

CRITICAL FIELDS OF THIN SUPERCONDUCTING FILMS. II. SPECULAR REFLECTION,  
VERY THIN FILMS

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The critical fields of thin films are found in the case of specular reflection of the electrons from the surface. Very thin films are considered, for which quantization of the electron-momentum projection on a direction perpendicular to the film plane is significant near the transition temperature. Pure films, as well as films containing arbitrary impurity concentrations, are investigated in detail. The analysis is performed for the entire temperature range.

**I**N an earlier paper<sup>[1]</sup> (referred to as **I**) we considered the critical fields of thin films (film thickness  $L$  much smaller than the BCS correlation parameter  $\xi_0 \sim v/T_c$ ) in the case of diffuse reflection of the electrons from the surface. There is undisputed interest in an analysis of the same problem under the assumption of specular reflection. This is all the more important because in many cases an appreciable contribution to the effect under consideration is made by electrons incident almost parallel to the surface of the film, and such electrons are more likely to be specularly reflected from the surface than diffusely, even when the surface is quite rough. Furthermore, as already indicated in **I**, the results obtained there hold for very thin films near the transition temperature, when the glancing angle of the electrons that make the main contribution to the considered effect is so small that quantization of the projection of the momentum on the direction perpendicular to the plane of the film becomes significant. In this case the quasiclassical-trajectory method used in **I** is not valid, but good quantum numbers for such electrons are the projection of the momentum on the plane of the film and the number  $n = 1, 2, \dots$  (for not too large  $n$ ) characterizing the motion in the direction perpendicular to the plane of the film. In other words, for such electrons  $p_z \sim 1/L$ , and since the surface roughnesses are much smaller than the thickness of the film, such electrons are insensitive to them and are always specularly reflected. Electrons for which  $p_z \gg 1/L$ , to the contrary, are quite sensitive to the surface roughnesses, but for them one can use successfully the quasiclassical approximation and the character of the surface can be taken

into account by introducing a corresponding reflection law.

The problem of the critical fields of thin films under specular reflection was considered earlier for the case of ideally pure films in the paper of Nambu and Tuan,<sup>[2]</sup> but many of their results are in error. In the direct vicinity of  $T_c$ , ideally pure films were considered also by Zaitsev.<sup>[3]</sup>

In Sec. 1 we calculate the kernel of the integral equation for the case of specular reflection, in Sec. 2 we find the critical fields in the entire temperature interval for ideally pure films of different thickness. Further, in Sec. 3 we consider films containing different impurity concentrations, and in Sec. 4 contaminated very thin films near the transition temperature (the results for such films far from  $T_c$  are given in Sec. 3 of **I**,<sup>[1]</sup> and in Sec. 3 of this paper respectively for diffuse and scattered reflection laws).

## 1. KERNEL OF INTEGRAL EQUATION IN THE SPECULAR CASE

As shown in Sec. 1, **I**, the problem of determining the critical field reduces to determining the maximum field for which there exists a nontrivial solution of the integral equation (1.15)**I**, whose kernel is constructed of the functions  $\Phi(\omega; z_1, z_2)$ . In the case of specular reflection of the electrons from the surface of the films, likewise, these functions are easiest to find by the trajectory method described there (see also the paper of de Gennes and Tinkham<sup>[4]</sup>):

<sup>1)</sup>In references to paper **I** we shall place the symbol **I** after the number of the corresponding formula or section from **I**.

$$\begin{aligned} \Phi(\omega; z_1, z_2) &= \frac{1}{2v} \int_0^1 \frac{d\mu}{\mu \operatorname{sh}(2\omega L/\mu v)} \\ &\times \left[ J_0 \left( \frac{\sqrt{1-\mu^2}}{\mu} eH \left( \frac{L^2}{2} - z_1^2 - z_2^2 \right) \right) \right. \\ &\times \operatorname{ch} \left( 2\omega \frac{z_1 + z_2}{\mu v} \right) + J_0 \left( \frac{\sqrt{1-\mu^2}}{\mu} eH (z_1^2 - z_2^2) \right) \\ &\left. \times \operatorname{ch} \left( 2\omega \frac{L - |z_1 - z_2|}{\mu v} \right) \right]. \end{aligned} \quad (1.1)^*$$

This expression<sup>2)</sup> is applicable so long as we can neglect the contribution of those electrons for which account must be taken of the quantization of the projection of the momentum on the direction perpendicular to the plane of the film. In other words, this expression is applicable when the contribution  $\mu = \cos \theta \leq 1/Lp_0$  is insignificant, as is the case when at least one of the dimensionless parameters  $eHL^2$  and  $\omega L/v$  is much larger than  $1/Lp_0$ . The oblique shading in Fig. 1 shows schematically the corresponding region of variation of these parameters where the quasi-classical trajectory method is applicable.

For different relations between the parameters, the function  $\Phi(\omega; z_1, z_2)$  takes the following form: in region I,  $\omega L/v \ll eHL^2, 1$ ,

$$\begin{aligned} \Phi(\omega; z_1, z_2) &= \frac{1}{4\omega L} \left[ \exp \left\{ -eH \left( \frac{L^2}{2} - z_1^2 - z_2^2 \right) \right\} \right. \\ &\left. + \exp \left\{ -eH |z_1^2 - z_2^2| \right\} \right]; \end{aligned} \quad (1.2a)$$

in region II,  $eHL^2 \ll \omega L/v \ll 1$ ,

$$\begin{aligned} \Phi(\omega; z_1, z_2) &= \Phi_0(\omega; z_1, z_2) \\ &- \frac{\pi^2}{128} (eH)^2 \frac{v}{\omega^2 L^2} \left[ \left( \frac{L^2/2 - z_1^2 - z_2^2}{\cos \pi(z_1 + z_2)/2L} \right)^2 \right. \\ &\left. + \left( \frac{z_1^2 - z_2^2}{\sin \pi(z_1 - z_2)/2L} \right)^2 \right]; \end{aligned} \quad (1.2b)$$

in region III,  $1, eHL^2 \ll \omega L/v$ ,

$$\Phi(\omega; z_1, z_2) = \Phi_0(\omega; z_1, z_2) - \frac{v^2}{6\omega^3} (eHz)^2 \delta_\omega(z_1 - z_2). \quad (1.2c)$$

Here  $\Phi_0(\omega; z_1, z_2)$  denotes the value of the function  $\Phi(\omega; z_1, z_2)$  in the absence of a magnetic field.

It is now necessary to calculate  $\Phi(\omega; z_1, z_2)$  in the region IV,

$$\min(\omega L/v, eHL^2) \ll 1/p_0L,$$

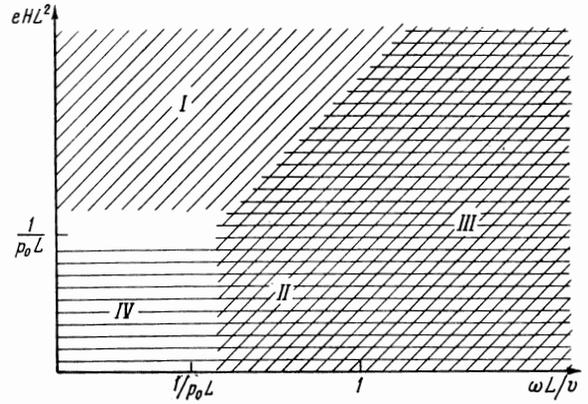


FIG. 1. Regions of applicability of different methods of calculating the functions  $\Phi(\omega)$ . The oblique shading shows the region of applicability of the quasiclassical method of trajectories, the horizontal shading—the region of applicability of perturbation theory. The numbers I–IV shows the regions of applicability of the corresponding limiting values of  $\Phi(\omega)$ .

where the trajectory method is not applicable. A different approach is needed here. We write, for an ideal plane-parallel plate in a homogeneous magnetic field, the Green's function that enters into the initial integral equation (1.1)I:

$$\begin{aligned} \mathfrak{G}_\omega(\mathbf{r}_1 - \mathbf{r}_2; z_1, z_2) &= \sum_n \int \frac{d\mathbf{p}}{(2\pi)^2} \frac{\exp\{i\mathbf{p}(\mathbf{r}_1 - \mathbf{r}_2)\} \psi_{p_n}(z_1) \psi_{p_n}^*(z_2)}{-i\omega - (p^2 - p_0^2)/2m - \xi_{p_n}^2}. \end{aligned} \quad (1.3)$$

In this formula,  $\mathbf{r} = \{x, y\}$  and  $\mathbf{p} = \{p_x, p_y\}$  lie in the plane of the plate, and  $\psi_n^p(z)$  are solutions of the one-dimensional Schrödinger equation

$$\left( -\frac{1}{2m} \frac{d^2}{dz^2} - \frac{ep_y}{m} Hz \right) \psi_{p_n}(z) = \xi_{p_n} \psi_{p_n}(z) \quad (1.4)$$

with boundary conditions

$$\psi_{p_n}(z) |_{z=\pm L/2} = 0.$$

(We have chosen, as usual, the vector potential  $A_x = Hz$ ,  $A_z = A_y = 0$ , and neglected in the equation the term quadratic in the field, since  $eHL^2 \ll Lp_0$  always.) In the case of interest to us ( $eHL^2 \ll 1/Lp_0$ ) Eq. (1.4) can be solved by perturbation theory. In fact, the region of applicability of perturbation theory is even broader, namely  $eHL^2 \ll n/Lp_0$ , so that it is applicable when the main contribution to  $\Phi(\omega; z_1, z_2)$  is made by  $\mu = \cos \theta = \pi n/Lp_0 \gg eHL^2$ , as is the case when  $\omega L/v \gg eHL^2$ . In Fig. 1, the region of applicability of perturbation theory is shown schematically by horizontal shading. Thus, in regions II and III we can apply equally

\*sh  $\equiv$  sinh, ch  $\equiv$  cosh.

<sup>2)</sup>In this paper we put, to simplify the formulas,  $c = 1$  (the velocity of light  $c$  enters in all formulas as part of the combination  $H/c$ ).

well either the quasiclassical-trajectory method or perturbation theory.<sup>3)</sup>

In practice it is preferable to calculate directly by perturbation theory the function  $\Phi(\omega; z_1, z_2)$  expressed in terms of the product of two Green's functions:

$$\Phi(\omega; z_1, z_2) = \frac{1}{2\pi\nu} \int dr \mathfrak{G}_\omega(r; z_1, z_2) \mathfrak{G}_{-\omega}(r; z_1, z_2). \quad (1.5)$$

In the absence of a magnetic field, the Green's function takes the form (1.3) with

$$\psi_n(z) = \sqrt{\frac{2}{L}} \sin \frac{\pi n}{L} \left( \frac{L}{2} + z \right), \quad \xi_n = \frac{1}{2m} \left( \frac{\pi n}{L} \right)^2. \quad (1.6)$$

It is easy to write the usual expansion of the product  $\mathfrak{G}_\omega \mathfrak{G}_{-\omega}$  up to second-order terms in the magnetic field, and obtain an expression for  $\Phi(\omega; z_1, z_2)$  for different relations between the parameters  $\omega L/v$  and  $eHL^2$ , including formulas (1.2b) and (1.2c). We note, however, that for weak fields ( $eHL^2 \ll 1$ ) the "gap"  $\Delta(z)$  does not vary along the thickness of the film. This is connected with the fact that the field-dependent contribution to the kernel of the integral equation is in this case a small addition to the field-independent kernel, which leads to a maximum temperature  $T_C$  ( $H=0$ ) just for the solution  $\Delta = \text{const.}$ <sup>4)</sup> We can therefore integrate  $\Phi(\omega; z_1, z_2)$  with respect to the coordinates:

$$\Phi(\omega) = \frac{1}{L} \int_{-L/2}^{+L/2} \int_{-L/2}^{+L/2} \Phi(\omega; z_1, z_2) dz_1 dz_2. \quad (1.7)$$

Then the corresponding expressions in the expansion greatly simplify from the very beginning, owing to the orthogonality of  $\psi_n(z)$  (Eq. (1.6)). As a result we obtain

$$\begin{aligned} \Phi(\omega) = & 1/2\omega - \frac{4}{\pi^3} (eH)^2 \frac{v^2 L}{\rho_0} \sum_{n_1, n_2=0}^{L\rho_0/\pi} \left[ 1 - \left( \frac{\pi}{2L\rho_0} \right)^2 ((2n_1 + 1)^2 \right. \\ & \left. + (2n_2 + 1)^2) \right] \left( \frac{1}{(2n_1 + 1)^2} - \frac{1}{(2n_2 + 1)^2} \right)^2 \\ & \times \left[ 4\omega^2 + \left( \frac{\pi}{L} \right)^4 \left( \frac{(2n_1 + 1)(2n_2 + 1)}{2m} \right)^2 \right]^{-1}. \end{aligned} \quad (1.8)$$

In limiting cases  $\Phi(\omega)$  takes the following form

Region IV ( $\omega L/v \ll 1/L\rho_0$ ). We can neglect here the first term in the denominator of (1.8) and the

second in the numerator; summing over  $n_1$  and  $n_2$ , we get

$$\Phi(\omega) = \frac{1}{2\omega} - \frac{\pi}{720} (eH)^2 \frac{\rho_0 L^3}{2\omega}. \quad (1.9a)$$

Region II ( $1/L\rho_0 \ll \omega L/v \ll 1$ ). Neglecting the second term of the numerator and replacing one of the summations by integration, we obtain

$$\Phi(\omega) = \frac{1}{2\omega} - \frac{31\zeta(5)}{16\pi^4} (eH)^2 \frac{vL^3}{\omega^2}. \quad (1.9b)$$

Region III ( $\omega L/v \gg 1$ ). We can neglect here the second term of the denominator

$$\Phi(\omega) = \frac{1}{2\omega} - \frac{1}{72} (eH)^2 \frac{L^2 v^2}{\omega^3}. \quad (1.9c)$$

## 2. IDEALLY PURE FILMS

We start the analysis with very strong fields,  $eHL^2 \sim 1$ . Then the main contribution to the kernel of the integral equation (1.5) is made by  $\Phi(\omega; z_1, z_2)$  from region I (Eq. (1.2a)). Accurate to a factor under the logarithm sign we obtain, in analogy with Sec. 2, I, the integral equation

$$\begin{aligned} \Delta(z_1) = & \frac{\ln(v/LT)}{\ln(v/LT_c)} \int_{-L/2}^{+L/2} \left[ \exp \left\{ -eH \left( \frac{L^2}{2} - z_1^2 - z_2^2 \right) \right\} \right. \\ & \left. + \exp \left\{ -eH |z_1^2 - z_2^2| \right\} \right] \Delta(z_2) \frac{dz_2}{2L}. \end{aligned} \quad (2.1)$$

Unlike the similar case for diffuse reflection, the kernel here is non-degenerate, so that a variational method must be used for the solution, choosing as the varied functions  $\exp[-\alpha z^2]$  and  $\exp[-\beta(L^2/4 - z^2)]$ . The maximum value of the field is reached when  $\alpha = \beta = eH$ , and the equation for its determination can be written in the form

$$f_1(eHL^2) = \frac{\ln(v/LT_c)}{\ln(v/LT)}. \quad (2.2)$$

The function  $f_1$  is very complicated, but it is easy to evaluate its behavior if one knows its limiting values:

$$f_1(x) = \begin{cases} \sqrt{\pi/2x} & , \quad x \gg 1 \\ 1 - 5x/24 & , \quad x \ll 1 \end{cases}. \quad (2.3)$$

Thus, as  $T \rightarrow 0$  the critical field increases like  $\ln^2(v/LT)$ , unlike the diffuse case, when the field increases like the first power of the logarithms. In addition, in strong fields ( $eHL^2 \gg 1$ ) the "gap" is

$$\Delta(z) \sim \exp(-eHz^2),$$

i.e., here, too, the "gap"  $\Delta(z)$  depends on  $z$ , but, unlike the diffuse case, it reaches a maximum at

<sup>3)</sup>We can solve Equation (1.4) in the quasiclassical approximation, and then return to expression (1.1) by using (1.3) and the definition (1.5) of the functions  $\Phi(\omega; z_1, z_2)$

<sup>4)</sup>This problem, as already indicated in I, can be formulated as the problem of determining the maximum temperature at which a non-trivial solution of the integral equation exists for a specified magnetic field  $H$ .

the center of the film and decreases towards the edges.

In expressions (2.2)I and (2.2) obtained by us for the critical field far from  $T_c$ , the main contribution is made by electrons for which  $\mu = \cos \theta \sim 1$ . For such electrons we expect more readily diffuse reflection than specular, so that apparently the result (2.2) is so far only of purely theoretical interest.

On approaching  $T_c$ , as follows from (2.3), the critical field decreases, and the "gap"  $\Delta(z)$  tends to a constant value. From expression (1.2a) for the kernel of the integral equation, we obtain with logarithmic accuracy an equation for the critical field:<sup>5)</sup>

$$eHL^2 = \frac{24}{5} \frac{\ln(T/T_c)}{\ln(eHL^2 v / LT_c)}. \quad (2.4)$$

This expression (compare with (2.5)I for diffuse reflection) is valid so long as the main contribution to the kernel is made by region I for  $\Phi(\omega; z_1, z_2)$ , i.e., under the condition

$$\max(1/Lp_0, L/\xi_0) \ll eHL^2 \ll 1.$$

Corresponding to this condition is a temperature in the region

$$1 \gg 1 - T/T_c \gg \max(L/\xi_0, 1/Lp_0).$$

In our case the main contribution to the effect is made by electrons incident at glancing angles,  $\mu \sim eHL^2$ , so that the nature of the reflection of such electrons from the surface can be assessed from the magnitude of the experimentally observed critical field.

With further increase in temperature, in the case of not very thin films ( $L^2 \gg \xi_0/p_0$ ), it is necessary to calculate the critical field with the expression for  $\Phi(\omega)$  from region II or with (1.9b). We then get

$$(eH)^2 = \frac{32\pi^3}{31\xi(5)} \frac{T_c}{L^3 v} \left(1 - \frac{T}{T_c}\right), \quad 1 - \frac{T}{T_c} \ll \frac{L}{\xi_0}, \quad L^2 \gg \frac{\xi_0}{p_0}. \quad (2.5)$$

The numerical coefficients in this formula is smaller by a factor 1.3 than the corresponding coefficient for diffuse reflection (2.6)I. The main contribution to the expression for the critical field is made here by electrons for which  $\mu = \cos \theta \sim L/\xi_0$ . If the films are sufficiently thin, we can apparently expect for such electrons specular reflection from surface more readily than diffuse reflection.

For very thin films ( $L^2 \ll \xi_0/p_0$ ), using the expression for  $\Phi(\omega)$  in region IV (1.9a), we can find the following expression for the critical field at  $1 - T/T_c \ll 1/Lp_0$ :

$$(eH)^2 = \frac{720}{\pi} \frac{1}{p_0 L^5} \frac{1 - T/T_c}{\ln(\pi\gamma v / 2L^2 p_0 T_c) + 1.9}. \quad (2.6)$$

The quantity under the logarithm sign in this formula can be readily calculated by first summing over  $\omega = \pi T(2n + 1)$  in the general expression (1.8) for  $\Phi(\omega)$ , and then summing over  $n_1$  and  $n_2$ .

This result (2.6) is valid regardless of the degree of roughness of the film surface.

Curves 1s and 1d of Fig. 2 (see below) illustrate the temperature dependence of the critical field of pure films for specular and diffuse reflection of electrons from the surface, respectively.

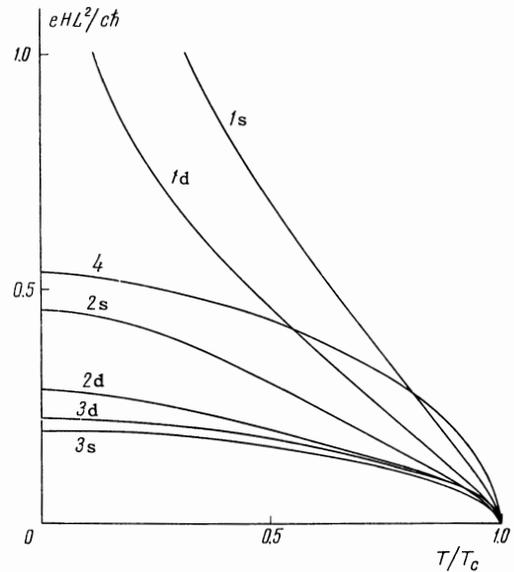


FIG. 2. Temperature dependence of the critical field for films with  $L = 0.01\xi_0$ ; curves 1—pure films, 2—case C ( $l \approx 0.3\xi_0$ ), 3—case D, 4—case E ( $l \approx 0.1L$ ).

### 3. CONTAMINATED FILMS

To find the critical fields of contaminated films in the region where the quasiclassical approach is valid, we use Eqs. (3.1)–(3.3)I of Section 3, I, from which we can obtain the functions  $\Phi_T(\omega; z_1, z_2)$  when  $\Phi(\omega; z_1, z_2)$  is known.

Constructing, as before, the kernel of the integral equation (1.15)I from the  $\Phi_T(\omega)$ , we obtain the following results for the critical fields of thin films containing different impurity concentrations.

A. Rather pure films ( $l \gg \xi_0$ ). In strong fields  $eHL^2 \sim 1$  and for  $T \ll 1/\tau$ , the critical field is determined, with logarithmic accuracy, from the equation

<sup>5)</sup>This result was obtained also by Ovchinnikov<sup>[5]</sup>.

$$f_1(eHL^2) = \frac{\ln(v/LT_c)}{\ln(v\tau/L)}. \quad (3.1)$$

we get

$$eHL^2 = \frac{192}{5\pi} T_c \tau (1 - T/T_c). \quad (3.6)$$

For  $T \gg 1/\tau$ , the critical field is determined by the same equations as for ideally pure films: (2.2), (2.4), (2.5), or (2.6). Thus, just as in the diffuse case, the field ceases to diverge at low temperatures when  $T \sim 1/\tau$ .

B.  $l \sim \xi_0$ ,  $1 - T/T_c \ll 1$ . Recognizing that the "gap" does not depend here on  $z$ , we obtain for

$$1 \gg eHL^2 \gg \max(L/\xi_0, L/l, 1/p_0L),$$

in analogy with (3.6)I, the following equation (accurate to a factor of the order of unity under the logarithm sign):

$$\left(1 - \frac{5}{24} eHL^2\right) \left(\ln \frac{T}{T_c} - \psi\left(\frac{1}{2}\right) + \psi\left(\frac{1}{2} + \frac{5}{98\pi} \frac{eHL^2}{T\tau}\right)\right) + \frac{5}{24} eHL^2 \ln \frac{eHLv}{T_c} = 0. \quad (3.2)$$

From this, when

$$1 \gg 1 - T/T_c \gg \max(L/\xi_0, \xi_0L/l^2, 1/p_0L)$$

we get

$$eHL^2 = \frac{24}{5} \frac{1 - T/T_c}{\ln(eHLv/T_c) + \pi/8T_c\tau}. \quad (3.3)$$

In the direct vicinity of  $T_c$ , depending on the relation between the parameters  $L/\xi_0$ ,  $L/l$ , and  $1/Lp_0$ , we obtain one of the expressions (2.5), (2.6), or (4.3).

C.  $\xi_0 \gg l \gg \sqrt{\xi_0 L}$ . From (3.2) we obtain in this limiting case, when

$$eHL^2 \gg \max(L/l, 1/Lp_0),$$

i.e.,<sup>6)</sup> when

$$1 - T/T_c \gg \max(L\xi_0/l^2, \xi_0/Lp_0l):$$

we have

$$\ln \frac{T}{T_c} = \psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{2} + \frac{5}{96\pi} \frac{eHL^2}{T\tau}\right). \quad (3.4)$$

In particular, when  $T \ll T_c$

$$eHL^2 = \frac{24\pi}{5\gamma} T_c \tau, \quad (3.5)$$

and when

$$1 \gg 1 - T/T_c \gg \max(L\xi_0/l^2, \xi_0/Lp_0l)$$

<sup>6)</sup>An error has crept into the estimate of the upper (temperature) limit of the applicability of the formulas (3.7), (3.12), and (3.14) of I. The limits of applicability of these formulas are the same as for formulas (3.3), (3.4), and (3.6) of the present paper.

Near the transition temperature ( $1 - T/T_c \ll L\xi_0/l^2$ ), for not very thin films ( $L^2 \gg l/p_0$ ), the critical field is determined by formula (2.5), and for very thin films ( $L^2 \ll l/p_0$ ) when  $1 - T/T_c \ll \xi_0/Lp_0l$ , it is determined by expression (4.3) obtained in Sec. 4 below for very thin films with impurities.

D. In the case when  $\sqrt{\xi_0 L} \gg l \gg L$  the critical field is determined by the following equations (for  $L^2 \gg l/p_0$ ):

$$\ln \frac{T}{T_c} = \psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{2} + \frac{31\zeta(5)}{16\pi^5} \frac{(eH)^2 L^3 v}{T}\right). \quad (3.7)$$

From this we obtain near absolute zero ( $T \ll T_c$ )

$$(eH)^2 = \frac{4\pi^5}{31\zeta(5)\gamma} \frac{T_c}{L^3 v}, \quad (3.8)$$

and near the transition temperature ( $1 - T/T_c \ll 1$ ) we return to formula (2.5).

E. (The Maki case.<sup>1,61)</sup> If the mean free path  $l \ll L$ , then the critical field does not depend on the manner of reflection of electrons from the surface and is determined by formulas (3.22)–(3.24)I.

Thus, in cases B–D the critical fields for diffuse and specular reflection differ only in numerical coefficient (for not very thin films, if  $L^2 p_0 \gg \min(\xi_0, l)$ ). The main contribution to the effect under consideration in cases A–C is made by the electrons for which  $\mu = \cos \theta \sim eHL^2$  (except for the temperature region directly adjacent to  $T_c$ ). In case D, and in cases A–C in the direct vicinity of the transition temperature, for not very thin films, the main contribution is made by electrons with  $\mu = \cos \theta \sim L/\min(\xi_0, l)$ .

Figure 2 illustrates the temperature dependence of the critical field for films of thickness  $L = 0.01\xi_0$  ( $\xi_0 = \gamma v/\pi^2 T_c \approx 0.18 v/T_c$ ) for different impurity concentrations. The letters s and d in this figure designate respectively specular and diffuse reflection.

The dependence of the critical field on the mean free path for films of the same thickness at  $T = 0$  is shown in Fig. 3.

#### 4. VERY THIN FILMS WITH IMPURITIES

We consider here very thin contaminated films in the case of a weak critical field, i.e., at temperatures close to the transition temperature. In this case the quasiclassical-trajectory method is

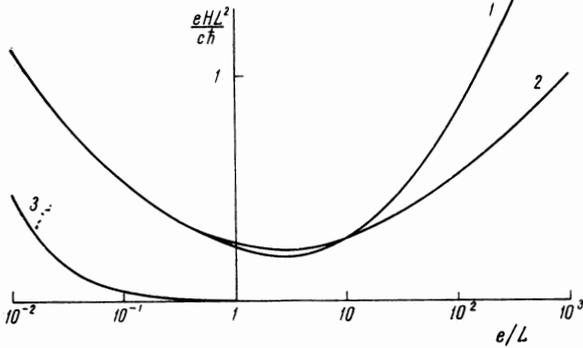


FIG. 3. Dependence of the critical field on the mean free path for films with  $L = 0.01\xi_0$ . Curve 1—specular reflection, 2—diffuse reflection, 3—dependence of the upper critical field  $H_{c2}$  of a bulky sample on the mean free path.

not applicable, but perturbation theory can be used successfully.

To solve this problem we use the diagram technique developed in the papers of Abrikosov and Gor'kov<sup>[7]</sup> and Edwards,<sup>[8]</sup> which make it possible to take into account the impurities by applying this technique to a plane parallel plate of small thickness.

To calculate the Green's function of a normal metal for a plate in the presence of impurities, it is necessary to sum the diagrams shown in Fig. 4.

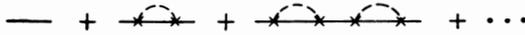


FIG. 4

Just as in the case of an infinite sample, diagrams with crossed dashed lines are of no significance if  $Lp_0 \gg 1$ . As a result it turns out that the Green's function  $\mathcal{G}_\omega^0$  of the sample with impurities in the absence of a magnetic field differs from the corresponding Green's function  $\mathcal{G}_\omega^0$  of the impurity-free sample in that  $\omega$  is replaced by  $\tilde{\omega} = \omega + 1/2\tau$ , where  $\tau$  is the free path time,<sup>7)</sup> defined, as usual, by:

$$\frac{1}{\tau} = \frac{nm p_0}{(2\pi)^2} \int |u(\mathbf{q})| d\mathbf{q}. \quad (4.1)$$

In the calculation of  $\Phi_\tau$  (Eq. (1.5)) it must be taken into account that in the weak fields under consideration the "gap"  $\Delta$  is constant along the

<sup>7)</sup>The Green's function of a sample of arbitrary shape, averaged over the positions of impurity atoms, will apparently have a similar form if we expand it in the eigenfunctions of the impurity-free sample.

thickness of the film, so that integration can be carried out with respect to  $z_1$  and  $z_2$ , thus greatly simplifying the subsequent calculations. The field-independent part of the function  $\Phi(\omega)$  is

$$\Phi_{0\tau}(\omega) = \frac{1}{2\pi v L} \int_{-L/2}^{+L/2} dz_1 dz_2 \int d\mathbf{r} \overline{\mathcal{G}_\omega^0(\mathbf{r}; z_1, z_2)} \overline{\mathcal{G}_{-\omega}^0(1; z_1, z_2)} = \frac{1}{2\omega}. \quad (4.2)$$

The bar denotes, as usual, averaging over all possible impurity positions. This result can be obtained by summing the "ladder" diagrams shown in Fig. 5. However, the same result can be reached in a much simpler way, by expressing the Green's function in terms of the eigenfunctions of the electron in the volume under consideration, with allowance for the presence of the impurities (see<sup>[9]</sup>), and then integrating over the volume. Thus, (4.2) depends neither on the form nor on the volume of the sample, nor on the presence of impurities.

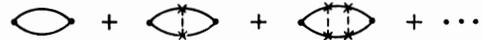


FIG. 5

The part of  $\Phi_\tau(\omega)$  that depends quadratically on the field is obtained by summing the diagrams shown in Fig. 6. As in the case of an infinite sample (see the paper by Gor'kov<sup>[10]</sup>), the contribution of diagrams with crossed dashed lines is negligible, as well as the contribution of diagrams where the dashed line joins two solid lines corresponding to Green's functions with identical  $\omega$ . Summation of diagrams of type a in Fig. 6 leads to the appearance of a factor  $\tilde{\omega}/\omega$  at each vertex where  $\mathcal{G}_\omega^0$  and  $\mathcal{G}_{-\omega}^0$  converge. The contribution of diagrams of type b, where the dashed line joins two solid lines located on opposite sides of a square, is insignificant when  $\tilde{\omega}L/v \ll 1$ . In the case  $\tilde{\omega}L/v \gg 1$ , corresponding to item E of the preceding section (the Maki case), we can employ successfully the trajectory method. Thus, when  $l \gg L$  we obtain for  $\Phi_\tau(\omega)$  an expression similar to (1.8), in which  $\omega$  should be replaced in the second term by  $\tilde{\omega}$ , and this term multiplied by  $(\tilde{\omega}/\omega)^2$ . The corresponding limiting expressions for region IV ( $\tilde{\omega}L/v \ll 1/Lp_0$ ) are obtained in similar fashion:

$$\Phi_\tau(\omega) = \frac{1}{2\omega} - \frac{\pi}{720} (eH)^2 \frac{p_0 L^3 \tilde{\omega}}{2\omega^2} \quad (4.2a)$$

and also for region II ( $1/Lp_0 \ll \tilde{\omega}L/v \ll 1$ ):

$$\Phi_\tau(\omega) = \frac{1}{2\omega} - \frac{31\zeta(5)}{16\pi^4} (eH)^2 \frac{L^3 v}{\omega^2}. \quad (4.2b)$$

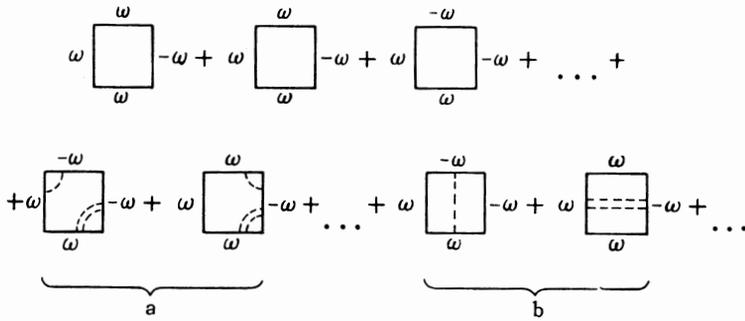


FIG. 6

From (4.2a) with  $l \gg \xi_0 \gg L^2 p_0$  we obtain the already known result (2.6) for pure very thin films, and for  $L^2 p_0 \ll l \ll \xi_0$  we have

$$(eH)^2 = \frac{5760}{\pi^2} \frac{\tau T_c}{\rho_0 l^5} (1 - T/T_c). \quad (4.3)$$

This result is valid so long as  $eHL^2 \ll 1/Lp_0$ , i.e., for temperatures  $1 - T/T_c \ll \xi_0/Lp_0 l$ . With increasing distance from  $T_c$ , the critical field increases and its value is determined by (3.6).

From (4.2b) we can obtain the already known result (2.5), which is valid for contaminated not very thin films,  $L/\min(l, \xi_0) \gg 1/p_0 L$ , in the temperature region  $1 - T/T_c \ll 1$ ,  $\max(L/\xi_0, L\xi_0/l^2)$ .

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