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SOLUTION OF THE TWO PLANE WAVE SCATTERING PROBLEM IN A NONLINEAR SCALAR FIELD THEORY OF THE BORN–INFELD TYPE

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The problem of scattering of two plane waves is formulated and solved exactly in classical nonlinear mesodynamics of the Born—Infeld type. It is found that the shape and direction of the plane waves do not change after scattering, only the arguments x − t and x + t are shifted by an amount that depends on the wave momenta. The solution is obtained on the basis of the Cauchy problem for the corresponding equation previously considered by the authors. It is shown that it may be a multi-valued function of x and t.

INTRODUCTION

HEISENBERG's proposal[1] to explain the matter by starting from the assumed existence of a single universal nonlinear field has again evoked interest in nonlinear fields. However, the principal difficulties in determining the exact solutions for the nonlinear equation do not make it possible to draw definite conclusions concerning the character of the solutions and concerning the physical consequences of these equations. Such questions as the existence of particle-like solutions or the possibility of specifying asymptotic solutions in the form of plane waves still remain open. The Thirring model[2,3], which can be solved exactly, did not turn out to be useful in this case, since it has led to a trivial S-matrix.

In the Thirties, Born[4] proposed his variant of nonlinear electro-dynamics, which led to important physical consequences, such as the finite nature of the self-energy of the electron, the scattering of light by light, and others. But further study of this variant of electrodynamics was hindered by the absence of exact solutions for the equations of motion of the electromagnetic field. This theory had the advantage over other nonlinear generalizations in that the velocity of the signal in it did not exceed the speed of light in vacuum[5].

In an earlier paper [6], we solved exactly the Cauchy problem for a two-dimensional scalar equation of the Born-Infeld type. In the present paper we solve the problem of scattering of two plane waves $\varphi(x-t)$ and $\varphi(x+t)$ in this theory. It turns out that in the interaction region the field function φ can be a non-unique function of x and t; the solution in this region is represented in parametric form. After the interaction, these two waves move further without change in shape or direction; all that happens is that the argument of each wave is shifted by the amount of the momentum of the other wave. This means that in this nonlinear theory no wave scattering takes place in the sense of a change in the direction and shape of the wave. The nonuniqueness of the solution with respect to x and t in the interaction region is an interesting and qualitatively new fact of this nonlinear theory. To understand it it is necessary, apparently, make use of some new physical notions.

In the first section of the article we reduce the problem of scattering of two plane waves in four-dimensional space-time to a two-dimensional problem. In the second section we solve the scattering problem on the basis of the already mentioned solution of the Cauchy problem. In the third section we discuss the singularities of the obtained solution, and in the fourth section we illustrate the scattering problem, using as an example two plane waves bounded in space.

1. The nonlinear equation of the Born-Infeld type for a scalar field $\varphi(x, y, z, t)$ is obtained by varying the Lagrangian

$$L = 1 - (1 + \varphi_x^2 + \varphi_y^2 + \varphi_z^2 - \varphi_t^2)^{1/2} \tag{1}$$

and has the following form:

$$\begin{aligned}(1 + \varphi_x^2 + \varphi_y^2 + \varphi_z^2 - \varphi_t^2)(\varphi_{xx} + \varphi_{yy} + \varphi_{zz} - \varphi_{tt}) \\ - \varphi_x^2\varphi_{xx} - \varphi_y^2\varphi_{yy} - \varphi_z^2\varphi_{zz} - \varphi_t^2\varphi_{tt} - 2\varphi_x\varphi_y\varphi_{xy} \\ - 2\varphi_x\varphi_z\varphi_{xz} - 2\varphi_y\varphi_z\varphi_{yz} + 2\varphi_t\varphi_x\varphi_{tx} \\ + 2\varphi_t\varphi_y\varphi_{ty} + 2\varphi_t\varphi_z\varphi_{tz} = 0.\end{aligned} \tag{2}$$

It is easy to verify that any plane wave

$$\varphi = \varphi(\mathbf{kr} - |k|t) \quad (3)$$

is a solution of Eq. (2). Consequently, the sum of two waves

$$\varphi = \varphi_1(\mathbf{k}_1\mathbf{r} - |k_1|t) + \varphi_2(\mathbf{k}_2\mathbf{r} - |k_2|t) \quad (4)$$

is also a solution of (2) in the space-time region where they do not overlap, i.e., in the region where either φ_1 or φ_2 vanishes.

Without loss of generality we can assume that k_{1y}, k_{2y} and k_{1z}, k_{2z} are equal to zero. It is possible always to arrive at this case by directing the time axis along the four-vector $k_1 + k_2$, and the x axis along $k_1 - k_2$, where

$$k_1 = (|k_1|, \mathbf{k}_1), \quad k_2 = (|k_2|, \mathbf{k}_2).$$

In the new system of coordinates we have

$$\mathbf{k}_1\mathbf{r} - |k_1|t = \sqrt{(k_1k_2)/2}(x' - t'),$$
$$\mathbf{k}_2\mathbf{r} - |k_2|t = -\sqrt{(k_1k_2)/2}(x' + t').$$

The transition to this system is accomplished by means of a Lorentz transformation, and since Eq. (2) is invariant to these transformations, we are justified in writing (4) in the form

$$\varphi = \psi_1(x - t) + \psi_2(x + t). \quad (5)$$

The solution (5) of Eq. (2) exists, for example, in the region $t < 0$ if $\psi_1(x) = 0$ for $x > -\delta$, and $\psi_2(x) = 0$ for $x < \delta$, where δ is any positive number. In the more general case the solution of Eq. (2) can satisfy only the limiting conditions

$$\lim_{u\to\infty} \varphi = \psi_2(v), \quad \lim_{v\to-\infty} \varphi = \psi_1(u), \quad (6)$$

where

$$u = x - t, \quad v = x + t. \quad (7)$$

It is obvious that in order to satisfy these limiting conditions we shall seek a solution of Eq. (2) not dependent on y or z, i.e., a function obeying the equation

$$(1 - \varphi_t^2)\varphi_{xx} + 2\varphi_t\varphi_x\varphi_{xt} - (1 + \varphi_x^2)\varphi_{tt} = 0. \quad (8)$$

We thus arrive at the Born-Infeld two-dimensional scalar model of the field, since Eq. (8) is obtained from the Lagrangian

$$L = 1 - (1 + \varphi_x^2 - \varphi_t^2)^{1/2}. \quad (9)$$

2. We have previously solved[6] the Cauchy problem for Eq. (8). The usual data are specified for $t = 0$

$$\varphi|_{t=0} = a(x), \quad \varphi_t|_{t=0} = b(x). \quad (10)$$

They are subject to the hyperbolicity condition $1 + a'^2(x) - b^2(x) > 0$. A solution of Eq. (8), satisfying the initial conditions (10), was obtained in parametric form

$$t = \frac{\beta - \alpha}{2} + \frac{1}{2}\int_\alpha^\beta H(\lambda)\,d\lambda,$$

$$x = \frac{\beta + \alpha}{2} + \frac{1}{2}\int_\alpha^\beta G(\lambda)\,d\lambda,$$

$$\varphi = \frac{a(\alpha) + a(\beta)}{2} + \frac{1}{2}\int_\alpha^\beta \pi(\lambda)\,d\lambda. \quad (11)$$

The quantities $\pi(x)$, $G(x)$, and $H(x)$ in (11) all have an important physical meaning. $\pi(x)$ is the canonical momentum of the field $\varphi(x, t)$ at $t = 0$:

$$\pi(x) = \left.\frac{\partial L}{\partial \varphi_t}\right|_{t=0} = \frac{b(x)}{[1 + a'^2(x) - b^2(x)]^{1/2}}, \quad (12)$$

$G(x)$ is the momentum density of the field $\varphi(x, t)$ at $t = 0$:

$$G(x) = -\left.\frac{\partial L}{\partial \varphi_t}\varphi_x\right|_{t=0} = -\pi(x)a'(x). \quad (13)$$

$H(x)$ is the energy density of the field $\varphi(x, t)$ at $t = 0$:

$$H(x) = \left[\frac{\partial L}{\partial \varphi_t}\varphi_t - L\right]_{t=0}$$
$$= [(1 + a'^2(x))(1 + \pi^2(x))]^{1/2} - 1. \quad (14)$$

Let us investigate the asymptotic behavior of solution (11), letting $v \to -\infty$ for fixed u, and $u \to \infty$ for fixed v; u and v are the isotropic coordinates (7). In both these cases $t = (v - u)/2 \to -\infty$. According to (11),

$$u = x - t = \alpha + \frac{1}{2}\int_\alpha^\beta [G(\lambda) - H(\lambda)]\,d\lambda,$$

$$v = x + t = \beta + \frac{1}{2}\int_\alpha^\beta [G(\lambda) + H(\lambda)]\,d\lambda,$$

$$\varphi = \frac{a(\alpha) + a(\beta)}{2} + \frac{1}{2}\int_\alpha^\beta \pi(\lambda)\,d\lambda. \quad (15)$$

We shall assume further that $a'(x)$ and $b(x)$ decrease sufficiently rapidly at infinity.

The parameters α and β were chosen to fit the Cauchy problem. To solve the problem of scattering of plane waves it is more convenient to use other parameters $\mu = \mu(\alpha)$ and $\nu = \nu(\beta)$. We shall arrive at them in natural fashion, by considering the limits of expressions (15) as $\beta \to -\infty$ and as $\alpha \to -\infty$. In the first case we obtain

$$\lim_{\beta\to-\infty} u = \alpha + \frac{1}{2}\int_{-\infty}^{\alpha}[H(\lambda) - G(\lambda)]d\lambda = \mu(\alpha),$$

$$\lim_{\beta\to-\infty} v = -\infty,$$

$$\lim_{\beta\to-\infty} \varphi = \frac{a(\alpha)}{2} - \frac{1}{2}\int_{-\infty}^{\alpha}\pi(\lambda)\,d\lambda = \psi_1(\mu). \tag{16}$$

In the second case

$$\lim_{\alpha\to\infty} u = \infty$$

$$\lim_{\alpha\to\infty} v = \beta - \frac{1}{2}\int_{\beta}^{\infty}[H(\lambda) + G(\lambda)]d\lambda = \nu(\beta),$$

$$\lim_{\alpha\to\infty} \varphi = \frac{a(\beta)}{2} - \frac{1}{2}\int_{\beta}^{\infty}\pi(\lambda)\,d(\lambda) = \psi_2(\nu). \tag{17}$$

It will be shown subsequently that the introduced functions ψ_1 and ψ_2 coincide with the limiting functions (6).

We note that $\mu = \mu(\alpha)$ and $\nu = \nu(\beta)$ are monotonically increasing functions, since $H(\lambda) > G(\lambda)$. Consequently, the transformation from the parameters α, β to the parameters μ, ν is mutually unique.

We express now the solution (15) in terms of the new parameters μ and ν and the functions ψ_1 and ψ_2. It is easily seen that

$$\psi_1(\mu) + \psi_2(\nu) = \varphi(\alpha, \beta) - \frac{1}{2}\int_{-\infty}^{\infty}\pi(\lambda)\,d\lambda. \tag{18}$$

The functions ψ_1 and ψ_2, as follows from (16) and (17), have the following limiting values:

$$\lim_{\nu\to\infty}\psi_2(\nu) = 0, \qquad \lim_{\mu\to-\infty}\psi_1(\mu) = 0$$

$$\lim_{\nu\to-\infty}\psi_2(\nu) = \lim_{\mu\to\infty}\psi_1(\mu) = \psi_0 = -\frac{1}{2}\int_{-\infty}^{\infty}\pi(\lambda)\,d\lambda, \tag{19}$$

since

$$\lim_{\alpha\to\pm\infty}\mu(\alpha) = \pm\infty, \qquad \lim_{\beta\to\pm\infty}\nu(\beta) = \pm\infty.$$

Thus

$$\varphi = \psi_1(\mu) + \psi_2(\nu) - \psi_0. \tag{20}$$

It remains for us to express u and v as functions of μ and ν. To this end we note first that in accordance with the definition (16) and (17) of the functions $\mu(\alpha)$ and $\nu(\beta)$, the quantities u and v, specified by formulas (15), can be represented in the form

$$u = \mu(\alpha) + \frac{1}{2}\int_{-\infty}^{\beta}[G(\lambda) - H(\lambda)]\,d\lambda,$$

$$v = \nu(\beta) + \frac{1}{2}\int_{\alpha}^{\infty}[G(\lambda) + H(\lambda)]\,d\lambda. \tag{21}$$

In the first of these integrals we make a change of variables $\sigma = \nu(\lambda)$, and in the second $\sigma = \mu(\lambda)$. Since according to (16) and (17)

$$\frac{d\mu}{d\lambda} = 1 + \frac{H(\lambda) - G(\lambda)}{2}, \qquad \frac{d\nu}{d\lambda} = 1 + \frac{H(\lambda) + G(\lambda)}{2}, \tag{22}$$

we get

$$u = \mu(\alpha) + \int_{-\infty}^{\nu(\beta)}\frac{G(\lambda) - H(\lambda)}{2 + H(\lambda) + G(\lambda)}\,d\sigma, \qquad \sigma = \nu(\lambda),$$

$$v = \nu(\beta) + \int_{\mu(\alpha)}\frac{G(\lambda) + H(\lambda)}{2 + H(\lambda) - G(\lambda)}\,d\sigma, \qquad \sigma = \mu(\lambda). \tag{23}$$

To express the quantities under the integral signs in (23) in terms of ψ_1 and ψ_2, we use (16) and (17) and find the derivatives

$$\frac{d\psi_1(\sigma)}{d\sigma} = \frac{a'(\lambda) - \pi(\lambda)}{2 + H(\lambda) - G(\lambda)}, \qquad \sigma = \mu(\lambda),$$

$$\frac{d\psi_2(\sigma)}{d\sigma} = \frac{a'(\lambda) - \pi(\lambda)}{2 + H(\lambda) + G(\lambda)}, \qquad \sigma = \nu(\lambda). \tag{24}$$

Using expressions (13) and (14), we obtain ultimately

$$u = \mu(\alpha) - \int_{-\infty}^{\nu(\beta)}\psi_2'^2(\sigma)\,d\sigma, \qquad v = \nu(\beta) + \int_{\mu(\alpha)}^{\infty}\psi_1'^2(\sigma)\,d\sigma. \tag{25}$$

Thus, we have obtained a solution of (8) in a new parametric representation

$$\varphi = \psi_1(\mu) + \psi_2(\nu) - \psi_0,$$

$$u = \mu - \int_{-\infty}^{\nu}\psi_2'^2(\sigma)\,d\sigma, \qquad v = \nu + \int_{\mu}^{\infty}\psi_1'^2(\sigma)\,d\sigma. \tag{26}$$

It is now easy to prove that this solution satisfies the limiting conditions (6). In fact, if $u \to \infty$, then also $\mu \to \infty$, then also $\mu \to \infty$, and ν becomes equal to v. Consequently

$$\lim_{u\to\infty}\varphi = \psi_2(v). \tag{27}$$

On the other hand, if $v \to -\infty$, then also $\nu \to -\infty$, and μ becomes equal to u. Consequently,

$$\lim_{v\to-\infty}\varphi = \psi_1(u). \tag{28}$$

We have thus solved the problem of scattering of two plane waves in nonlinear mesodynamics of the Born-Infeld type.

Let us see how our solution changes when $u \to -\infty$ and $v \to \infty$. In both these cases $t = (v-u)/2 \to \infty$. It is obvious that when $u \to -\infty$ then also $\mu \to -\infty$, and ν becomes equal to $v - H_1$, where

$$H_1 = \int_{-\infty}^{\infty} \psi_1'^2(\sigma)\, d\sigma. \tag{29}$$

Consequently

$$\lim_{u\to-\infty} \varphi = \psi_2(v - H_1) - \psi_2(-\infty). \tag{30}$$

If $v \to \infty$, then also $\nu \to \infty$, and μ becomes equal to $u + H_2$, where

$$H_2 = \int_{-\infty} \psi_2'^2(\sigma)\, d\sigma. \tag{31}$$

Consequently

$$\lim_{v\to\infty} \varphi = \psi_1(u + H_2) - \psi_1(\infty). \tag{32}$$

Thus, two plane waves created at $t = -\infty$ as a result of collision with each other, go over at $t = \infty$ also into two plane waves of the same form, but with shifted arguments. The values H_1 and $-H_2$ by which the arguments are shifted are equal to the momenta of the first and second waves, respectively. Indeed, according to the canonical definition, the energy density of a field with Lagrangian (1) is equal to

$$H = \frac{\varphi_t^2}{(1+\varphi_x^2+\varphi_y^2+\varphi_z^2-\varphi_t^2)^{1/2}} + (1+\varphi_x^2+\varphi_y^2+\varphi_z^2-\varphi_t^2)^{1/2} - 1,$$

and the momentum density of this field is

$$G^k = -\frac{\varphi_t\varphi_k}{(1+\varphi_x^2+\varphi_y^2+\varphi_z^2-\varphi_t^2)^{1/2}}. \tag{34}$$

In particular, for a plane wave (3) we obtain for the energy and momentum similar expressions as in the linear case. Putting $\varphi = \psi_1(x-t)$, we obtain $H = \psi_1'^2$, $G^x = \psi_1'^2$, $G^y = G^z = 0$; putting $\varphi = \psi_2(x+t)$, we obtain $H = \psi_2'^2$, $G^x = -\psi_2'^2$, $G^y = G^z = 0$, thus proving the statements made above concerning the quantities (29) and (31). If we so desire, we can consider the collision process in the center-of-inertia system, where $H_1 = H_2$.

3. The obtained solution (26) is in general a multiply-valued function of u and v. Indeed, let us consider the dependence of u and v on μ and ν:

$$u = \mu - B(\nu), \quad v = \nu + A(\mu), \tag{35}$$

where

$$A(\mu) = \int_{\mu}^{\infty} \psi_1'^2(\sigma)\, d\sigma, \quad B(\nu) = \int_{-\infty}^{\nu} \psi_2'^2(\sigma)\, d\sigma. \tag{36}$$

Eliminating ν from (35), we obtain the equation

$$\mu = u + B(v - A(\mu)), \tag{37}$$

which defines μ as a function of u and v. The right side of this equation is a monotonically increasing function of μ, since its derivative is equal to

$$\psi_2'^2(v - A(\mu))\psi_1'^2(\mu) \geqslant 0. \tag{38}$$

Since

$$0 \leqslant A(\mu) \leqslant H, \tag{39}$$

we get

$$u + B(v) \leqslant u + B(v - A(\mu)) \leqslant u + B(v + H_1). \tag{40}$$

Consequently, all the roots of (37) lie within the limits (40). From continuity considerations it is obvious that the number of these roots is odd.

If at all points

$$\psi_1'^2(\mu)\psi_2'^2(\nu) < 1, \tag{41}$$

then there is only one root of (37), and the solution (26) is a single-valued function of u and v.

We can write for the solution the approximate expression

$$\varphi = \psi_1(u + B(v)) + \psi_2(v - A(u)). \tag{42}$$

The smaller the upper bound of the product $\psi_1'^2(\mu)\psi_2'^2(\nu)$, the more accurate this expression.

On the other hand, if condition (41) is violated, then Eq. (37) can have three or more roots. Substituting one of these roots into the second equation of (35), we get ν, and then from (26) also φ.

When condition (41) is violated, the Jacobian of the transformation

$$\frac{\partial(u,v)}{\partial(\mu,\nu)} = 1 - \psi_1'^2(\mu)\psi_2'^2(\nu) \tag{43}$$

is not different from zero everywhere. If the Jacobian (43) is equal to zero, then either

$$1 - \psi_1'(\mu)\psi_2'(\nu) = 0, \tag{44}$$

or

$$1 + \psi_1'(\mu)\psi_2'(\nu) = 0. \tag{45}$$

Let us see what are the characteristic features of the corresponding points of the surface (26) in the space φ, u, v. We set up the equation for a plane tangent to this surface and obtain parametric equations of the tangent surface by differentiating the function (26):

$$d\varphi = \psi_1'(\mu)\,d\mu + \psi_2'(\nu)\,d\nu,$$
$$du = d\mu - \psi_2'^2(\nu)\,d\nu, \quad dv = d\nu - \psi_1'^2(\mu)\,d\mu. \quad (46)$$

From this we get

$$[1 - \psi_1'(\mu)\psi_2'(\nu)]\,d\varphi = \psi_1'(\mu)\,du + \psi_2'(\nu)\,dv. \quad (47)$$

Thus, a tangent plane exists at each point of the surface (26). At the points (44) it becomes parallel to the φ axis. It remains for us to ascertain its position at the point (45). As shown earlier [6], Eq. (8) provides the space φ, y, v with the structure of a Minkowski space with a metric

$$ds^2 = dt^2 - dx^2 - d\varphi^2 = -d\varphi^2 - du\,dv. \quad (48)$$

The scalar product of the vectors $\{d\varphi, du, dv\}$ and $\{N_\varphi, N_u, N_v\}$ is

$$-N_\varphi d\varphi - {}^1\!/_2 N_v du - {}^1\!/_2 N_u dv, \quad (49)$$

so that the plane (47) is orthogonal to the vector

$$N_\varphi = [\psi_1'(\mu)\psi_2'(\nu) - 1],$$
$$N_u = {}^1\!/_2\psi_2'(\nu), \quad N_v = {}^1\!/_2\psi_1'(\mu). \quad (50)$$

The scalar square of the vector (50) is equal to

$$(N, N) = -[1 + \psi_1'(\mu)\psi_2'(\nu)]^2 \leqslant 0. \quad (51)$$

If (N, N) < 0, then the plane (47) intersects the isotropic cone

$$d\varphi^2 + du\,dv = 0 \quad (52)$$

along two straight lines. At these points the solution (26) is hyperbolic. On the other hand, if (N, N) = 0, which occurs precisely at the point (45), then the plane (47) is tangent to the isotropic cone (52) and the solution (26) becomes parabolic.

We can thus conclude that in the case of a nonlinear equation there can exist several values of the field at a given space-time point. This does not fit into the modern concept of a physical field.

4. To make the collision process more illustrative, let us consider the case when the incident waves are bounded in space. We assume that $\psi_1(x)$ is equal to zero outside the interval $-1 < x < 0$, and $\psi_2(x)$ outside the interval $0 < x < 1$. The equality of the widths of these waves does not limit the generality of the problem, in view of the arbitrariness of the functions ψ_1 and ψ_2 in these intervals. We note that the derivatives of these functions $\psi_1'(x)$ and $\psi_2'(x)$ also vanish outside the corresponding intervals. Let us examine our solution (26) for these limiting functions.

Let us break up the (μ, ν) plane into nine regions, as shown in Fig. 1, and let us see how the solution (26) is written for each region. In regions I, II, III, and IV, the solution is equal to zero, i.e., there are

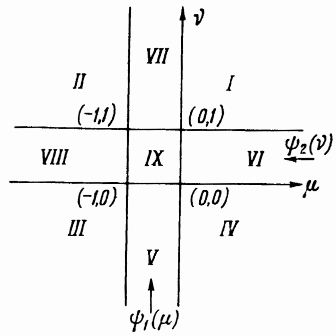

FIG. 1

no incident or scattered waves:

I. $\mu > 0, \quad \nu > 1, \quad u = \mu - H_2, \quad v = \nu, \quad \varphi = 0;$

II. $\mu < -1, \quad \nu > 1, \quad u = \mu - H_2, \quad v = \nu + H_1, \quad \varphi = 0;$

III. $\mu < -1, \quad \nu < 0, \quad u = \mu, \quad v = \nu + H_1, \quad \varphi = 0;$

IV. $\mu > 0, \quad \nu < 0, \quad u = \mu, \quad v = \nu, \quad \varphi = 0.$

In region V there propagates the incident wave $\psi_1(\mu)$, and in region VI the incident wave $\psi_2(\nu)$; in fact,

V. $-1 < \mu < 0, \quad \nu < 0, \quad u = \mu,$

$$v = \nu + \int_\mu^0 \psi_1'^2(\sigma)\,d\sigma, \qquad \varphi = \psi_1(\mu) = \psi_1(u);$$

VI. $\mu > 0, \quad 0 < \nu < 1, \quad u = \mu - \int_0 \psi_2'^2(\sigma)\,d\sigma, \quad v = \nu,$

$$\varphi = \psi_2(\nu) = \psi_2(v).$$

In regions VII and VIII, the following waves propagate after scattering:

VII. $-1 < \mu < 0, \quad \nu > 1, \quad u = \mu - H_2,$

$$v = \nu + \int_\mu^0 \psi_1'^2(\sigma)\,d\sigma, \quad \varphi = \psi_1(\mu) = \psi_1(u + H_2);$$

VIII. $\mu < -1, \quad 0 < \nu < 1, \quad u = \mu - \int_0^\nu \psi_2'^2(\sigma)\,d\sigma,$

$$v = \nu + H_1 \quad \varphi = \psi_2(\nu) = \psi_2(v - H_1).$$

Greatest interest attaches to region IX, where the interaction of two incident waves takes place. In this region we have

IX. $-1 < \mu < 0, \quad 0 < \nu < 1, \quad u = \mu - \int_0^\nu \psi_2'^2(\sigma)\,d\sigma,$

$$v = \nu + \int_\mu^0 \psi_1'^2(\sigma)\,d\sigma,$$

$$\varphi = \psi_1(\mu) + \psi_2(\nu) - \psi_0.$$

In this region both waves ψ_1 and ψ_2 are present, and μ and ν are not expressed explicitly in terms of u and v.

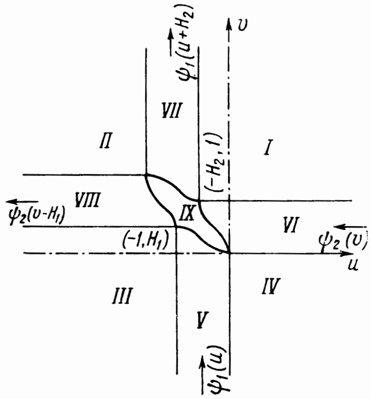

FIG. 2

We now consider the scattering picture in terms of the variables u and v (see Fig. 2). The first eight regions of the (μ, ν) plane map on the corresponding eight regions of the u, v plane as follows:

I. $u > -H_2, \quad v > 1, \quad \varphi = 0;$
II. $u < -1 - H_2, \quad v > 1 + H_1, \quad \varphi = 0;$
III. $u < -1, \quad v < H_1, \quad \varphi = 0;$
IV. $u > 0, \quad v < 0, \quad \varphi = 0.$

The regions of the incident waves are bounded by the following conditions

$$\text{V.} \quad -1 < u < 0, \quad v < \int_u^0 \psi_1'^2(\sigma)\,d\sigma, \quad \varphi = \psi_1(u);$$

$$\text{VI.} \quad u > -\int_0^v \psi_2'^2(\sigma)\,d\sigma, \quad 0 < v < 1, \quad \varphi = \psi_2(v).$$

The regions of the scattered waves are bounded by the conditions

$$\text{VII.} \quad -1 - H_2 < u < -H_2, \quad v > 1 + \int_{u+H_2}^0 \psi_1'^2(\sigma)\,d\sigma,$$
$$\varphi = \psi_1(u + H_2);$$

$$\text{VIII.} \quad u < -1 - \int_0^{v-H_1} \psi_2'^2(\sigma)\,d\sigma, \; v < 1 + H_1,$$
$$\varphi = \psi_2(v - H_1).$$

The result of scattering recalls here the picture of two bodies that collide at right angles and continue to move in their previous directions after collision, but they shift perpendicular to their motion, the first by an amount H_2 and the second by an amount $-H_1$, i.e., by the magnitude of the momentum of the striking body.

The interaction region IX is expressed in terms of μ and ν, and the function φ, as already indicated, can be a non-unique function of u and v. If $\psi_1'^2(\mu)\psi_2'^2(\mu) < 1$, then region IX of the (μ, ν) plane is uniquely mapped on to region IX of plane u, v. In the opposite case the region IX may become mapped on a larger region of the u, v plane.

In conclusion, the authors thank D. I. Blokhintsev for a useful discussion of the problems touched upon here.

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