THE QUANTIZATION OF THE GENERAL THEORY OF RELATIVITY BY MEANS OF INTEGRALS IN CURVED FUNCTIONAL SPACE

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The formalism of the "integral over all histories" is applied to the quantization of the gravitational field. It is shown that the functional space of all possible space-times has a curvature so that it is impossible to make the transition to operators and a dynamical theory. The quantum theory of a weak gravitational field is a theory based on a functional space tangent to the curved space. If the interaction of the gravitational field with other matter is excluded one obtains the classically defined metric, and the S matrix formalism can be applied. When the gravitational field is quantized the quantum theory of all other fields becomes necessarily of the S matrix type.

1. INTRODUCTION

THE problem of the quantization of gravitation is an important problem of physics not so much because of the practical necessity of including gravitation in various processes, but rather because the establishment of a closed quantum theory of all fields including the gravitational field (i.e., of a quantum theory of space-time-matter) is necessary for further progress in our understanding of the physical essence of the quantization procedure, the connection of this procedure with the properties of space-time, and also for a deeper insight in the properties of matter. The space-time relations play an essential role in the quantization formalism; for example, the operators of observables must commute if the points at which they are taken are separated by a space-like interval. But this concept is closely connected with the metric, with the geometry of space-time. If the quantization of gravitation consists in turning the metric tensor into an operator, then the question arises how one can preserve the concept of a space-like interval.

There exist a number of formal schemes for the quantization of fields, which can be divided into two groups: 1) the operator method (canonical formalism,[1] the dynamical principle of Schwinger,[2] etc.) and 2) the Feynman method of the integral over all possible configurations of fields.[3-5] All these schemes have been employed by various authors and in various modifications for the quantization of the gravitational field, but despite a sufficient number of papers, the problem cannot be regarded as solved in principle. This is so, first, just because of the variety of versions which differ from one another in essential points, and second, because the overwhelming majority of the papers use, explicitly or implicitly, some auxiliary space-time for the quantization—either a Minkowski space (cf., for example, [6-8]) or a Galilei-Newton space (for example, [9-11]), which is not in accord with the principles of the general theory of relativity, and is possibly also principally useless.

The quantization by means of functional integrals is carried out in the following way: one considers a functional space of all possible configurations of the field to be quantized, $y(x^i)$, in space-time and assumes that each of these configurations is realized with a probability amplitude proportional to $\exp\{iS[y(x)]/h\}$, where $S[y]$ is the classical action functional. Each configuration of the field $y(x^0, x^1, x^2, x^3)$ is a point in the functional space and the action is a function of this point. The distance between the points $y_1(x)$ and $y_2(x)$ is defined as

$$l^2 = N\int |y_1(x) - y_2(x)|^2 \sqrt{-g}\, d_4x. \tag{1}$$

It is invariant with respect to the transformation

$$y'(x) = y(x) + \eta(x). \tag{2}$$

Thus the transformation (2) transforms the functional space into itself—it represents a motion of this space, and owing to the commutativity of transformations with different $\eta(x)$, it is a translation. It follows from this that the functional space under consideration is flat (Euclidean). To every two points of this space $y_1(x)$ and $y_2(x)$ one can assign a vector $a(x)$:

$$a(x) = y_1(x) - y_2(x). \tag{3}$$

The vectors are invariants of the translation group (2). The value of the functions $a(x)$ is the projection of the vector a onto the locus determined by the point x. Thus each point of the four-dimensional space-time determines a locus. A measure of the density of the loci is the four-dimensional volume of the space-time region. The functional space of the vectors $a(x)$ can be divided into orthogonal subspaces by dividing the four-dimensional space into regions (of dimensionality 0 to 4) each of which determines the loci on which each subspace is constructed. As a consequence, the functional integral over all space-time becomes a product of functional integrals in the corresponding subregions. In particular, subdividing the space-time into non-intersecting space-like hyperplanes (SLH), we obtain a representation of it in the form of a product of functional integrals in open space-time regions and in the SLH which bound them.

If a functional of the probability amplitude for the distribution of the field on the hypersurface $t = -\infty$ is given, this same functional is on the hypersurface $t = +\infty$ determined by the propagation operator

$$\hat{G}[+\infty, y'; -\infty, y''] = N^{-1}\int_{-\infty}^{\infty} \exp\left\{\frac{i}{h} S[y(x); y', y'']\right\}\delta y(x). \tag{4}$$

Here N is a normalization factor; integrals of the type $\int \delta y(x)$ are understood in the sense of functional integrals. Then

$$\Psi[+\infty, y'] = \int_{-\infty}^{\infty} \hat{G}[+\infty, y'; -\infty, y'']\Psi[-\infty, y'']\delta y''. \tag{5}$$

Since the functional space is flat, the integral (4) separates into a product of functional integrals:

$$\int_{-\infty}^{\infty} = \int_{-\infty}^{\sigma_1} \times \int_{\sigma_1} \times \int_{\sigma_1}^{\sigma_2} \times \ldots \times \int_{\sigma_n} \times \int_{\sigma_n}^{+\infty}. \tag{6}$$

We can thus also introduce a functional of the probability amplitude (state vector) on intermediate hypersurfaces:

$$\Psi(\sigma) = \hat{G}(-\infty, \sigma)\Psi(-\infty);$$

$$\hat{G}(\sigma, \sigma') = N^{-1}(\sigma, \sigma')\int_{\sigma}^{\sigma'} \exp\left\{\frac{i}{h} S[y]\right\}\delta y. \tag{7}$$

The multiplication is understood in the operator sense, i.e., it implies an integration over the functional space of the functions on the hypersurface. The operators are defined in the following manner:

$$\hat{y}(x) = N^{-1}\int_{-\infty}^{\infty} y(x)\exp\left\{\frac{i}{h} S[y]\right\}\delta y(x)\hat{G}(+\infty; -\infty). \tag{8}$$

The invariance of the functional space with respect to the transformations (2) implies the invariance of the volume element δy with respect to the shift[2]

$$\delta(y + \eta) = \delta y. \tag{9}$$

This leads to the equations of motion in operator form:[12]

$$\begin{aligned}
\int \exp\left\{\frac{i}{h} S[y]\right\}\delta y
&\equiv \int \exp\left\{\frac{i}{h} S[y+\eta]\right\}\delta[y+\eta] \\
&\approx \int \exp\left\{\frac{i}{h} S[y] + \frac{i}{h}\int \frac{\delta S}{\delta y}\eta(x)\,d\Omega\right\}\delta y \\
&\approx \int \left(1 + \frac{i}{h}\int \frac{\delta S}{\delta y}\eta\,d\Omega\right)\exp\left\{\frac{i}{h} S[y]\right\}\delta y \\
&= \int \exp\left\{\frac{i}{h} S[y]\right\}\delta y \\
&+ \frac{i}{h}\int \eta(x)\,d\Omega \int \frac{\delta S}{\delta y(x)}\exp\left\{\frac{i}{h} S[y]\right\}\delta y.
\end{aligned} \tag{10}$$

Since $\eta(x)$ is arbitrary, we have

$$\int \frac{\delta S}{\delta y(x)}\exp\left\{\frac{i}{h} S[y]\right\}\delta y = 0. \tag{11}$$

As a consequence we obtain the operator form of the principle of least action:

$$\widehat{\delta s} / \delta y(x) = 0. \tag{12}$$

Thus the quantization by means of functional integrals in a Euclidean functional space is equivalent to the operator form of quantum field theory. But, as we have seen, the definition of a flat functional space requires a definite system of unit vectors, i.e., the geometry of space-time must be given. Therefore the authors who have attempted to quantize the gravitational field in operator form (like an ordinary field) were forced to take recourse, explicitly or implicitly, to some auxiliary space-time—the space of quantization. Misner[12] has tried to avoid this by constructing a functional space only on a topological four-dimensional manifold without a definite metric; however, he was led to a dynamical theory in which there is no dynamics. Thus, for example, the state vector is the same on all hypersurfaces; the Hamiltonian, which is identically equal to zero, leads to operators which are also equal at all points of space-time.

It will be shown below that the functional space in which the functional integrals for the quantization of the gravitational field are defined, is curved, so that the transition to a dynamical theory is impossible. Isn't this an enormous difficulty, so unnatural that one should search for other methods in developing a quantum theory of gravitation? It turns out that functional spaces with curvature are no less rare or artificial than the usual Riemannian spaces. The simplest example of such a space is the manifold of relativistic time-like trajectories in a two-dimensional pseudo-Euclidean plane (x, τ) joining the points A(0, 0) and B(0, 1). The interval in a plane is given by the quadratic form

$$ds^2 = d\tau^2 - dx^2. \tag{13}$$

If in this functional space, we define a group of infinitesimal motions by the infinitesimal transformations

$$\tau' = \tau + \varepsilon[u(\tau + x) + v(\tau - x)], \quad u(0) = u(1) = 0,$$

$$x' = x + \varepsilon[u(\tau + x) - v(\tau - x)], \quad v(0) = v(1) = 0, \tag{14}$$

which transform this space into itself (i.e., transform each time-like curve leading from A to B into another such curve), we easily see that the generators of this group of transformations do not commute:

$$L_1 = u_1(\xi)\frac{\partial}{\partial\xi} + v_1(\eta)\frac{\partial}{\partial\eta}, \quad L_2 = u_2(\xi)\frac{\partial}{\partial\xi} + v_2(\eta)\frac{\partial}{\partial\eta};$$

$$\xi = \tau + x, \quad \eta = \tau - x;$$

$$[L_1L_2] = \left(u_1\frac{du_2}{d\xi} - u_2\frac{du_1}{d\xi}\right)\frac{\partial}{\partial\xi} + \left(v_1\frac{dv_2}{\partial\eta} - v_2\frac{dv_1}{\partial\eta}\right)\frac{\partial}{\partial\eta} \neq 0. \tag{15}$$

If we set $\tau = ct$ and let c go to ∞, then the transformations have the form

$$t' = t + \varepsilon \lim_{c\to\infty}\frac{1}{c}\left[u\left(t + \frac{x}{c}\right) + v\left(t - \frac{x}{c}\right)\right] \to t,$$

$$x' = x + \varepsilon \lim_{c\to\infty}\left[u\left(t + \frac{x}{c}\right) - v\left(t - \frac{x}{c}\right)\right] \to x + \varepsilon\eta(t),$$

$$\eta(t) = u(t) - v(t). \tag{16}$$

The generators of this group commute with each other:

$$L = \eta(t)\frac{\partial}{\partial x}, \quad [L_1L_2] = 0. \tag{17}$$

In this case (in the nonrelativistic limit) the space of trajectories becomes flat, and it was with the help of this space that Feynman[3] first constructed a quantum mechanics of nonrelativistic particles in the form of integrals over all paths. Because of the curvature of the space of relativistic trajectories, there exists no quantum mechanics of relativistic spinless particles.

The next step in constructing a quantum theory of gravitation is the study of the curved functional space of all possible space-like hypersurfaces in four-dimensional space-time (the space of SLH).

2. THE SPACE OF SLH

A hypersurface in a four-dimensional space with the topology and the signature of a Minkowski space will be given by defining the four-dimensional coordinates x^i of its points as functions of three variables—the coordinates of the hypersurface:

$$x^i = x^i(u^\alpha), \quad i = 0, 1, 2, 3, \quad \alpha = 1, 2, 3. \tag{18}$$

All SLH are isomorphous to the three-dimensional Euclidean space and can be obtained, one from the other, by successive infinitesimal transformations:

$$\bar{x}^i(u) = x^i(u) + \delta x^i(u). \tag{19}$$

In the manifold of the SLH we can introduce a topology by defining an ϵ neighborhood: the SLH $\bar{\Sigma}$, connected with the SLH Σ by the transformation (19) lies in the ϵ neighborhood of the latter if $|\delta x^i(u)| < \epsilon$ for all i and all values of the variables u^α. Thus the set of all SLH forms a topological space whose points are different SLH. The functionals of the functions on these hypersurfaces are functions of the points of this space.

The set of SLH which depends on a single parameter t forms a one-dimensional path in this manifold.

The various derivatives $\partial x^i/\partial u^\alpha$ form a rectangular 3×4 matrix. Its third-rank minors are components of the covariant pseudovector n_i. These components are pseudoscalars with respect to transformations of the variables u^α. Using the n_i one can define the four-vector of the volume element of the hypersurface directed along its normal and with a modulus equal to the volume element:

$$d\sigma_i = \sqrt{-g}n_i du^1du^2du^3, \quad d\sigma = \sqrt{-gg^{ij}n_in_j}du^1du^2du^3. \tag{20}$$

Let us now introduce the notion of distance between two points in the space of the SLH as the mean distance between the corresponding hypersurfaces:

$$dl^2 = \lim_{\sigma\to\infty} \frac{1}{\sigma} \int [\delta x^n(u)]^2 d\sigma = \lim_{\sigma\to\infty} \frac{1}{\sigma} \int \frac{(\delta x^i n_i)^2}{n^2} d\sigma. \quad (21)$$

Here δx^i is the field of the displacement vector on account of which one SLH goes over into another, and δx^n is its normal component. Since the transition from one hypersurface to another is due only to δx^n, only this component takes part in the definition of the distance. Therefore we may take δx^i as being directed along the normal:

$$\delta x^i = \frac{g^{ij} n_j}{n} \delta x^n. \quad (22)$$

For infinitesimal transitions along the normal only transitions between SLH are possible. A transition out of the light cone is excluded since the normal to it lies on it.

If we introduce the notation

$$\delta x^n(u) = \nu(u) dl, \quad (23)$$

we conclude from (21)

$$\lim_{\sigma\to\infty} \frac{1}{\sigma} \int \nu^2(u) d\sigma = 1. \quad (24)$$

If we consider the space of real scalar functions of three variables u^α with the scalar product

$$(\varphi\psi) = \lim_{\sigma\to\infty} \frac{1}{\sigma} \int \varphi(u)\psi(u) d\sigma \quad (25)$$

and the norm

$$\|\varphi\|^2 = \lim_{\sigma\to\infty} \frac{1}{\sigma} \int \varphi^2(u) d\sigma, \quad (26)$$

we see that this space is tangent to the space of SLH. The quantity $\nu(u)$ is a unit vector in this space and defines the directions along which the transitions from one SLH to another can be accomplished. The scalar functions on the hypersurfaces are (contravariant) vectors in the space of SLH. Formula (25) defines the metric of the space of SLH.

We can introduce the operation of functional variation in the transition from one SLH to another; to this end we must introduce functional Christoffel symbols. A repeated variation determines the functional curvature tensor. The corresponding flat space is the space of SLH in space-time with a velocity of light equal to zero. Here any hypersurface without a tangent time axis is space-like. If we choose the spatial coordinates x^1, x^2, x^3 as the variables u^α and interpret the vector space of the functions of these coordinates as the manifold of all possible dependencies of the coordinate x^0 on x^1, x^2, x^3, then the whole space coincides with the tangent space. The normal to each hypersurface is directed along the time axis, and the volume element is always equal to $dx^1dx^2dx^3$, i.e., the metric is independent of the point. If $c \neq 0$, not all functions $x^0(x^1, x^2, x^3)$ are admissible, and the metric changes from one SLH to another.

If the hypersurface is tangent to the light cone in some point, then $n = 0$ in this point although $n^i \neq 0$, and if in this point $\delta x^i n_i \neq 0$, then $dl^2 = \infty$. Hence a hypersurface which is tangent to the light cone is metrically infinitely far from a SLH which does not pass through the tangent point (although topologically they may lie in the same ϵ neighborhood with a small ϵ). In particular, the light cones are infinitely far from all SLH and form a limiting manifold in the space under consideration.

With the help of the space of SLH we can construct a classical field theory in a covariant Hamiltonian form. Moreover, it plays an essential role in the representation of quantum field theory in the form of functional integrals, for the SLH constitute the points between which the propagation operator is defined. However, these questions lie outside the scope of our basic aim.

3. FUNCTIONAL SPACE OF ALL POSSIBLE SPACE-TIMES

The quantization of the general theory of relativity by the method of functional integrals consists in the following postulate:[12] in contrast to the classical theory, where only those space-time configurations exist in reality which satisfy the Einstein equations, the quantum theory assumes that the physical space-time can be an arbitrary Riemannian space with the signature $(-3, +3)$ which is realized with a probability amplitude proportional to $e^{iS/h}$, where $S = \kappa^{-2} \int R\, d\Omega$:

$$A[g] = N^{-1} \exp\left\{\frac{i}{h\varkappa^2} \int R d\Omega\right\}. \quad (27)$$

The normalization constant is determined from the requirement

$$\int A^* A \delta \ \{\text{all possible space-times}\} = 1. \quad (28)$$

It is now necessary to construct a measure in the space of all possible space-times. First it must be noted that by successive infinitesimal deformations of space-time we only reach spaces with the same topology. Spaces which are not isomorphous to the Minkowski space must either be excluded altogether or introduced in a special way (for example, by taking the sum of functional integrals over spaces of different topological types). For the time being we shall assume that the functional space is formed of spaces which are isomorphous to the Minkowski space. As is known,[15] any such space can be regarded as a four-dimensional hypersurface with the signature $(-1, +3)$ in a ten-dimensional plane space with the signature $(-1, +9)$ or $(-7, +3)$, so that all normals are either space- or time-like. The transition from one space-time to another is effected by shifts in six normals, i.e., is described by six functions of the four variables x^i.

However, different points in such a functional space can have the same internal geometry—these are the ones which are obtained from each other by shifts and deformations in the ten-dimensional space. An internal transition (i.e., one independent of the imbedding) can be described with the help of the tensor $\delta g_{ij}(x)$ which characterizes an infinitesimal deformation of space-time. However, in order to preserve the previous number of degrees of freedom, the components of this tensor must be related via four additional conditions. Apart from this there is still an arbitrariness in the choice of the coordinate system in space-time.

A concrete description of the functional space thus obtained can be given in the following manner: for each given metric one can define an infinitesimal region of the functional space near the space-time with the metric tensor $g^0_{ij}(x)$:

$$g_{ij}(x) = g_{ij}{}^0(x) + \delta g_{ij}(x), \tag{29}$$

where δg_{ij} is restricted by four conditions. This procedure can be carried out in covariant form.[16, 17] The volume $\sqrt{-g^0}\, dx^0 dx^1 dx^2 dx^3$ defines the "number" of the unit vectors in the space of δg_{ij} in each region of space-time. However, the δg_{ij} do not vary between $-\infty$ and $+\infty$, as other quantized fields do but only within an infinitesimal range of values. Therefore, the functional space is flat with respect to such infinitesimal δg_{ij} (tangent space). The entire functional space can be covered by such infinitesimal regions. It is a curved functional space since a system of unit vectors is given in each point and there are no common unit vectors for the whole space. The metric is given by the infinitesimal deviations δg_{ij}, as in the linearized theory of gravitation. In the neighborhood of each space-time we construct a linearized theory of gravitation.

What is the essential physical difference between our scheme and other attempts to quantize the general theory of relativity by means of functional integrals (refs. [5, 12, 13], etc.)? It lies in the fact that in the other approaches the integration goes over the functional space of the tensor g_{ij} or over the quantities a_i^α connected with it: $g_{ij} = \epsilon_{\alpha\beta} a_i^\alpha a_j^\beta$, where $\epsilon_{\alpha\beta}$ is the Minkowski tensor. This functional space is built on some space-time—the space of quantization. This results in a dynamical theory with a given and unalterable space of quantization. But then the tensor g_{ij}, which is varied, practically loses the meaning of a metric tensor, although it serves for contractions. In our method, on the other hand, we integrate over all possible types of space-time without introducing any auxiliary spaces of quantization. The metric tensor determines the geometry of each space-time on which the functional spaces for the other fields as well as its infinitesimal deformations are constructed. It should be noted that in such a description there is no question as to what should be quantized—g_{ij} or a_i. The space-time itself is quantized (varied).

4. NONGRAVITATIONAL FIELDS

The quantization of nongravitational fields [$y(x)$—in general a multi-component field] in the quantization of gravitation is carried out in the following way: a) all possible types of space-time are realized with the probability (27); b) in each space-time all possible configurations of fields are realized with the probability amplitude

$$A_g[y] = N_g{}^{-1} \exp\left\{\frac{i}{h} S[y, g]\right\}, \tag{30}$$

where $S[y, g]$ is the classical action, and the index g indicates a definite space-time. Thus the probability amplitude for the whole combination is given by

$$A[y, g] = A[g] A_g[y] = (NN_g)^{-1} \exp\left\{\frac{i}{h} \int \left[\frac{1}{\varkappa^2} R + Z\right] d\Omega. \right. \tag{31}$$

In other words, in each space-time one must define a quantum field theory and "sum" over these with the probability amplitude of the corresponding space-time. In the description of the quantum theory of interacting fields we shall use the method of adiabatic switching on of the interaction of the gravitational field with other matter. Since the classical equations of gravitation have the form

$$R_{ij} - {}^1\!/_2 g_{ij} R = \varkappa^2 T_{ij}, \tag{32}$$

the switching off of the interaction corresponds to the vanishing of the gravitational constant κ. But in the functional integral κ enters only in the combination $\kappa^2 h$ (h is the Planck constant); therefore the vanishing of κ leads to the same result as the vanishing of h in the gravitational terms, which, by the correspondence principle, leads to the classical equations of gravitation in empty space:

$$R_{ij} - {}^1/_2 g_{ij} R = 0. \tag{33}$$

For vanishing κ we thus obtain the classical space-time [determined by (33)] and in it, a quantum field theory, since the vanishing of κ does not affect the nongravitational terms. We are left, moreover, with a field of free linear gravitons. Indeed, let us consider Gupta's[7] expansion of the Lagrangian of the gravitational field in powers of κ by writing

$$g_{ij} = \varepsilon_{ij} + \kappa h_{ij}. \tag{34}$$

Here ϵ_{ij} is a metric which satisfies (33) or equations of gravitation with given macroscopic masses (i.e., a macroscopic metric). It is usually simplest to choose it to be flat. As κ tends to zero, (34) leads to $g_{ij} = \epsilon_{ij}$, i.e., the metric is unaffected by changes in h_{ij}. However, the Lagrangian of the field expressed through the tensor h_{ij}, does not vanish since the term which is quadratic in h_{ij} does not contain the gravitational constant κ. This corresponds to a field of free linear gravitons; however, the tensor h_{ij} which describes these has no longer the meaning of an addition to the metric, which is uniquely defined.

Thus we shall assume that for $t \to \pm\infty$, $\kappa \to 0$ the space-time goes over into a flat space and that there exists a field of free gravitons described by a linearized theory of gravitation.[6-8] Switching on of κ between $t = -\infty$ and $t = +\infty$ does not lead to the inclusion of the next terms in Gupta's expansion, but to a variation over all spaces which become flat at infinity. In each of these spaces, one can introduce an S_g matrix through the integral (30) or by some other means. It will be a functional of the space-time. The complete S matrix is obtained with the help of the probability amplitudes for each space-time:

$$\hat{S} = N^{-1} \int \hat{S}_g \exp\left\{\frac{i}{h\kappa^2} \int R d\Omega\right\} \delta g. \tag{35}$$

Here δg is the volume element of the functional space introduced in the preceding section. Analogously we define the propagation function between the hypersurfaces $t = -\infty$ and $t = +\infty$ as the propagation function averaged (in the amplitude) with respect to space-time:[18]

$$G(\mathbf{r}, \mathbf{r}') = \frac{\int G(\mathbf{r}, \mathbf{r}'; g) \exp\left(\frac{i}{h\kappa^2} \int R d\Omega\right) (S_g)_0 \delta g}{\int \exp\left(\frac{i}{h\kappa^2} \int R d\Omega\right) (S_g)_0 \delta g}. \tag{36}$$

For n fermions interacting only via the gravitational field (for example, neutrinos) we have

$$G(\mathbf{r}_1 \ldots \mathbf{r}_n;\ \mathbf{r}_1' \ldots \mathbf{r}_n') = \frac{\int G(\mathbf{r}_1 \ldots \mathbf{r}_n; \mathbf{r}_1' \ldots \mathbf{r}_n'; g) (S_g)_0 \exp\left(\frac{i}{\kappa^2 h} \int R d\Omega\right) \delta g}{\int (S_g)_0 \exp\left(\frac{i}{\kappa^2 h} \int R d\Omega\right) \delta g} \tag{37}$$

where

$$G(\mathbf{r}_1, \ldots \mathbf{r}_n;\ \mathbf{r}_1' \ldots \mathbf{r}_n';\ g) = \det |G(\mathbf{r}_i, \mathbf{r}_j';\ g)|. \tag{38}$$

Thus the fermions do not interact in each separate space-time; but it is impossible to write the complete function (37) in the form (38), which indicates that there is an interaction between the particles.

If there are gravitons at $\pm\infty$ the S matrix is defined in the following way (for one graviton at $-\infty$ and none at $+\infty$):

$$\hat{S}_{ij}(\mathbf{r}) = \lim_{\substack{t \to -\infty \\ \kappa(-\infty) \to 0}} N^{-1} \int \frac{[g_{ij}(\mathbf{r}, t) - \varepsilon_{ij}]}{\kappa} \hat{S}_g \exp\left(\frac{i}{\kappa^2 h} \int R d\Omega\right) \delta g. \tag{39}$$

Analogously we can introduce the complete S matrix including gravitons.

We note that if we had defined the functional space of the gravitational and nongravitational fields in a given space of quantization, then we could, in principle, carry out the integration over δg in (35) or (39) (in analogy to the elimination of the boson field in [18]) and we would be left with the interacting nongravitational fields. But then they would be given in a definite space of quantization and the gravitation would not be equivalent to a curvature of space-time. Thus the quantization in a space of quantization is inconsistent with the spirit of Einstein's theory of relativity.

5. IMPOSSIBILITY OF A DYNAMICAL THEORY

In the quantization of the gravitational field the corresponding functional space cannot, because of its curvature, be written as a direct product of subspaces defined by some regions in space-time. Therefore, it is impossible to write the integral of (31), which carries $\Psi(-\infty)$ into $\Psi(+\infty)$, in the form of a product of independent propagation operators between certain hypersurfaces, and thus the state vector can only be introduced at $t = \pm\infty$. Indeed, the state vectors are given on the system of SLH. But to which space-time can this system of SLH belong if the space-time is varied? Perhaps it belongs to all spaces at the same time? Otherwise one must introduce an auxiliary space-time, distinguished from all other space-times, in which the system of SLH can be defined.

Let us see whether it is impossible to establish a one-to-one correspondence between the SLH in different space-times. To this end there must first be a correspondence between the boundary points of the corresponding spaces of the SLH—the light cones. Let us assume that we have established such a correspondence for two spaces and have introduced such a system that corresponding points have the same coordinates.[17] Then we have on the light cones

$$g_{ij}^{(1)}\,dx^i\,dx^j = 0, \qquad g_{ij}^{(2)}\,dx^i\,dx^j = 0. \tag{40}$$

Here each solution of one equation must be a solution of the other. This implies $g_{ij}^{(1)} = \lambda g_{ij}^{(2)}$. Since this must hold for all x^i, this means that spaces (1) and (2) are conformal. Since four-dimensional Riemannian spaces are in general not conformal to one another, this correspondence is not possible for all spaces—the spaces of the SLH of different space-times are not topologically isomorphous to each other. This means that, whatever coordinates common to all spaces we introduce, no hypersurface equation will describe a space-like hypersurface in all space-times simultaneously; by the causality principle no state vector can therefore be defined on it.

In each space-time we can introduce operators of nongravitational fields $\hat{y}(x)$ whose dynamics is described by a quantum field theory in the given space-time. The formal averaging [summing with the weight function $\exp\{(i/\kappa^2 h)\int R d\,\Omega\}$ over all space-times does not, because of the absence of a unique correspondence between their points, lead to a uniquely defined operator for different correspondences; one and the same operator in one space-time corresponds to different operators in another space-time. Hence the introduction of coordinate-dependent operators in the quantization of the gravitational field is devoid of physical meaning and the only characteristic of the system is the S matrix. The introduction of a metric operator is meaningless for the same reasons. The theory becomes in essence an S matrix theory. This is in agreement with Heisenberg's point of view, which has been actively developed in the last decade. A specific physical verification of this result may be seen in the conclusion of Bronshteĭn[6] that the components of the gravitational field are principally unobservable according to the quantum theory of measurement. Bronshteĭn concluded in his article that it is necessary to alter the quantum theory of gravitation radically. The transition to an S matrix theory does just this.

6. CONCLUSION

Thus the Feynman formalism is the only useful one of the existing formalisms for the quantization of the gravitational field regarded as the quantization of space-time. This leads to the conclusion that this method of quantization is in some sense better than the others, having a deeper physical meaning. Up to now, as long as the functional space of the quantized field is flat, this method of quantization can be used to derive others, for example Schwinger's dynamical principle, and all these methods are equivalent. But as soon as the space of quantization becomes curved, the operator dynamical principles of quantization become useless and the theory is in essence an S matrix theory.

As to a practical account of gravitational effects: as long as they are small their contribution to the integral of (31) gives only space-times which are nearly flat, i.e., a small region of functional space. But in the small this space is flat (coincides with the tangent space). The theory corresponding to the tangent space is the quantum theory of the weak gravitational field with an auxiliary Minkowski space of quantization.[6-8] Therefore one may use this theory to take account of weak gravitational effects.

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