THE GENERAL FORM OF THE SOLUTION OF THE CHEW-LOW EQUATIONS FOR PION-NUCLEON SCATTERING

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General solutions for the Chew-Low equations for the S, P, D and other partial waves are found. The character of the nonuniqueness of the solutions is discussed. It is shown that a strong dependence of the S-phase shifts in pion-nucleon scattering on the magnitude of the total isospin of the system can be understood in the framework of unitarity and crossing symmetry.

INTRODUCTION

INITIALLY, the Chew-Low[1] equations have been derived for the symmetric interaction of charged pseudoscalar mesons (pions) with a fixed source (nucleon). These equations are of the form

$$h_i(\omega)=\frac{\lambda_i}{\omega}+\frac{1}{\pi}\int_\mu^\infty\left(\frac{\operatorname{Im}h_i(\omega')}{\omega'-\omega}+\frac{A_{ij}\operatorname{Im}h_j(\omega')}{\omega'+\omega}\right)d\omega', \quad (1)$$

where $\omega=\sqrt{q^2+\mu^2}$ is the energy of the meson,

$$h_j(\omega)=\frac{e^{i\delta_j(\omega)}\sin\delta_j(\omega)}{q^3u^2(q^2)},$$

$\delta_j(\omega)$ is the real phase shift for scattering in the j state, $u(q^2)$ is the Fourier transform of the source function, and A_{ij} is the crossing matrix.

The Chew-Low equations are not necessarily related to a certain interaction Hamiltonian. They can be derived as static limits of rigorously proved relativistic dispersion relations.[2] Thus, in the paper by Chew, Goldberger, Low, and Nambu (CGLN)[3] it was shown how to derive the Chew-Low equations for pion-nucleon scattering for fixed momentum transfer. In addition, the authors derived similar equations for the S-waves of pion-nucleon scattering:

$$h_1(\omega)=\frac{1}{3}(1+2\omega)a_1+\frac{2}{3}(1-\omega)a_3+\frac{q^2}{\pi}\int_\mu^\infty\left\{\frac{\operatorname{Im}h_1(\omega')}{\omega'-\omega}+\frac{-\operatorname{Im}h_1(\omega')+4\operatorname{Im}h_3(\omega')}{3(\omega'+\omega)}\right\}\frac{d\omega'}{q'^2}$$

$$h_3(\omega)=\frac{1}{3}(2+\omega)a_3+\frac{1}{3}(1-\omega)a_1+\frac{q^2}{\pi}\int_1^\infty\left\{\frac{\operatorname{Im}h_3(\omega')}{\omega'-\omega}+\frac{2\operatorname{Im}h_1(\omega')+\operatorname{Im}h_3(\omega')}{\omega'+\omega}\right\}\frac{d\omega'}{q'^2}, \quad (2)$$

equations which had also been derived somewhat earlier by Oehme.[4] The equations for higher (D, F, etc.) waves in pion-nucleon-scattering can also be obtained by the method of CGLN.[3] These examples differ in the number of functions $h_i(\omega)$ and in the form of the crossing matrix A_{ij}. We shall give below a solution of the Chew-Low equations for all partial waves of pion-nucleon scattering by a unique method.

FORMULATION OF THE PROBLEM

Equations (1) define analytic functions $h_i(\omega)$ with the following properties:

1. $h_i(\omega)$ is analytic in the complex plane of the variable ω with cuts along the real axis on the intervals $(-\infty, -1]$ $[+1, +\infty)$;
2. $h_i^*(\omega)=h_i(\omega^*)$;
3. $\operatorname{Im} h_i(\omega+i0)=q^{2l+1}u^2(q^2)|h_i(\omega+i0)|^2$—the unitarity condition;
4. $h_i(-\omega)=A_{ij}h_j(\omega)$—the crossing condition,
5. $h_i(\omega)$ has a pole of first order at the origin, with $\operatorname{Res} h_i(\omega)=\lambda_i$;
6. $h_i(\omega)\to 0$ for $|\omega|\to\infty$ and $\operatorname{Im}\omega\neq 0$.

Making use of the Cauchy theorem and of the properties 1–6, one can derive Eqs. (1). As to Eq. (2), the corresponding functions $h_i(\omega)$ satisfy the properties 1–4, but do not possess poles at the origin and do not vanish as $|\omega|\to\infty$ and $\operatorname{Im}\omega\neq 0$. Therefore, the properties 1–4 are common for Eqs. (1) and (2) and these properties can be considered as fundamental. Thus, we shall look for analytic functions which satisfy the conditions 1–4. Let us consider in more detail the condition 4. The crossing matrix A has the property

$$A^2=E, \quad (3)$$

i.e., by applying crossing symmetry twice, one returns to the original function. It follows that the eigenvalues λ of this matrix equal ± 1. Another form of writing Eq. (3) is obvious:

$$A = \sqrt{E}. \tag{4}$$

In addition, the matrix A satisfies two more equations, namely

$$\sum_j A_{ij} = 1, \tag{5}$$

$$\sum_i c_i A_{ij} = c_j, \tag{6}$$

where c_i is a dimension of the representation with index i. These properties of the crossing matrix follow from its definition. We shall explain, for example, the meaning of the condition (5): it means that the S-matrix, without interaction, satisfies the property of crossing symmetry.

We go over from the function $h_i(\omega)$ (scattering amplitude) to the matrix element of the S-matrix:

$$S_i(\omega) = e^{2i\delta_i(\omega)} = 1 + 2iq^{2l+1}u^2(q^2)h_i(\omega). \tag{7}$$

We choose that branch of $q = \sqrt{\omega^2 - 1}$ for which $\operatorname{Im} q > 0$, which defines the physical sheet. From (7) and (5) it follows that condition 4 is satisfied for the functions $S_i(\omega)$. Going over from the function $h_i(\omega)$ to $S_i(\omega)$ will of course modify the properties 5 and 6 somewhat. The source function $u^2(q^2)$ will introduce new singularities and will change the behavior at infinity. However, this transition does not have any influence on the fundamental properties which now can be rewritten as:

1. $S_i(\omega)$ are analytic functions in the complex ω-plane with cuts along the real axis over the intervals $(-\infty, -1][+1, +\infty)$;
2. $S_i^*(\omega) = S_i(\omega^*)$;
3. $|S_i(\omega + i0)|^2 = 1$ for $\omega > 1$;
4. $S_i(-\omega) = A_{ij}S_j(\omega)$.

SOLUTIONS OF SPECIAL CASES

The form of the functions which satisfy the fundamental properties depends on their number ($i = 1, 2, \ldots, N$) and for given N can depend on different matrices A. As the simplest example we consider the matrix A appearing in (2):[5, 6]

$$A = \frac{1}{3}\begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix}. \tag{8}$$

This matrix corresponds to angular momenta 1 and ½ of the scattered particles and of the source, respectively, and satisfies the equations (5) and (6). First we find out what limitations are imposed on the column S_i by the condition of crossing symmetry 4, and for this purpose we decompose $S_i(\omega)$ into a symmetric part $s_i(\omega)$, and an antisymmetric part $a_i(\omega)$. Then

$$A_{ij}s_j(\omega) = s_i(\omega), \quad A_{ij}a_j(\omega) = -a_i(\omega). \tag{9}$$

Since the eigenvalues of A are ± 1, we have

$$s_i(\omega) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} s(\omega), \quad a_i(\omega) = \begin{pmatrix} -2 \\ 1 \end{pmatrix} a(\omega)$$

$$\text{and } S_i(\omega) = \begin{pmatrix} s(\omega) - 2a(\omega) \\ s(\omega) + a(\omega) \end{pmatrix}. \tag{10}$$

Equations (10) yield the most general form for the column $S_i(\omega)$ satisfying the condition 4. We note that the functions s and a have the following properties:

$$s^*(\omega) = s(\omega^*), \quad a^*(\omega) = a(\omega^*). \tag{11}$$

The unitarity condition leads to

$$|s(\omega) - 2a(\omega)|^2 = |s(\omega) + a(\omega)|^2 = 1 \quad \text{for } \omega > 1. \tag{12}$$

Equations (12) define two unit circles with centers at the points $2a(\omega)$ and $-a(\omega)$. It is obvious that not for all values of $s(\omega)/a(\omega)$ will the circles intersect, i.e., will Eqs. (12) be compatible. From (12) it follows

$$|s/a - 2|^2 = |s/a + 1|^2, \tag{13}$$

$$|s + a|^2 = 1. \tag{14}$$

In the s/a-plane, Eq. (13) represents two circles of the same radius, which intersect at points determined by the equation

$$s/a + (s/a)^* = 1. \tag{15}$$

The condition (15) is a consequence of (13) and determines the ratio s/a for $\omega > 1$. Because the function $s(\omega)/a(\omega)$ is odd, the right hand side of (15) changes sign when $\omega < -1$. We put $s(\omega)/a(\omega) = \Phi(\omega)$. Then the definition of $s_i(\omega)$ implies that in (15) $\Phi^{(+)} = \Phi(\omega + i0)$ and from (11) it follows that $(s(\omega)/a(\omega))^* = \Phi^{(-)}(\omega) = \Phi(\omega - i0)$.

Finally,

$$\Phi^{(+)}(\omega) + \Phi^{(-)}(\omega) = \pm 1 \begin{cases} \omega > 1 \\ \omega < -1 \end{cases}. \tag{16}$$

Equation (16) is the well known inhomogeneous linear boundary problem of Riemann, for which solution methods are to be found, for instance, in [7]. The result is:

$$\Phi(\omega) = w(\omega) + i\sqrt{\omega^2 - 1}\,\beta(\omega),$$

$$\beta^*(\omega) = \beta(\omega^*), \quad \beta(\omega) = -\beta(-\omega),$$

$$w(\omega) = \frac{1}{\pi}\arcsin\omega. \tag{17}$$

The degree of arbitrariness is determined by the condition that $\Phi(\omega)$ be bounded at the ends of the cuts integrable on the cut, which leads to a special form for $\beta(\omega)$. Since the functions $S_i(\omega)$ in the fundamental conditions are meromorphic,[8] in our formulation of the problem $\beta(\omega)$ is an arbitrary odd meromorphic function of ω. Since the ratio $s(\omega)/a(\omega)$ is known (cf. Eq. (17)), the solution of (14) reduces to determining, for example, the function $a(\omega)$.

We shall look for the solution as a function of w. Then

$$i\sqrt{\omega^2-1}\beta(\omega) = \cos\pi w\beta(\sin\pi w) = \beta_0(w).$$

It is easy to see that

$$\beta_0(w) = \beta_0(w+1),$$

$$\beta_0(w) = -\beta_0(-w),\ s(\omega)/a(\omega) = w+\beta_0(w). \quad (18)$$

We introduce a new function $\varphi(w)$:

$$\varphi(w) = s(w)+a(w). \quad (19)$$

From the relations (14), (19), and (11) it follows that $\varphi(w)\varphi(w^*) = 1$. Making use of the definition of w (17), we derive the functional equation for $\varphi(w)$:

$$\varphi(w)\varphi(1-w) = 1. \quad (20)$$

In addition $a(w) = \varphi(w)/[w+\beta_0(w)+1]$ is an odd function of w and consequently also of ω. Finally, we have two functional equations for determining $\varphi(w)$

$$\varphi(w)\varphi(1-w) = 1;$$

$$\frac{\varphi(w)}{w+\beta_0(w)+1} = -\frac{\varphi(-w)}{-w-\beta_0(w)+1}. \quad (21)$$

In order to solve the functional equations (21) it is sufficient to take their logarithms; then the first yields $\varphi(w) = e^{g(w-1/2)}$, where g(z) is an arbitrary antisymmetric function of its argument. The final solution has the form

$$\varphi(w) = \frac{w+\beta_0(w)}{w+\beta_0(w)-1}\exp h(w-1/2), \quad (22)$$

where

$$h(w)+h(-w) = 0;\quad h(w)+h(w+1) = 0,$$

or

$$S(w) = \begin{pmatrix} w+\beta_0(w)-2 \\ w+\beta_0(w)+1 \end{pmatrix}\frac{w+\beta_0(w)}{[w+\beta_0(w)]^2-1}$$
$$\times\exp h(w-1/2). \quad (23)$$

Without discussing the properties of the solution (23), we generalize it to the case of a 2 × 2 matrix A corresponding to the addition of angular momenta l and $1/2$. We shall construct it starting from (4)–(6). From (4) it follows that A can be represented in the form

$$A = \begin{pmatrix} x & (1-x^2)/y \\ y & -x \end{pmatrix}, \quad (24)$$

where x and y are independent parameters. The condition (5) leads to the expression

$$A = \begin{pmatrix} x & 1-x \\ 1+x & -x \end{pmatrix}. \quad (25)$$

In our case $(l, 1/2)$ we have $c_i = (2l, 2l+2)$; finally,

$$A_l = \frac{1}{2l+1}\begin{pmatrix} -1 & 2l+2 \\ 2l & 1 \end{pmatrix}. \quad (26)$$

The expression (8) is a special case of (26) for $l = 1$. Similarly we obtain

$$S(\omega) = \begin{pmatrix} s_l(\omega)-\dfrac{l+1}{l}a_l(\omega) \\ s_l(\omega)+a_l(\omega) \end{pmatrix} \quad (27)$$

and $\Phi_l(\omega) = s_l(\omega)/a_l(\omega)$ is subject to the equation

$$\Phi_l^{(+)}(\omega)+\Phi_l^{(-)}(\omega) = \pm\frac{1}{l}\begin{cases}\omega>1\\ \omega<-1\end{cases}. \quad (28)$$

It is obvious that the general solution of (28) has the form

$$\Phi_l(\omega) = l^{-1}w(\omega)+i\sqrt{\omega^2-1}\beta(\omega). \quad (29)$$

We will also, as for the case $l = 1$, look for solutions as functions of the variable w. We introduce a new function

$$\varphi_l(w) = s_l(w)+a_l(w) \quad (30)$$

and write a system similar to (21) in the following form:

$$\varphi_l(w)\varphi_l(1-w) = 1;$$

$$\frac{\varphi_l(w)}{l^{-1}w+\beta_0(w)+1} = -\frac{\varphi_l(-w)}{-l^{-1}w-\beta_0(w)+1}. \quad (31)$$

For any integer l the solution of (31) is given by

$$\varphi_l(w) = \prod_{n=1}^{l}\frac{w+l\beta_0(w)-1/2+(-1)^{n-1}[1/2+(l-n)]}{w+l\beta_0(w)-1/2-(-1)^{n-1}[1/2+(l-n)]}$$
$$\times\exp h(w-1/2). \quad (32)$$

We finally find that

$$S_l(w) = \begin{pmatrix} \dfrac{w+l\beta_0(w)-(l+1)}{w+l\beta_0(w)+l} \\ 1 \end{pmatrix}\varphi_l(w). \quad (33)$$

The solution for arbitrary l has a structure similar to the case $l = 1$, therefore we shall consider the analytic properties of $S_1(w)$.

ANALYSIS OF THE FUNCTION $S_1(w)$

We consider in more detail the properties of the solution $S_1(w)$ (Eq. (23)). We find the form of $h(w = 1/2)$ as a function of ω, where $h(w)$ has the properties (22). First of all we note that

$$\frac{\arcsin\omega + \arccos\omega}{\pi} = 1/2, \qquad (34)$$

i.e. $h(w - 1/2)$ depends on $\pi^{-1}\arccos\omega$. We make use of the fact that the function $h(w)$ has a period 2 and is odd in w, and represent it as a Fourier series:

$$h(w) = \sum_m a_m \sin\pi(2m+1)w,$$

where $a_m^* = a_m$. Making use of (34), we show that

$$h(\omega) = \sqrt{1-\omega^2}\sum_m b_m\omega^{2m}. \qquad (35)$$

Thus, $h(\omega)$ is an analytic function on a two sheeted Riemann surface. We note that only $\exp h(w - 1/2)$ occurs in the solution (23), therefore we introduce the notation

$$D(\omega) = \exp h(w - 1/2). \qquad (36)$$

For $D(\omega)$ we have

$$|D(\omega)|^2 = 1, \quad \omega \geqslant 1, \quad D(-\omega) = D(\omega). \qquad (37)$$

The condition (37) determines $D(\omega)$ as a Blaschke function[9] of the variable η:

$$\eta = (\omega + iq)/\omega,$$

namely,

$$D[\omega(\eta)] = \eta^\lambda \prod_k \frac{|\eta_k|}{\eta_k}\frac{\eta_k - \eta}{1 - \eta_k^*\eta}. \qquad (38)$$

Here λ is the order of the pole or of the zero at the origin and η_k is the set of zeroes of $D(\omega)$ which is symmetric with respect to the axes $\mathrm{Im}\eta = 0$ and $\mathrm{Re}\eta = 0$ on both sheets of the Riemann surface. The position of each of the zeroes is determined by two parameters, which correspond to the two parameters of the CDD-pole.[10] It follows that a nonuniqueness of the type of a CDD-pole appears even in more complicated cases and is the same in both partial waves.

The remainder of $S_1(w)$ depends on an arbitrary meromorphic function $\beta_0(w)$ and must be unitary on the right cut in the ω-plane. On the left cut $\omega < -1$ we have $|e^{2i\delta}| \neq 1$, as follows from unitarity and crossing symmetry. Thus, for instance, $|\exp(2i\delta_3(-\omega))| \leq 1$. Equality is possible only when $\delta_1(\omega) - \delta_3(\omega) = \pi n$, but it can be seen from (23) that $\delta_1(\omega) - \delta_3(\omega)$ depends on ω. The analysis of this part is conveniently carried out in the complex w-plane. The poles and zeroes of $S_1(w)$ are determined by the equation

$$w + \beta_0(w) + n = 0, \quad \text{where } n = 0, \pm1, -2. \qquad (39)$$

Making use of the periodicity of $\beta_0(w)$ (18), Eq. (39) can be brought to the form

$$w + n + \beta_0(w+n) = 0, \qquad (40)$$

from which it follows that in order to determine the zeroes of (39) it is sufficient to solve one equation for $n = 0$:

$$w + \beta_0(w) = 0. \qquad (41)$$

We consider the ratio

$$[w + \beta_0(w)]/[w + \beta_0(w) - 1],$$

which occurs in $S_3(w)$. It is obvious that the zeroes of the denominator are displaced by unity with respect to the zeroes of the numerator. In other words the poles and zeroes of this ratio are symmetric with respect to the line $\mathrm{Re}w = 1/2$, which is the map of the right hand cut in the ω-plane (the cut $\omega > 1$). The expression for $S_1(w)$ contains the additional factor

$$[w + \beta_0(w) - 2]/[w + \beta_0(w) + 1],$$

The set of zeroes and poles of this factor is also symmetric about the line $\mathrm{Re}\, w = 1/2$. This symmetry is a consequence of the unitarity on the cut $\omega > +1$. It is easy to see also that (23) satisfies the crossing condition 4 and consequently is nonunitary on the cut $\omega < -1$ or on its image in the w-plane, the line $\mathrm{Re}\, w = -1/2$.

It follows from (41) that the set of zeroes, W, has the following properties:

(a) The set W is symmetric with respect to the origin.

(b) The set W is symmetric with respect to the line $\mathrm{Im}\, w = 0$.

As a consequence of (a) and (b), the set W is symmetric also with respect to the axis $\mathrm{Re}\, w = 0$. Consequently, in order to determine it, it is necessary to find all zeroes of (41) which are situated in the first quadrant of the w-plane including the boundary.

A further specification of the properties of the set W is possible only if one makes certain assumptions about $\beta_0(w)$. Below it will be sufficient to assume that $\beta_0(w)$ has the form

$$\beta_0(w) = \cos\pi w \frac{P(\sin\pi w)}{Q(\sin\pi w)},$$

which corresponds to the assumption that

$$\beta(\omega) = P(\omega) / Q(\omega).$$

P and Q are polynomials in ω with real coefficients, one being an odd function of ω (cf. (22)). Then (41) reduces to the form

$$wQ\,(\sin \pi w) + \cos \pi w P\,(\sin \pi w) = 0. \quad (42)$$

A determination of the zeroes of (42) is given in the paper by Pontryagin[11] (Sec. 2).

S-WAVES IN PION-NUCLEON SCATTERING

The crossing matrix (cf. (2)) has the form (8), and the functions $S_i(w)$ can be expanded for small $q = \sqrt{w^2 - 1}$ in the series

$$S_i(\omega) = 1 + a_i q + \ldots \quad (43)$$

The quantities a_i are scattering lengths and their values are known from experiment:[12]

$$a_1 - a_3 = 0.259,$$
$$a_1 + 2a_3 = -0.005 \pm 0.0065. \quad (44)$$

We shall put below $a_1 + 2a_3 = 0$. The strong dependence of the quantity a_i on the isospin has not been understood for a long time. At present it is being explained in terms of a pion-pion interaction in the $T = J = 1$ state, i.e., with the existence of the ρ-meson.[13] We shall show that this result follows from the analytic properties for $S_i(\omega)$.

In order to ensure the validity of the expansion (43), we set in (23) $h(w - 1/2) = 0$ and let

$$\beta_0 = \frac{\beta}{\cos \pi w \sin \pi w}, \quad (45)$$

which corresponds to $\beta(w) = \beta/\cos^2 \pi w \sin \pi w$, where β is a parameter. Such a choice of $\beta(w)$ is related to the fact that it guarantees the right behavior of (43) and qualitatively agrees with the experimental data on the S-phase shift in pion-nucleon scattering. The expressions for $\tan \delta_i$ have the form

$$\tan \delta_1(\omega) = \frac{4\omega q\,[\beta + \omega q \pi^{-1} \ln(\omega + q)]}{4\,[\beta + \omega q \pi^{-1} \ln(\omega + q)]^2 + 3\omega^2 q^2}$$

$$\tan \delta_3(\omega) = -\frac{1}{2}\,\frac{\omega q}{\beta + \omega q \pi^{-1} \ln(\omega + q)}$$

or

$$a_1 = 1/\beta, \quad a_3 = -1/2\beta, \quad a_1 + 2a_3 = 0.$$

Thus, the strong dependence of a_i on the isospin can be obtained within the framework of unitarity and crossing symmetry. A derivation of the equality $a_i \approx a_3$ has been given in [14] on the basis of crossing symmetry alone. The unitarity condition leads to the fact that all expansions with respect to the variable ω which follow from crossing symmetry have a convergence radius equal to unity and therefore one cannot derive on their basis conclusions on the magnitudes of the scattering lengths.

SOLUTION OF EQUATIONS OF CHEW-LOW TYPE

The results given above allow one to construct directly a solution for the Chew-Low equations. The crossing matrix for the Chew-Low equation is the direct product of two matrices A of type (8). One of them corresponds to the addition of isospins of the meson and nucleon (1, $1/2$) and the second corresponds to the addition of the angular momenta of the meson (1) and nucleon ($1/2$):

$$A_{CL} = A_T \times A_J = \frac{1}{9}\begin{bmatrix} 1 & -4 & -4 & 16 \\ -2 & -1 & 8 & 4 \\ -2 & 8 & -1 & 4 \\ 4 & 2 & 2 & 1 \end{bmatrix}. \quad (46)$$

It is obvious that the solution, i.e., the functions which satisfy the fundamental conditions, have the form

$$S_{11}(w) = S_1(w)\tilde{S}_1(w), \quad S_{13}(w) = S_1(w)\tilde{S}_3(w),$$

$$S_{31}(w) = S_3(w)\tilde{S}_1(w), \quad S_{33}(w) = S_3(w)\tilde{S}_3(w), \quad (47)$$

Here $S_i(w)$ and $\tilde{S}_i(w)$ are solutions of (23) with different functions $\beta_0(w)$ and $\tilde{\beta}_0(w)$.

The functions $h(w)$ and $\tilde{h}(w)$ can be unified into a single function, since they both possess the same analytic structure and appear as factors in (47). The solution (47) depends on three arbitrary functions, two of which are meromorphic with properties (17) and (18) and the third is a Blaschke product. The usually discussed 3×3 Chew-Low matrix is obtained from (46) with the additional assumption $S_{13} = S_{31}$ which is equivalent to $\beta_0(w) = \tilde{\beta}_0(w)$ or $S_i(w) = \tilde{S}_i(w)$. In this case the solution depends on two arbitrary functions.

The scattering of pions on nucleons in D, F, ... states is also automatically solved by this method. Indeed, the crossing matrix has the form

$$A_{CL,l} = A_T \times A_l \quad (48)$$

and the solution

$$S_{2T,2J}(w) = S_{2T}(w) \times S_{2J}(w),$$

where S_{2T} is defined by (23) and S_{2J} by (39). We note that in the direct product (48) one of the columns can be replaced by units. The construction of functions with concrete properties will be carried out in a subsequent paper.

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APPENDIX

The system (31) reduces to the solution of the functional equation

$$g(w)+g(w+1)=\ln\frac{w+\beta_0(w)+{}^1/_2+l}{w+\beta_0(w)+{}^1/_2-l}, \quad (A.1)$$

where

$$g(w) = -g(-w) \text{ and } \beta_0(w) = -\beta_0(-w),$$

$$\beta_0(w) = \beta_0(w+1).$$

We look for the solution of (A.1) in the form

$$g(w)=\ln\frac{w+\beta_0(w)+\alpha_1}{w+\beta_0(w)-\alpha_1}+g_\alpha(w).$$

The choice of this form corresponds to the fact that $\varphi(w)$ is a meromorphic function. Setting $\alpha_1 = {}^1/_2 + (l-1)$, we obtain

$$g_{\alpha_1}(w)+g_{\alpha_1}(w+1)=-\ln\frac{w+\beta_0(w)+{}^1/_2+(l-1)}{w+\beta_0(w)+{}^1/_2-(l-1)}.$$

Let

$$g_{\alpha_1}(w)=-\ln\frac{w+\beta_0(w)+\alpha_2}{w+\beta_0(w)-\alpha_2}+g_{\alpha_2}(w).$$

Then for $\alpha_2 = {}^1/_2 + (l-2)$ we have

$$g_{\alpha_2}(w)+g_{\alpha_2}(w+1)=\ln\frac{w+\beta_0(w)+{}^1/_2+(l-2)}{w+\beta_0(w)+{}^1/_2-(l-2)}.$$

For intergral l we obtain (32) by a finite number of steps.

For nonintegral l the solution of (A.1) is also meaningful. In this case the form of the matrix A is not related to a concrete transformation group, with respect to which the interaction is assumed to be invariant. Therefore, nonintegral l are just a method of parametrization in Eq. (25) where $x = -1/(2l+1)$. It is easy to see that an iteration process of the solution (A.1) does not depend on the form of $\beta_0(w)$. For simplicity we assume $\beta_0(w)$. Then successive iterations lead to the conclusion that g(w) has zeroes in the points $w = (-1)^n\alpha_n$ and poles in the points $w = (-1)^{n-1}\alpha_n$, where $\alpha_n = {}^1/_2 + (l-n)$. In addition, for integral l the form of g(w) is known. Starting from this fact, one can construct g(w) for arbitrary l:

$$g(w)=\ln\left[\cot\frac{\pi}{2}(w-{}^1/_2)\times\frac{\Gamma(-(w+{}^1/_2+l)/2+1)\,\Gamma(-(-w-{}^1/_2+l)/2)}{\Gamma((w-{}^1/_2-l)/2+1)\,\Gamma(-(w-{}^1/_2+l)/2)}\right],$$

and therefore, also the functions $S_1(w)$ and $S_3(w)$.

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