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THE GREEN'S FUNCTION FOR A COMPOUND PARTICLE

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The question of the possibility of constructing the Green's function for a compound particle in the scalar-field model is considered. It is shown that a Green's function can be constructed only in the case when there is a contact interaction and the Bethe-Salpeter equation is without meaning.

1. INTRODUCTION

OWING to the increase of the number of observed particles there is great interest in ascertaining what is the difference between compound particles and elementary particles. The answer to this question obviously depends on the theoretical framework in which the discussion is conducted. We say that in a pure dispersion treatment (analyticity plus unitarity) all particles appear on an equal footing. From the field-theory point of view, on the other hand, some of the particles—those to which independent fields are ascribed—can be called elementary, and the others are compound. As is well known, these two approaches do not contradict each other. The suspicion may arise that the difference between elementary and compound particles in field theory is a formal one and that in fact the equations of field theory can be formulated in such a way that compound particles could play the role of the elementary particles. In this case there should exist in the theory, besides the usual Green's functions and vertex parts, Green's functions and vertex parts for the compound particles. It is essential that these quantities should have meaning off the mass shells—this is the departure from a pure dispersion treatment.

There are different ways to construct Green's functions of compound particles.[1-4] In analogy with the elementary particles, however, it is reasonable to start from certain interpolating local field operators corresponding to the compound particles. The Green's functions that are constructed will then play the usual role in scattering theory. The interpolating local operators have been suggested earlier by Zimmerman.[5] In the present note we investigate whether the Green's function of a compound particle can be constructed by means of these operators. We shall consider the case in which all of the particles (both fundamental and compound) are scalar. There are two cases for which there are different answers: a) when there is a contact interaction of the type $\lambda\varphi^4$ between the compound particles, and b) when there is no such interaction.

In the first case a Green's function for the compound particle can be constructed. In this case, however, no Bethe-Salpeter equation can be written for the compound particle, and the formation of the compound particle occurs in about the same way as in nonrelativistic theory with a δ-function potential. The mass of the compound particle is here a parameter of the theory rather than an object of calculation.

In case b) no Green's function exists for the compound particle. Here the Bethe-Salpeter equation can be written, and this case is reminiscent of ordinary nonrelativistic theory. We note that it is case b) that has been treated in papers by one of the writers[1, 2] from a different point of view but with the same conclusion that it is impossible to construct the Green's function of the compound particle.

To make matters clearer we shall consider first the nonrelativistic theory and then the relativistic theory.

2. THE NONRELATIVISTIC THEORY

Let there be scalar particles of mass m described by a complex field $\varphi(x)$, which interact with a potential $V(x)$ and form a bound state of mass M. Following Zimmerman, we introduce the local field operator

$$\Psi(X) = \lim_{\xi\to 0}\frac{T\{\varphi(X+\xi)\varphi(X-\xi)\}}{F_{0,c}(\xi)}, \tag{1}$$

where

$$F_{0,c}(\xi) = (2\pi)^{3/2}\langle 0|T\{\varphi(\xi)\varphi(-\xi)\}|\Phi_0\rangle,$$

and Φ_0 is the bound state at rest. The operator $\Psi(X)$ is the interpolating field operator for the compound particle, in the scheme of Lehmann, Symanzik, and Zimmerman.[5-7]

It is natural to define the Green's function of the compound particle as

$$G(X-Y) = \langle 0|T\{\Psi(X)\Psi^+(Y)\}|0\rangle = \lim_{\substack{\xi\to 0\\ \eta\to 0}} \frac{G(X-Y;\,\xi,\eta)}{F_{0,c}(\xi)F_{0,c}^*(\eta)} \tag{2}$$

where

$$G(X-Y;\,\xi,\eta) = \langle 0|T\{\varphi(X+\xi)\varphi(X-\xi)\varphi^+(Y+\eta)\varphi^+(Y-\eta\}|0\rangle \tag{3}$$

is the two-particle Green's function for the φ-particles.

In nonrelativistic theory the two-particle Green's function in momentum space satisfies the equation

$$G(\varepsilon;\mathbf{k},\mathbf{k}') = G_0(\varepsilon,\mathbf{k})\delta^3(\mathbf{k}-\mathbf{k}') + G_0(\varepsilon,\mathbf{k})\int d^3k''V(\mathbf{k}-\mathbf{k}'')G(\varepsilon;\mathbf{k}'',\mathbf{k}'), \tag{4}$$

where $\mathbf{k}$ and $\mathbf{k}'$ are the values of the relative momentum of the particles before and after scattering, ϵ is the energy of the relative motion, and

$$G_0(\varepsilon,\mathbf{k}) = \frac{1}{\varepsilon - \mathbf{k}^2/m + i0}. \tag{5}$$

The bound-state wave function $F_0(k)$ satisfies the Schrödinger equation

$$F_0(k) = G_0(-\varepsilon_0,\mathbf{k})\int d^3k''V(\mathbf{k}-\mathbf{k}'')F_0(\mathbf{k}''). \tag{6}$$

According to Eq. (2) the Green's function for the compound particle is of the form

$$G(\varepsilon) = \frac{\int d^3k\, d^3k'\, G(\varepsilon;\mathbf{k},\mathbf{k}')}{\left|\int d^3kF_0(\mathbf{k})\right|^2}. \tag{7}$$

We replace the requirement that the passage to the limit ξ, $\eta \to 0$ be simultaneous in the numerator and denominator of (2) with the requirement that passage to the limit of infinite limits of integration be simultaneous in the numerator and denominator of (7).

We begin with the case in which there is no δ-function term in $V(x)$, and consequently $|V(\mathbf{k})| \to 0$ for $|\mathbf{k}| \to \infty$. More exactly, we assume, as usual, that

$$\int d^3k|V(\mathbf{k})|^2 < +\infty \tag{8}$$

Then

$$\lim_{\xi\to 0} F_0(\xi) = \int F_0(\mathbf{k})d^3k < +\infty. \tag{9}$$

We introduce into the treatment the function

$$\Phi(\varepsilon,\mathbf{k}) = \int d^3k'G(\varepsilon;\mathbf{k},\mathbf{k}'). \tag{10}$$

It obviously obeys the equation

$$\Phi(\varepsilon,\mathbf{k}) = G_0(\varepsilon,\mathbf{k}) + G_0(\varepsilon,\mathbf{k})\int d^3k'V(\mathbf{k}-\mathbf{k}')\Phi(\varepsilon,\mathbf{k}'). \tag{11}$$

For $|\mathbf{k}| \to \infty$ the second term in (11) falls off at least as well as $|\mathbf{k}|^{-(r+\alpha)}$, where $\alpha > 1/2$. Therefore we can neglect it in comparison with the first term. In the integration in (7) this first term gives a linearly diverging quantity. Then by (9) we find that in the case in which there is no δ-function potential the Green's function of the compound particle does not exist (diverges linearly).

Now let

$$V(\mathbf{x}) = \lambda\delta(\mathbf{x}) + V_1(\mathbf{x}), \tag{12}$$

or in momentum space

$$V(\mathbf{k}) = \lambda + V_1(\mathbf{k}), \quad \int |V_1(\mathbf{k})|^2 d^3k < +\infty. \tag{13}$$

Equations (4), (6), and (10) now have no meaning. Therefore in analogy with G and Φ we introduce functions $G_1(\epsilon;\,\mathbf{k},\,\mathbf{k}')$ and $\Phi_1(\epsilon,\,\mathbf{k})$ which obey the same equations but with the potential $V_1(\mathbf{k})$, for example

$$\Phi_1(\varepsilon,\mathbf{k}) = G_0(\varepsilon,\mathbf{k}) + G_0(\varepsilon,\mathbf{k})\int V_1(\mathbf{k}-\mathbf{k}')\Phi_1(\varepsilon,\mathbf{k}')d^3k. \tag{14}$$

We also define

$$R_1(\varepsilon) = i\int \Phi_1(\varepsilon,\mathbf{k})d^3k. \tag{15}$$

It is then not hard to show that

$$G(\varepsilon;\mathbf{k},\mathbf{k}') = G_1(\varepsilon;\mathbf{k},\mathbf{k}') + \Phi_1(\varepsilon,\mathbf{k})\frac{\lambda}{[1+i\lambda R_1(\varepsilon)]}\Phi_1(\varepsilon,\mathbf{k}'). \tag{16}$$

To verify this it suffices to substitute (16) in the equation (4) with the potential (13). The function $R_1(\epsilon)$ diverges linearly, since for $|\mathbf{k}| \to \infty$ we have $\Phi_1 \sim G_0(\epsilon,\,\mathbf{k})$. To give meaning to (16) it is necessary to carry out a renormalization. We proceed in the following way. If G has a pole corresponding to the bound state at $\epsilon = -\epsilon_0$, then this pole must arise only owing to the denominator $1 + i\lambda R_1(\epsilon)$. The point is that if a pole with respect to the variable ϵ arises in G_1, then it appears simultaneously in Φ_1 and R_1 also. It is easy to verify that the complete function G will then not have a pole.

Accordingly,

$$1 + i\lambda R_1(-\varepsilon_0) = 0. \tag{17}$$

This relation allows us to rewrite (16) in the "renormalized" form:

$$G(\varepsilon; \mathbf{k}, \mathbf{k}') = G_1(\varepsilon; \mathbf{k}, \mathbf{k}') - i\Phi_1(\varepsilon, \mathbf{k}) \frac{1}{[R_1(\varepsilon) - R_1(-\varepsilon_0)]} \Phi_1(\varepsilon, \mathbf{k}'). \tag{18}$$

This new expression has a meaning.

Comparing (18) with the usual expansion of the Green's function in the neighborhood of a pole corresponding to a bound state, we find that in our case the quantity that plays the role of the wave function of the bound state is

$$F_0(\mathbf{k}) = A^{1/2}\Phi_1(-\varepsilon_0, \mathbf{k}), \tag{19}$$

where

$$A = \operatorname*{Res}_{\varepsilon=-\varepsilon_0} [R_1(\varepsilon) - R_1(-\varepsilon_0)]^{-1}.$$

We emphasize that here, unlike the normal case, the function $F_0(\mathbf{k})$ satisfies an inhomogeneous equation of the Schrödinger type [Eq. (14)]. The quantity ϵ_0 is not by any means determined from the equation for $F_0(\mathbf{k})$, but is a parameter of this equation.

Substituting (17) and (19) in (7), we get as the expression for the Green's function of the compound particle

$$G(\varepsilon) = \frac{-i}{A} \frac{R_1(\varepsilon)}{R_1(-\varepsilon_0)} \frac{1}{[R_1(\varepsilon) - R_1(-\varepsilon_0)]}. \tag{20}$$

Here the two expressions $R_1(\epsilon)$ and $R_1(-\epsilon_0)$ are linearly divergent, but their ratio exists and is equal to unity. Therefore the desired Green's function of the compound particle can be constructed in the form

$$G(\varepsilon) = \frac{-i}{A} [R_1(\varepsilon) - R_1(-\varepsilon_0)]^{-1}. \tag{21}$$

It is not hard to verify that in the case $V_1(\mathbf{x}) = 0$ the Green's function of the compound particle is of the form

$$G(\varepsilon) = \frac{1}{2\sqrt{\varepsilon_0}(\sqrt{\varepsilon} + i\sqrt{\varepsilon_0})}. \tag{22}$$

3. THE RELATIVISTIC THEORY

In the relativistic case the two-particle Green's function of the φ-particles satisfies the Bethe-Salpeter equation

$$G(p, k, k') = G_0(p, k)\delta^4(k - k') + G_0(p, k) \int I(p, k, k'') G(p, k'', k') d^4k'', \tag{23}$$

where

$$G_0(p, k) = \Delta\left(\frac{p+k}{2}\right)\Delta\left(\frac{p-k}{2}\right),$$

$\Delta(q)$ is the one-particle Green's function of the φ-particle with four-momentum q, I(p, k, k′) is the relativistic interaction, p is the four-momentum of the center of mass, and k and k′ are the relative four-momenta before and after scattering.

The wave function $F_{p,c}(k)$ of the bound state satisfies the homogeneous Bethe-Salpeter equation

$$F_{p,c}(k) = G_0(p, k) \int I(p, k, q) F_{p,c}(q) d^4q. \tag{24}$$

For the existence of the Bethe-Salpeter equation (24) it is necessary that the kernel decrease to zero or oscillate rapidly when k or q increases to infinity. If we exclude oscillations, which lead to an essential singularity of the kernel at infinity, it follows that $F_{p,c}(k)$ falls off more rapidly than $G_0(p, k)$ for $k \to \infty$. As is well known, all of the calculations can be made in the Euclidean metric.[8]

Let us assume for concreteness that the kernel I has the property

$$\int d^4k d^4q G_0(p, k) I(p, k, q) G_0(p, q) < +\infty.$$

Then it is not hard to show that

$$\int d^4k F_{p,c}(k) < +\infty.$$

We again introduce the function Φ(p, q) with the definition (10). It satisfied the inhomogeneous Bethe-Salpeter equation

$$\Phi(p, q) = G_0(p, q) + G_0(p, q) \int I(p, q, k) \Phi(p, k) d^4k. \tag{25}$$

It is clear that $\Phi \sim G_0$ for $q \to \infty$.

Turning to the Green's function of the compound particle, we find that the denominator in Eq. (7) is finite, and the numerator increases logarithmically. Consequently the Green's function of the compound particle does not exist.

Now suppose that there is a contact interaction of the type $\lambda_0(\bar{\varphi}\varphi)^2$. The kernel I will contain a term $2i\lambda$ ($\lambda = Z_2^2\lambda_0$ is the renormalized coupling constant, and Z_2 is the renormalization constant for φ).

Taking crossing symmetry into account, we represent the kernel in the following form (see a paper by Rowe,[4]):

$$I(p, k, q) = i\lambda + I_1(p, k, q). \tag{26}$$

In what follows we shall show that the kernel I_1 is well enough behaved for k and $q \to \infty$ so that the Bethe-Salpeter equation with this kernel has meaning. We emphasize that in perturbation theory based on the constant λ the kernel I_1 contains nondecreasing terms, and it acquires the necessary decreasing properties only after renormalization.

We introduce as before the Green's function G_1 which satisfies the usual equation but with the kernel I_1. As for the function Φ_1, we introduce it in analogy with (14), but with an additional factor needed for the additional renormalization which appears in the relativistic theory owing to crossing symmetry:

$$\Phi_1(p,k) = Z_1 G_0(p,k) + G_0(p,k) \int I_1(p,k,q)\Phi_1(p,q)\,d^4q. \tag{27}$$

We determine the factor Z_1 from the condition $\Phi_1 = G_0$ when $(p+k)^2 = 4m^2$ and $p^2 = M^2$, so that

$$\Phi_1(p,k) = G_0(p,k) + G_0(p,k) \int I_1(p,k,q)\Phi_1(p,q)\,d^4q - \left(G_0(p,k)\int I_1(p,k,q)\Phi_1(p,q)\,d^4q\right)\Bigg|_{\substack{(p\pm k)^2=4m^2\\ p^2=M^2}}. \tag{28}$$

As in the nonrelativistic theory, the complete Green's function G can be represented in the form

$$G(p,k,q) = G_1(p,k,q) - i\Phi_1(p,k)\left[\frac{\lambda Z_1^{-2}}{1+\lambda Z_1^{-2}R(p)}\right]\Phi_1(p,q), \tag{29}$$

where

$$R(p) = -iZ_1 \int \Phi_1(p,k)\,d^4k.$$

The pole corresponding to the bound state appears owing to a zero in the denominator of (29). As in the nonrelativistic case, the poles arising from G_1 and Φ_1 cancel each other. Therefore we can rewrite (29) in analogy with (18):

$$G(p,k,q) = G_1(p,k,q) - i\Phi_1(p,k)\frac{1}{[R(p) - R(p)\,|_{p^2=M^2}]}\Phi_1(p,q). \tag{30}$$

If we assume that $\Phi_1 \sim G_0$ for $k \to \infty$, then R(p) diverges logarithmically. The difference $R(p) - R(p)|_{p^2=M^2}$ will be finite, and for $p^2 \to \infty$ it will behave like $\ln p^2$.

Let us now return to I_1. This kernel includes the part of the interaction that comes from $\lambda_0(\overline{\varphi}\varphi)^2$, and other interactions between the fundamental particles. It is not hard to see that the first part can be represented formally in terms of a Yukawa interaction of the φ-particles and the compound particles, with the vertex Φ_1 and the propagation function $[R(p) - R(p)|_{p^2=M^2}]^{-1}$ for the compound particles (cf. [4]). Consequently, for k or $k' \to \infty$ we have the respective behaviors $I_1 \sim (\ln k^2)^{-1}$ and $I_1 \sim (\ln k'^2)^{-1}$, and the equations for G_1 and Φ_1 have meaning. Furthermore it is not hard to see that for $k \to \infty$ we have $\Phi_1 \sim G_0$, and consequently the entire scheme is selfconsistent.

The relativistic wave function $F_{p,c}(k)$ of the compound particle is obviously determined by the value of the residue of the pole of G at $p^2 = M^2$:

$$F_{p,c}(k) = A^{1/2}\Phi_1(p,k)\,|_{p^2=M^2}, \tag{31}$$

where

$$A = \operatorname*{Res}_{p^2=M^2}[R(p) - R(p)\,|_{p^2=M^2}]^{-1}.$$

As in the nonrelativistic case, when there is a contact interaction $F_{p,c}(k)$ satisfies an inhomogeneous wave equation.

By using (30) and (31) we can write the Green's function of the compound particle in a form analogous to (20):

$$\Delta'(p) = \frac{-i}{A}\frac{R(p)}{R(p)\,|_{p^2=M^2}}[R(p) - R(p)\,|_{p^2=M^2}]^{-1}. \tag{32}$$

Owing to the logarithmic divergence of R(p), we can replace the ratio $R(p)/R(p)|_{p^2=M^2}$ by unit. Therefore

$$\Delta'(p) = \frac{-i}{A}\frac{1}{R(p) - R(p)\,|_{p^2=M^2}}. \tag{33}$$

Accordingly, when there is an interaction of the type $\lambda\varphi^4$ the Green's function of the compound particle exists. In this case, however, we cannot write a Bethe-Salpeter equation. From the very beginning the mass of the compound particle must be artificially introduced into the function $\Delta'(p)$ as a subtractive constant.

We also point out that $\Delta'(p)$ behaves very poorly at infinity (it decreases logarithmically). This corresponds to the condition $Z_3 = 0$ for the renormalization constant for the wave function of the compound particle. It is important that the approximate replacement $\Delta'(p) \approx -i/(p^2 - M^2)$, which actually means that the compound nature of the M-particle is ignored, is possible in this case, but only for not too large values of p^2. This fact can be significant in passage beyond the mass shell in theories of the "bootstrap" type.

4. CONCLUSION

Accordingly, we have examined one of the possible methods for constructing the Green's function of a compound particle, namely by means of the interpolating field of the compound particles, which was proposed by Zimmerman. The following are the main results.

1. When there is no interaction of the type $\lambda\varphi^4 [|F_{p,c}(0)| < +\infty]$ the Green's function constructed in this way does not exist; this is in agreement with the results of [1,2].

2. In the case in which there is a contact interaction $[F_{p,c}(0) = \infty]$ the Green's function can be constructed by the method of Zimmermann (the method used in [1,2] cannot be applied here). The existence of the Green's function in this case is associated with the exceptional character of the contact interaction and with the nonexistence of a homogeneous Bethe-Salpeter equation for the wave functions of the bound state. The mass of the compound particle enters the theory as an external parameter, and it is impossible to express it in terms of the parameters of the elementary particles; a parameter whose meaning is that of the mass of the compound particle would be present even in theories in which there is no bound state.

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