessary for the beam electrons to traverse the length of the installation.[9] More amenable to experiment is a different formulation of the problem, namely, investigation of the stationary particle distribution function and the noise occurring upon stationary injection of a weak beam of electrons into a half-space occupied by a plasma normally to the separation boundary.[2)]

We shall reckon the coordinate x from the boundary inside the plasma. When $x = 0$ the electron velocity distribution function takes the form shown in Fig. 1 ($x = 0$ corresponds to $t = 0$ in the temporal problem). The state of the plasma-beam system, occurring following stationary injection, is described by equations that are derived from the system (1)–(3) by replacing in Eq. (1) $\partial f/\partial t$ by $v\partial f/\partial x$, and in Eq. (2) $\partial\epsilon_k/\partial t$ by $(\partial\omega/\partial k)(\partial\epsilon_k/\partial x)$. The latter expression describes the change in the spectral density of the noise energy due to the drift of the noise along the x axis with group velocity $v_{gr} = \partial\omega/\partial k$. The resultant system of equations is similar to that investigated in the preceding sections. In particular, there exists a quasilinear integral

$$[f(V,x)-f(V,0)]\,V = \frac{\partial}{\partial V}\left[\frac{v_{gr}}{v_0}\left(w(V,x)-w(V,0)\right)\right]. \tag{36}$$

In this formula and henceforth, the group velocity is calculated at the point $k = \omega_0/v$, and is therefore a function of the variable V. It is assumed that v_{gr} is determined by the properties of the entire plasma and does not depend on the coordinate x. With the aid of (36) we can eliminate the distribution function from the system of quasilinear equations and obtain an equation for $w(V, x)$:

$$\frac{v_{gr}}{v_0}\frac{\partial w}{\partial \bar{x}} = V^2 w\frac{\partial}{\partial V}\left(\frac{1}{V}\frac{\partial}{\partial V}\left[\frac{v_{gr}}{v_0}\left(w-w(V,0)\right)\right]\right) + V^2 w\frac{\partial f(V,0)}{\partial V} \tag{37}$$

in lieu of (8). The dimensionless coordinate $\bar{x}$ is given by $\bar{x} = \omega_0 x n_1/v_0 n_0$. In lieu of the boundary condition (10) we obtain the following relation:

$$w(1,\bar{x}) - w(1,0) = \frac{\pi v_0}{v_{gr}(1)}. \tag{38}$$

The system (36)–(37) with boundary condition (38) has a self-similar solution described by Eq. (13) for the function

$$\varphi(\xi) = \frac{w(V,\bar{x})}{\pi} - \frac{w(V,0)}{\pi}.$$

The solution of (13), for large values of $\ln(\varphi/\varphi_0)$ is of the form (15). However, now the variable ξ is

$$\xi = \frac{1-V}{\sqrt{\pi\bar{x}}},$$

and the constant ξ_0 is determined from the boundary condition $\varphi(0) = \pi v_0/v_{gr}(1)$, which follows from (38). The distance x over which the width of the beam becomes equal to Δv is

$$x = \frac{1}{2\pi}\left(\frac{\Delta v}{v_0}\right)^2\frac{n_0}{n_1}\frac{v_{gr}(v_0)}{\omega_0}\ln\frac{1}{\varphi_0}.$$

Self-similarity holds so long as $\Delta v \ll v_0$, i.e., in the region

$$x \ll \frac{n_0}{n_1}\frac{v_{gr}}{\omega_0}\ln\frac{1}{\varphi_0}.$$

Using the analogy between the temporal and spatial problems, we can obtain a solution which is valid for all values of Δv and x.

Figures 1 and 4 show plots of the beam-electron distribution function and of the spectral energy density of the Langmuir oscillations for the case when $\ln(\epsilon_k/\epsilon_0) \gg 1$. The distribution function is shown for three values of the dimensionless coordinate $\bar{x} = \omega_0 x n_1/v_0 n_0$. The boundary $u(x)$ of the distribution function in velocity space is determined from the equation

$$\frac{1}{u^2} + 2\ln u - 1 = \frac{4\pi\bar{x}}{\ln(1/\varphi_0)}\cdot\frac{2}{3}\frac{V_0^2}{c_e^2},$$

2)The initial nonstationary phase of this process was considered by Fainberg and Shapiro [10].

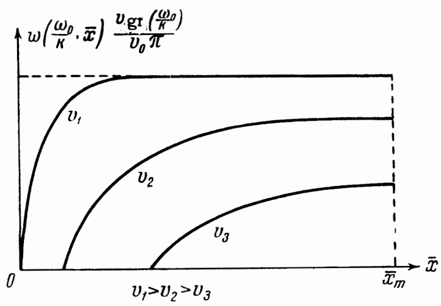

FIG. 4

If $\partial\omega/\partial k = {}^3/_2\, c_e^2 k \omega_0^{-1}$. The spectral density, measured in units of $n_1 m v_0^2/2$ as a function of the dimensionless coordinate $\bar{x}$, is given for three values of the phase velocity. Its analytic expression is

$$w = \frac{\pi v_0}{v_{gr}(v)} \frac{V^2 - u^2}{1 - u^2}.$$

Beyond the point

$$\bar{x}_m \approx \ln\frac{1}{\varphi_0}\left(x = \frac{v_0}{\omega_0}\frac{n_0}{n_1}\ln\frac{1}{\varphi_0}\right)$$

the growth of the oscillations ceases, since the distribution function becomes stable. Figures 1 and 4 will agree with experiment if the oscillations are absorbed on the wall opposite the injector.

We have purposely not written out the explicit expression for the group velocity. For a half-space the group velocity is $v_{gr} = {}^3/_2\, c_e^2 k \omega_0^{-1}$. Under real experimental conditions, when the beam diameter is comparable with the wavelength and the plasma is situated in a magnetic field, the group velocity can be determined by these factors and can be much larger than ${}^3/_2\, c_e^2 k \omega_0^{-1}$. In addition, when $\int \epsilon_k dk > n_0 T_e$ (and the formulas presented above are still valid), the group velocity depends on the noise energy.[11] From the requirement that the functions f and w vary slowly over a wavelength, which is necessary for the applicability of the quasilinear theory, it follows that the results obtained above are valid when $n_1 v_0 / n_0 v_{gr} \ll 1$.

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[1] A. A. Vedenov, E. P. Velikhov, and R. Z. Sagdeev, Nuclear Fusion, Suppl. **2**, 465 (1962).

[2] W. Drummond and D. Pines, ibid. Suppl. **3**, 1049 (1962).

[3] A. A. Vedenov and R. Z. Sagdeev, UFN **73**, 701 (1961), Soviet Phys. Uspekhi **4**, 332 (1961).

[4] A. A. Vedenov, Introduction to the Theory of Weakly Turbulent Plasma, in: Voprosy teorii plazmy (Problems in Plasma Theory) **3**, Gosatomizdat, 1963.

[5] L. D. Landau and E. M. Lifshitz, Mekhanika sploshnykh sred (Mechanics of Continuous Media), Gostekhizdat, 1953.

[6] E. Kamke, Differentialgleichungen Losungsmetoden und Losungen, Pt. I, Chelsea, 1948.

[7] Ya. B. Faĭnberg, Atomnaya énergiya **11**, 313 (1961).

[8] V. D. Shapiro and V. I. Shevchenko, JETP **42**, 1515 (1962), Soviet Phys. JETP **15**, 1053 (1962).

[9] I. F. Kharchenko, Ya. B. Faĭnberg, et al. Nuclear Fusion Suppl. **2**, 521 (1962).

[10] Ya. B. Faĭnberg in V. D. Shapiro, in: Vzaimodeĭstvie puchkov zaryazhennykh chastits s plazmoĭ (Interaction of Beams of Charged Particles with a Plasma), Kiev, 1965.

[11] A. A. Vedenov, A. V. Gordeev, and L. I. Rudakov, Paper at Culham conference, 1965.

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