EXTREMAL VALUE OF THE DIFFERENTIAL CROSS SECTION OF ELASTIC SCATTERING

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For given total cross sections a conditional extremum is determined for the differential cross section of elastic scattering. The ratio of the extremal value for backward scattering to the value for 90° grows as the square root of the laboratory energy (for constant total cross sections).

WE derive below a formula for the extremal value of the differential cross section of elastic scattering under an arbitrary angle at arbitrary energy. The result is expressed in terms of the experimentally known total cross sections. The case of high energies is considered in detail. In particular, we show that for $k \to \infty$ (k is the wave number in the center-of-mass system) the experimentally measured ratio of the real part of the forward scattering amplitude to the imaginary part must vanish, and that the ratio of the extremal value of the cross section for 90° to that for 180° also vanishes, both as k^{-1}.

For a derivation of the formula we search for a conditional extremum of the elastic differential cross section under the constraint of fixed (experimental) values of the total cross sections for interactions (σ_t) and elastic scattering (σ_{el}). We assume that one can retain only a finite number L of partial waves in the sums which express the amplitudes and cross sections in terms of partial amplitudes and cross sections. (We shall justify this assumption below and determine L for large energies.) Then the Lagrange multiplier method for conditional extrema leads to the following algebraic system of linear equations for the partial amplitudes $f_l^{(e)}$, which realize the extremum of the differential cross section:

$$\operatorname{Re} f_l^{(e)} = \alpha \operatorname{Re} f^{(e)}(\theta) P_l(\theta), \quad l = 0, 1, 2, \ldots, L, \tag{1}$$

$$\operatorname{Im} f_l^{(e)} = \alpha \operatorname{Im} f^{(e)}(\theta) P_l(\theta) + \beta, \quad l = 0, 1, 2, \ldots, L. \tag{2}$$

Here α and β are Lagrange multipliers, determined by the two constraints, namely σ_{el} and σ_t expressed in terms of $f_l^{(e)}$ and $f(\theta)$ is the elastic scattering amplitude

$$f(\theta) = \sum_{l=0}^{L} (2l+1) f_l P_l(\theta) \tag{3}$$

where, owing to the unitarity condition

$$0 \leqslant k|f_l| \leqslant 1. \tag{4}$$

The system (1) is homogeneous and the system (2) is nonhomogeneous, both with the same determinants. The determinant does not vanish. Therefore the system (1) has only the zero solution

$$\operatorname{Re} f_l^{(e)} = 0, \quad l = 0, 1, 2, \ldots, L. \tag{5}$$

From (2), (3), and (5) we obtain the following formula for the extremal value of the elastic differential cross section

$$\sigma^{(e)}(\theta) = \left\{ \left[a_0 \frac{\sigma_{el}}{4\pi} - \left(\frac{k\sigma_t}{4\pi} \right)^2 \right]^{1/2} \left[\frac{a_2}{a_0} - \frac{a_1^2}{a_0^2} \right]^{1/2} + \frac{k\sigma_t}{4\pi} \frac{a_1}{a_0} \right\}^2, \tag{6}$$

where we have used the notations

$$a_0 \equiv \sum_{l=0}^{L} (2l+1), \qquad a_1 \equiv \sum_{l=0}^{L} (2l+1) P_l(\theta),$$

$$a_2 \equiv \sum_{l=0}^{L} (2l+1) P_l^2(\theta). \tag{7}$$

It is clear from (6) that

$$(L+1)^2 \geqslant k^2 \sigma_t^2 / 4\pi\sigma_{el}. \tag{8}$$

Therefore, one can set[1]

$$(L+1)^2 = (k^2\sigma_t^2 / 4\pi\sigma_{el})(1 + \Delta^2), \quad \Delta^2 \geqslant \delta^2 \geqslant 0 \tag{9}$$

(δ is the ratio of the real and imaginary parts of the forward scattering amplitude). Substituting (9) into (6) we have

$$X^{(e)}(\theta) \equiv \frac{16\pi^2 \sigma^{(e)}(\theta)}{k^2 \sigma_t^2} = \frac{1}{a_0^2} [(-1)^{L+1} |\Delta| (a_0 a_2 - a_1^2)^{1/2} + a_1]^2, \tag{10}$$

where the sign in front of the square root has been chosen so as to make the backward scattering cross section vanish in the limit of infinite energy.

Use of sufficient conditions for extrema shows that we are dealing here with a minimum. We also note that the solutions obtained here for the partial amplitudes $f_l^{(e)}$ satisfy the unitarity requirement (4) with ample reserves. Thus, at high energies, unitarity is satisfied if $\sigma_t > \sigma_{el}$.

We consider several special cases of Eq. (10) at high energies.

A. The diffraction peak region. For $\theta \ll k^{-1}$, (10) implies

$$X(\theta) \geqslant X^{(e)}(\theta) \approx 1 + at + {}^5/_{12}a^2t^2,$$
$$a \equiv \sigma_t^2/16\pi\sigma_{el}, \quad t \equiv -2k^2(1-\cos\theta) \approx -k^2\theta^2, \tag{11}$$

where t is the square of the momentum transfer.

B. Backward scattering. In this case the general formula yields:

$$X(180°) \geqslant [-|\Delta|(1-a_0^{-1})^{1/2} + a_0^{-1/2}]^2. \tag{12}$$

If we now require that the backward scattering cross section vanish for $k \to \infty$, we obtain the following estimate for the experimentally measurable ratio δ of the real and imaginary parts of the forward scattering amplitude:

$$\delta^2 \leqslant \Delta^2 \to 4\pi\sigma_{el}/k^2\sigma_t^2, \quad k \to \infty, \tag{13}$$

which is not in disagreement with experiments[2,3].

C. Scattering under 90°. In this case we have at high energies

$$\frac{\sigma^{(e)}(180°)}{\sigma^{(e)}(90°)} \approx \frac{k\sigma_t}{4\pi}\left(\frac{\pi^3}{\sigma_{el}}\right)^{1/2}. \tag{14}$$

We remark in conclusion that for the special case of backward scattering the extremal value has been discussed previously.

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