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INTERACTION OF MAGNONS WITH PHOTONS AND PHONONS

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Interaction processes between magnons and photons or phonons, in which virtual magnons are involved, are considered. A diagram technique involving magnon causal functions is used in the calculations. The characteristics of photon decay into a photon and a magnon are calculated. It is suggested that the process may be detected by irradiating ferrites with a laser light source. A quantum interpretation of the indirect phonon instability is given and a calculation of the corresponding matrix elements for an arbitrary direction of motion and for all three phonon polarizations is presented.

1. INTRODUCTION

THE quantum theory of the interaction of magnons with magnons or photons with phonons, based on the quantization method of Holstein and Primakoff,[1] has been developed by a number of authors.[2-10] Processes have been studied which are connected with the relaxation of magnons and with instabilities of a different type. The processes chiefly studied corresponded to the Feynman diagrams with a single vertex. In the present work, we consider effects of higher order involving virtual magnons. Causal magnon functions are introduced in Sec. 2. The interaction of magnons with photons is calculated in Sec. 2 with the help of these functions by second-order perturbation theory. In the last section, the so-called indirect transverse instability is interpreted from the quantum viewpoint.[11] This instability is connected with the formation of phonons, and a calculation is given of this effect for an arbitrary angle of propagation and for all types of phonon polarization.

2. MAGNON CAUSAL FUNCTIONS

According to Holstein and Primakoff,[1] the magnon function can be represented in the following way:

$$a(x) = V^{-1/2}\sum_{\mathbf{k}}(u_{\mathbf{k}}c_{\mathbf{k}}e^{-i(kx)} + v_{\mathbf{k}}{}^*c_{\mathbf{k}}{}^+e^{i(kx)}), \qquad (2.1)$$

where c_k and c_k^+ are the annihilation and creation operators of magnons, V is the volume of space, x is the four-dimensional coordinate, (kx) the four-dimensional scalar product: $(kx) \equiv \Omega_k t - \mathbf{k}\cdot\mathbf{r}$,

$$\Omega_{\mathbf{k}} = (A_{\mathbf{k}}{}^2 - |B_{\mathbf{k}}|^2)^{1/2};$$

$$u_{\mathbf{k}} = [(A_{\mathbf{k}} + \Omega_{\mathbf{k}})/2\Omega_{\mathbf{k}}]^{1/2},$$

$$v_{\mathbf{k}} = -(B_{\mathbf{k}}/|B_{\mathbf{k}}|)[(A_{\mathbf{k}} - \Omega_{\mathbf{k}})/2\Omega_{\mathbf{k}}]^{1/2}. \qquad (2.2)$$

The functions A_k and B_k characterize the properties of the material and, in the general case, can take into account the anisotropy of the shape of the sample and the anisotropy of the crystalline lattice. (The values of A_k and B_k for a uniaxial crystal are given, for example, in the review.[12]) We give the values of these functions in the simplest case, where the anisotropies can be neglected:

$$A_{\mathbf{k}} = \gamma(H_0 + D\mathbf{k}^2 + 2\pi M_0\sin^2\theta_{\mathbf{k}}),$$

$$B_{\mathbf{k}} = 2\pi\gamma M_0\sin^2\theta_{\mathbf{k}}\exp(2i\varphi_{\mathbf{k}}), \qquad (2.3)$$

H_0 is the constant magnetic field, M_0 the constant magnetization, D a constant characterizing the exchange interaction, γ the gyromagnetic ratio, and θ_k and φ_k the angles of orientation of the vector **k**.

We introduce the magnon causal functions, which are defined as the mean over the vacuum of the chronological products:

$$M_{--}(x_1 - x_2) = \langle T\{a(x_1)a(x_2)\}\rangle_0,$$

$$M_{-+}(x_1 - x_2) = \langle T\{a(x_1)a^+(x_2)\}\rangle_0,$$

$$M_{+-}(x_1 - x_2) = \langle T\{a^+(x_1)a(x_2)\}\rangle_0$$

$$M_{++}(x_1 - x_2) = \langle T\{a^+(x_1)a^+(x_2)\}\rangle_0. \qquad (2.4)$$

Calculation according to Eqs. (2.4), (2.1), and (2.2) for arbitrary values of A_k and B_k lead to the following expressions for the functions $M(x_1 - x_2)$:

$$M(x_1 - x_2) = (2\pi)^{-4} \int d^4k M(k) \exp\{-ik(x_1 - x_2)\},$$

$$k \equiv (\Omega, \mathbf{k}); \tag{2.5}$$

$$\begin{aligned} M_{--}(k) &= iB_{\mathbf{k}}^* / (\Omega_{\mathbf{k}}^2 - \Omega^2 - i0), \\ M_{-+}(k) &= -i(A_{\mathbf{k}} + \Omega) / (\Omega_{\mathbf{k}}^2 - \Omega^2 - i0), \\ M_{+-}(k) &= -i(A_{\mathbf{k}} - \Omega) / (\Omega_{\mathbf{k}}^2 - \Omega^2 - i0), \\ M_{++}(k) &= iB_{\mathbf{k}} / (\Omega_{\mathbf{k}}^2 - \Omega^2 - i0). \end{aligned} \tag{2.6}$$

The following identities have been used in the derivation:

$$\frac{1}{\Omega_{\mathbf{k}}} \exp\{-i\Omega_{\mathbf{k}} |t_1 - t_2|\} = \frac{1}{\pi i} \int_{-\infty}^{\infty} d\Omega \frac{1}{\Omega_{\mathbf{k}}^2 - \Omega^2 - i0} \exp\{-i\Omega(t_1 - t_2)\},$$

$$\frac{|t_1 - t_2|}{t_1 - t_2} \exp\{-i\Omega_{\mathbf{k}} |t_1 - t_2|\} = \frac{1}{\pi i} \int_{-\infty}^{\infty} d\Omega \frac{\Omega}{\Omega_{\mathbf{k}}^2 - \Omega^2 - i0} \exp\{-i\Omega(t_1 - t_2)\}.$$

Equation (2.5) was obtained for an infinitely large volume V. If we assume V to be finite, then the integral must be replaced by the sum:

$$(2\pi)^{-3} \int d^3k \to V^{-1} \sum_{\mathbf{k}}.$$

3. INTERACTION OF PHOTONS WITH MAGNONS IN SECOND ORDER PERTURBATION THEORY

The Hamiltonian of the magnetic dipole interaction can be written in the form

$$W = -[{}^1/_2 (2\mu M_0)^{1/2} (a^+ H_{(-)} + a H_{(+)}) - \mu a^+ a H_z], \tag{3.1}$$

($\mathbf{H}$ is the operator of the variable magnetic field, $\frac{1}{2}\mu$ is the Bohr magneton, and $H_{(\pm)} = H_x + iH_y$.

Matrix elements proportional to μ^2 correspond to the ordinary scattering of the photon by the magnon. However, of more interest are the processes shown graphically in Fig. 1, inasmuch as their matrix elements are proportional to the larger quantity $\mu(2\mu M_0)^{1/2}$. (The solid lines in the graphs correspond to magnons, the dashed lines to photons, the direction of the process is from right to left.) Processes are considered below which are shown in Figs. 1a and 1b, with the emission of a magnon and a photon of very low frequency. Similarly, one can consider processes (Figs. 1c and 1d) with the absorption of a magnon. Thus, we shall be interested in the following part of the product of the operators $W(x_1)W(x_2)$:

$$-\mu(2\mu M_0)^{1/2} [a^+(x_1) H_{(-)}(x_1) a^+(x_2) a(x_2) H_z(x_2) + a(x_1) H_{(+)}(x_1) a^+(x_2) a(x_2) H_z(x_2)]. \tag{3.2}$$

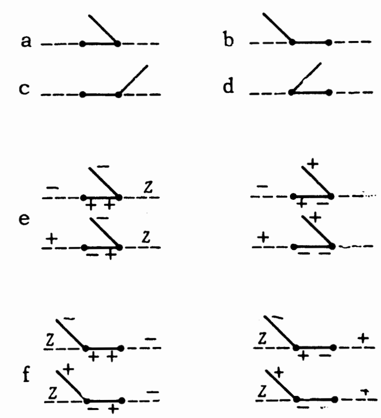

FIG. 1. Diagrams of the interaction of photons with magnons.

Here we have taken it into account that

$$\int d^4x_1 \int d^4x_2 T\{W_1(x_1) W_2(x_2)\} = \int d^4x_1 \int d^4x_2 T\{W_2(x_1) W_1(x_2)\}$$

$$(W = W_1 + W_2). \tag{3.3}$$

When the expression for the matrix element is determined from the form of a graph, certain specific features, which are characteristic for the given interaction, must be considered along with the ordinary rules. To each process (for example, graph 1a) there corresponds several matrix elements (graphs 1e), and the signs + and − for the internal magnon lines, which correspond to the indices of the causal functions, are opposite to the signs of the adjoining lines (+ and − of the external lines correspond to the Fourier components of the functions a^+ and a for the magnon, and $H_{(+)}$, $H_{(-)}$ for the phonon lines, and z corresponds to H_z). In the calculation of the coefficient, it is necessary to take into account the factor which arises in the summation of the matrix elements with permuted coordinates (in the given case, the result is doubled).

Thus, for the diagrams 1e and 1f we get the following matrix element:

$$\begin{aligned} \frac{1}{2}\pi\gamma(2\gamma M_0/\hbar)^{1/2} \{ [& h_{\lambda(-)}^{+}(k') (M_{++}(k') v_{\mathbf{k}-\mathbf{k}'}^* + M_{+-}(k') u_{\mathbf{k}-\mathbf{k}'}^*) \\ & + h_{\lambda(+)}^{+}(k') (M_{-+}(k') v_{\mathbf{k}-\mathbf{k}'}^* + M_{--}(k') u_{\mathbf{k}-\mathbf{k}'}^*)] H_{1z} \\ & + h_{\lambda z}^{+}(k') [(v_{\mathbf{k}-\mathbf{k}'}^* M_{++}(k) + u_{\mathbf{k}-\mathbf{k}'}^* M_{-+}(k)) H_{1(-)} \\ & + (v_{\mathbf{k}-\mathbf{k}'}^* M_{+-}(k) + u_{\mathbf{k}-\mathbf{k}'}^* M_{--}(k)) H_{1(+)}] \} \\ & \times (n_{\mathbf{k}-\mathbf{k}'} + 1)^{1/2} \delta(\omega - \omega' - \Omega_{\mathbf{k}-\mathbf{k}'}); \end{aligned} \tag{3.4}$$

$n_{\mathbf{k}-\mathbf{k}'}$ is the number of magnons, $h_\lambda(k')$ is the matrix element of the Fourier amplitude of the radiated field, $k' \equiv (\omega', \mathbf{k}')$, λ is the index of polarization. The matrix element of the Fourier amplitude of the absorbed field ($k \equiv (\omega, \mathbf{k})$), divided by $V^{1/2}$, is replaced by the classical value $\frac{1}{2}\mathbf{H}_1$, inasmuch as the number of photons of the absorbed field is assumed to be large; in the general case of arbitrary polarization, the amplitude of the magnetic field H_1 is a complex quantity.

Upon a shift in the frequency of the radiated field, we get the following expression, with the help of Eqs. (2.2) and (2.3):

$$\omega - \omega' = \Omega_{\mathbf{k}-\mathbf{k}'} = \gamma[H_0 + D(\mathbf{k}-\mathbf{k}')^2 + 4\pi M_0 \sin^2\theta_{\mathbf{k}-\mathbf{k}'}]^{1/2} \times [H_0 + D(\mathbf{k}-\mathbf{k}')^2]^{1/2}. \quad (3.5)$$

The angular dependence of the frequency shift is greatly simplified if one can neglect the exchange term. Let $H_0 \sim 10^3$; then, in the optical band of interest to us (for example, $|\mathbf{k}| \sim 10^5$ cm^{-1}), the exchange term can be neglected if $D \lesssim 10^{-9}$ Oe-cm^2. In this case, the frequency shift is of the order $\gamma H_0 \sim 10^{10}$ sec^{-1} ($\gamma = 1.8 \times 10^7$ Oe^{-1}-sec^{-1}; it is assumed that the term with M_0 does not increase the order of the quantity $\omega - \omega'$). Consequently, $(\omega - \omega')/\omega \ll 1$. The latter inequality is clearly valid also in those cases in which one cannot neglect the exchange term. Therefore, it can be assumed that the vectors $\mathbf{k}$ and $\mathbf{k}'$ are identical in length, and one can set

$$(\mathbf{k}-\mathbf{k}')^2 = 4|\mathbf{k}|^2 \sin^2\frac{1}{2}\alpha \quad (3.6)$$

in Eq. (3.5), where α is the angle between $\mathbf{k}$ and $\mathbf{k}'$; this assumption is valid for the condition

$$\alpha \gg \frac{|\mathbf{k}| - |\mathbf{k}'|}{|\mathbf{k}|} \sim 10^{-5}. \quad (3.7)$$

The angle $\theta_{\mathbf{k}-\mathbf{k}'}$ characterizes the direction of the momentum of the emitted magnon relative to the vector $\mathbf{M}_0$. In particular, let $\mathbf{M}_0 \perp \mathbf{k}$; then, for observation in a plane perpendicular to $\mathbf{M}_0$ ($\mathbf{k}' \perp \mathbf{M}_0$), $\theta_{\mathbf{k}-\mathbf{k}'} = \pi/2$ and the angular dependence is determined by the exchange term alone. If the same $\mathbf{k}'$ lies in the plane ($\mathbf{k}, \mathbf{M}_0$), then $\theta_{\mathbf{k}-\mathbf{k}'} = \alpha/2$ under the condition that the inequality (3.7) is satisfied. For $\alpha = 0$, $\theta_{\mathbf{k}-\mathbf{k}'} = \pi/2$: thus, for small α, a jump in the frequency difference is observed, from γH_0 to $\gamma[H_0(H_0 + 4\pi M_0)]^{1/2}$.

The probability of the process is determined by the square of the modulus of the matrix element (3.4). According to the estimate made above, the frequency of the virtual magnon (the optical range) is much greater than the frequency of the free magnon for the given value of the momentum. Therefore, one can neglect the quantities M_{--} and M_{++} in (3.4), while one can take the values i/ω and $-i/\omega$ for M_{-+} and M_{+-}. We also take into account the relation

$$\mathbf{h}_\lambda^+(k') = -i(2\pi\hbar\omega')^{1/2}(n_\lambda' + 1)^{1/2}\mathbf{e}_\lambda^*,$$

where n_λ' is the number of photons with momentum $\mathbf{k}'$ and polarizations $\lambda = 1, 2$; $\mathbf{e}_\lambda$ are the complex unit vectors of the polarization. The square of the δ function, after the usual transition $\delta(0) \to t/2\pi$, is transformed to the expression $(t/2\pi) \times (\delta(\omega - \omega' - \Omega_{\mathbf{k}-\mathbf{k}'})$, then the finite lifetime of the magnon is taken into account and the δ function is replaced by a nonsingular function which is assumed to be Lorentzian and which has the value τ/π at resonance, where τ is the relaxation time of the magnon. Finally, from the probability of the process under consideration, which is proportional to $(n_{\mathbf{k}-\mathbf{k}'} + 1)(n_\lambda' + 1)$, it is necessary to subtract the probability of the reverse process (the photon ω', absorbing a magnon, is transformed into the photon ω), which is proportional to $n_{\mathbf{k}-\mathbf{k}'}n_\lambda'$. As a result, omitting the factors which characterize the angular distribution and polarization of the radiation, we get the following estimate of the total probability of the process per unit time:

$$\pi\gamma^3 M_0 \tau H_1^2 (n_\lambda' + n_{\mathbf{k}-\mathbf{k}'}) / 2\omega. \quad (3.8)$$

For an illustration of the calculation of the angular distribution and polarization, we consider a special case: $\mathbf{M}_0 \perp \mathbf{k}$, the variable field $\mathbf{H}_1$ is linearly polarized and directed along $\mathbf{M}_0$ (the z axis). The corresponding factor is equal to

$$|u^*_{\mathbf{k}-\mathbf{k}'}(e^*_\lambda)_{(-)} - v^*_{\mathbf{k}-\mathbf{k}'}(e^*_\lambda)_{(+)}|^2,$$
$$\mathbf{e}_1 = (\mathbf{f}_1 + i\mathbf{f}_2)/\sqrt{2}, \quad \mathbf{e}_2 = \mathbf{e}_1^*; \quad (3.9)$$

$\mathbf{f}_1$, $\mathbf{f}_2$, and $\mathbf{k}'/|\mathbf{k}'|$ form a right-handed triplet of real orthogonal unit vectors. Let $\mathbf{k}'$ lie in the plane perpendicular to $\mathbf{M}_0$; in this case, $\mathbf{f}_2$ can be conveniently directed along the z axis. Then (3.9) does not depend on λ, that is, the radiation in the perpendicular plane is not polarized. By directing the x axis along $\mathbf{k}'$, we obtain [under the condition (3.7)] the relation $\varphi_{\mathbf{k}-\mathbf{k}'} = (\pi - \alpha)/2$ and for the factor (3.9) we find the value

$$(A_{\mathbf{k}-\mathbf{k}'} + |B_{\mathbf{k}-\mathbf{k}'}|\cos\alpha)/2\Omega_{\mathbf{k}-\mathbf{k}'} \quad (\theta_{\mathbf{k}-\mathbf{k}'} = \pi/2). \quad (3.10)$$

A separate calculation for $\alpha = 0$ leads to the result $(A_{\mathbf{k}-\mathbf{k}'} - |B_{\mathbf{k}-\mathbf{k}'}|)/2\Omega_{\mathbf{k}-\mathbf{k}'}$, that is, for small α the probability undergoes a jump (if M_0 is sufficiently large).

As calculation shows, the radiation in the other directions can be polarized.

The effect considered in this section can evidently be observed by irradiation of ferrites, which are transparent for a given frequency, by laser radiation which is highly monochromatic and very intense. For an estimate of the probability (3.8), selecting the parameters $\omega \sim 10^{15}\ \mathrm{sec}^{-1}$, $M_0 \sim 10^2$ Gauss, $\tau \sim 10^{-9}$ sec, we get $10^{-1}\ H_1^2(n'_\lambda + n_{\mathbf{k}-\mathbf{k}'})$. Favorable possibilities for observations are given by the effect of the increase in the particles—here there is an analogy with the induced Raman and Brillouin scattering (see, for example, [13]), the increase being a threshold effect that arises in this case if the probability of the process exceeds the probability of relaxation.

Further, more detailed estimates are connected with the conditions of the arrangement of the experiment and with the problem of the stability of the substance relative to the powerful radiation. By considering the possibility of accumulation of magnons, one can consider processes with absorption of magnons and radiation of the anti-Stokes component $\omega'' > \omega$. In the case of ferrites, it is of interest to elucidate the role of the exchange branch of the energy spectrum of the magnons.

4. INTERACTION OF MAGNONS WITH PHONONS

In this section, the quantum interpretation of the processes of interaction of magnons with phonons is presented. This leads to the so-called indirect phonon instability, which was discussed earlier from the classical point of view.[11] Auld considered the decay of a spin wave, which corresponds to a homogeneous type of precession, into two longitudinally polarized phonons. The calculations were carried out for a special case of the azimuthal angle of phonon propagation. As was pointed out,[11] such a calculation is sufficient in the case of isotropy of the magnetostrictive interaction (equality of the constants b_1 and b_2).

The method with the use of the causal functions of Sec. 2 is applied below to the calculation of a more general variant, corresponding to an arbitrary angle of propagation and to all three types of phonon polarization.

The Hamiltonian of the magnetostrictive interaction has the form

$$(b_1\mu / 2M_0)\,[(a^2 + a^{+2})(u_{xx} - u_{yy}) + 2a^+a(u_{xx} + u_{yy} - 2u_{zz})] + (b_2 / M_0)\,[i\mu(a^2 - a^{+2})u_{xy} + (2\mu M_0)^{1/2}(au_{(+)z} + a^+u_{(-)z})], \qquad (4.1)$$

where x, y, z are the cubic axes, the direction of M_0 coincides with z, b_1 and b_2 correspond to the constants of the research of Auld,[11] u_{ij} is the deformation tensor, defined by the operator expression:

$$u_{ij} = \frac{1}{2} iV^{-1/2} \sum_{\mathbf{k},\lambda} (\hbar/2\rho\omega_{k\lambda})^{1/2}\, k v_{\mathbf{k}\lambda;\, ij}\,(b_{\mathbf{k}\lambda}e^{-i(kx)} - b^+_{\mathbf{k}\lambda}e^{i(kx)}),$$

$$(kx) \equiv \omega_{k\lambda}t - \mathbf{kr}, \quad v_{\mathbf{k}\lambda,\, ij} = (\varepsilon_{\mathbf{k}\lambda,\, i}k_j + \varepsilon_{\mathbf{k}\lambda,\, j}k_i)\, / \, k, \qquad (4.2)$$

$b_{\mathbf{k}\lambda}$ and $b^+_{\mathbf{k}\lambda}$ are the annihilation and creation operators of the phonon, λ is the index of polarization ($\lambda = 3$ corresponds to longitudinal polarization, $\lambda = 1, 2$, to transverse); the polarization vector $\epsilon_{\mathbf{k}1}$ lies in the $(\mathbf{z}, \mathbf{k})$ plane, while $\epsilon_{\mathbf{k}1,\,z} = -\cos\theta$;

$$\omega_{k1} = \omega_{k2} = k(c_{11} / \rho)^{1/2}, \quad \omega_{k3} = k(c_{44} / \rho)^{1/2};$$

c_{11}, c_{44} are the elastic constants of the interaction ρ the density of the medium, $u_{(\pm)z} \equiv u_{xz} \pm iu_{yz}$. We introduce the expressions $v_{\mathbf{k}\lambda,\, ij}$:

$$\begin{aligned}
&v_{1,\,xx} = \sin 2\theta \cos^2\varphi, \quad v_{1,\,yy} = \sin 2\theta \sin^2\varphi, \quad v_{1,\,zz} = -\sin 2\theta,\\
&v_{1,\,xy} = {}^1/_2 \sin 2\theta \sin 2\varphi, \quad v_{1,(+)z} = (v_{1,\,(-)z})^* = \cos 2\theta e^{i\varphi},\\
&v_{2,\,xx} = -\sin\theta \sin 2\varphi, \quad v_{2,\,yy} = \sin\theta \sin 2\varphi, \quad v_{2,\,zz} = 0,\\
&v_{2,\,xy} = \sin\theta \cos 2\varphi, \quad v_{2,\,(+)z} = (v_{2(-)z})^* = i\cos\theta\, e^{i\varphi},\\
&v_{3,\,xx} = 2\sin^2\theta \cos^2\varphi, \quad v_{3,\,yy} = 2\sin^2\theta \sin^2\varphi,\\
&v_{3,\,zz} = 2\cos^2\theta,\\
&v_{3,\,xy} = \sin^2\theta \sin 2\varphi, \quad v_{3,\,(+)z} = (v_{3,\,(-)z})^* = \sin 2\theta\, e^{i\varphi},
\end{aligned} \qquad (4.3)$$

θ and φ are the angles of the vector $\mathbf{k}$.

We note that by using the part of the operator (4.1) that is linear in a and a^+, it is comparatively simple to obtain the dispersion equation for the mixed magneto-elastic excitations, which have been repeatedly derived by classical methods. In what follows, it is assumed that the frequencies under consideration are far from points of intersection of the magnon and phonon dispersion characteristics; therefore, one can speak of pure magnon and phonon states.

The indirect phonon instability of Auld is connected with the processes shown in the diagrams of Figs. 2a and 2b (the wavy lines are phonons, the

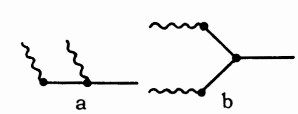

FIG. 2. Diagrams of the interaction of magnons with phonons.

incoming magnon line corresponds to homogeneous precession: $k = 0$). The vertices of first and second order with phonon lines are determined by the linear and quadratic terms in (4.1). The three magnon vertices of the graph 2b correspond to the interaction of the magnetic moment with the demagnetizing field of the spin waves, which is responsible for the ordinary transverse instability. It is convenient to write down this reaction in the form

$$-\pi\mu(2\mu M_0)^{1/2}[a^+a\nabla_z\nabla^{-2}(\nabla_-a^+ + \nabla_+a) + (a^+\nabla_- + a\nabla_+)\nabla^{-2}\nabla_z a^+a],$$
$$\nabla_z = \partial/\partial z, \quad \nabla_\pm = \partial/\partial x \pm i\partial/\partial y. \tag{4.4}$$

We shall assume that the pumping is brought about by a transverse variable magnetic field with frequency ω and with circular polarization,[11] and we represent $M_x + iM_y$ for homogeneous precession in the form $Ae^{-i\omega t}$. If the pumping occurs exactly at resonance, then A can be regarded as a real number; in the general case, A is complex. In the calculation of the elements of the scattering matrix, the corresponding homogeneous precession operator a ought to remain unpaired, the matrix element of this operator being replaced by $A(2\mu M_0)^{1/2}$. Therefore, terms in a^{+2} can be neglected in (4.1) from the very beginning.

The matrix element of the second order process (Fig. 2a) involves components with b_2^2 and b_1b_2: from the terms of first order, we obtain the expression

$$2(v_{\mathbf{k}\lambda,(+)z}M_{--} + v_{\mathbf{k}\lambda,(-)z}M_{+-})v_{-\mathbf{k}\lambda,xy}, \tag{4.5}$$

from terms of second order—

$$2(v_{\mathbf{k}\lambda,(+)z}M_{--} + v_{\mathbf{k}\lambda,(-)z}M_{+-})(v_{-\mathbf{k}\lambda,xx} - v_{-\mathbf{k}\lambda,yy}) + 2(v_{\mathbf{k}\lambda,(+)z}M_{-+} + v_{\mathbf{k}\lambda,(-)z}M_{++}) \times (v_{-\mathbf{k}\lambda,xx} + v_{-\mathbf{k}\lambda,yy} - 2v_{-\mathbf{k}\lambda,zz}). \tag{4.6}$$

By M here are understood the Fourier components (2.6) for $k = (\tfrac{1}{2}\omega, \mathbf{k})$. It is convenient to compute the remaining factor separately; here, it is necessary to take into account the doubling of the result, due to the summation of the matrix elements with permuted coordinates, and the doubling resulting from the possibility of two variants of the diagram 2a: the momenta of first and second phonons are equal to $\mathbf{k}$ and $-\mathbf{k}$ or $-\mathbf{k}$ and $\mathbf{k}$.

We shall consider here the case of identical polarization of both phonons.

We write the final expression for the matrix elements in the following form:

$$\frac{i\pi A\gamma b_2k^2\delta(2\omega_{\mathbf{k}\lambda} - \omega)}{2M_0^2\rho\omega_{\mathbf{k}\lambda}(\Omega_{\mathbf{k}}^2 - \omega^2/4)}(n_{\mathbf{k}\lambda}+1)^{1/2}(n_{-\mathbf{k}\lambda}+1)^{1/2}e^{i\varphi}\sin 2\theta f_\lambda,$$

$$f_1 = \cos 2\theta[b_1(4\Omega_1 + \omega) + i(b_2 - b_1)(\Omega_1 - \tfrac{1}{2}\omega)e^{-2i\varphi}\sin 2\varphi],$$
$$f_2 = -[b_2 + i(b_1 - b_2)e^{-2i\varphi}\sin 2\varphi](\Omega - \tfrac{1}{2}\omega),$$
$$f_3 = 2[b_1(2\Omega_1\cos 2\theta + \omega\cos^2\theta) - i(b_2 - b_1)(\Omega_1 - \tfrac{1}{2}\omega)e^{-2i\varphi}\sin 2\varphi\sin^2\theta],$$
$$\Omega = A_{\mathbf{k}} + |B_{\mathbf{k}}|, \quad \Omega_1 = A_{\mathbf{k}} - |B_{\mathbf{k}}|, \quad B_{\mathbf{k}} = |B_{\mathbf{k}}|e^{2i\varphi}; \tag{4.7}$$

$n_{\mathbf{k}\lambda}$ is the number of phonons.

In the calculation of the matrix element of third order (Fig. 2b) it is necessary to keep in Eq. (4.1) only components linear in a and a^+, while in (4.4), it must be kept in mind that one of the a corresponds to homogeneous precession (we denote the corresponding part of a by a_0); therefore, (4 4) is transformed into

$$-\pi\mu(2\mu M_0)^{1/2}a_0(2a^+\nabla_-\nabla^{-2}\nabla_z a^+ + a\nabla_+\nabla^{-2}\nabla_z a^+ + a^+\nabla_+\nabla^{-2}\nabla_z a). \tag{4.8}$$

In the calculation of the chronological product of the three operators of interaction, the functions a and a^+ which enter into the product

$$[a(x_1)u_{(+)z}(x_1) + a^+(x_1)u_{(-)z}(x_1)] \times [a(x_2)u_{(+)z}(x_2) + a^+(x_2)u_{(-)z}(x_2)],$$

must be connected with the functions $a(x_3)$ and $a^+(x_3)$ in Eq. (4.8). As a result, the following sum is obtained:

$$v_{\mathbf{k}\lambda,(+)z}v_{-\mathbf{k}\lambda,(+)z}[2e^{-i\varphi}M_{-+}M_{-+} + e^{i\varphi}(M_{--}M_{-+} + M_{-+}M_{--})] + v_{\mathbf{k}\lambda,(-)z}v_{-\mathbf{k}\lambda,(-)z}[2e^{-i\varphi}M_{++}M_{++} + e^{i\varphi}(M_{+-}M_{++} + M_{++}M_{+-})] + v_{\mathbf{k}\lambda,(+)z}v_{-\mathbf{k}\lambda,(-)z}[2e^{-i\varphi}M_{-+}M_{++} + e^{i\varphi}(M_{--}M_{++} + M_{-+}M_{+-})] + v_{\mathbf{k}\lambda,(-)z}v_{-\mathbf{k}\lambda,(+)z}[2e^{-i\varphi}M_{++}M_{-+} + e^{i\varphi}(M_{+-}M_{-+} + M_{++}M_{--})]. \tag{4.9}$$

In Eq. (4.9), $M = M(\tfrac{1}{2}\omega, \mathbf{k})$; in the derivation, it was taken into account that

$$M(\tfrac{1}{2}\omega, -\mathbf{k}) = M(\tfrac{1}{2}\omega, \mathbf{k}), \ \nabla_-\nabla^{-2}\nabla_z \to \tfrac{1}{2}\sin 2\theta e^{-i\varphi}, \quad \nabla_+\nabla^{-2}\nabla_z \to \tfrac{1}{2}\sin 2\theta e^{i\varphi},$$

the factor $\tfrac{1}{2}\sin 2\theta$ was not included in Eq. (4.9). As in Eqs. (4.5), (4.6), only the factor $v_{\pm\mathbf{k}\lambda,ij}$ of the deformation tensor remains. By analogy with the previous case, it is necessary in the calculation of the total factor, to take into account the coefficients 3 (permutation of the coordinates in the cross terms) and 2 (two variants of the values of the momenta: $\mathbf{k}$, $-\mathbf{k}$ and $-\mathbf{k}$, $\mathbf{k}$). Moreover, the result should be doubled because of the two possibilities of coupling (for example, $a(x_1)$ with the first function $a^+(x_3)$, $a(x_2)$ with the second, and conversely). We note that (4.9) has a definite structure—the first indices in the causal functions are opposite to the indices $\pm y$ of the functions $v_{\mathbf{k}\lambda}$. Therefore, the next three terms are automatically obtained from the first.

The sum (4.9) is somewhat simplified by means of the equation

$$v_{\mathbf{k}\lambda,(-)z} = v_{\mathbf{k}\lambda,(+)z}e^{-2i\varphi} \quad (\lambda = 1, 3),$$
$$v_{\mathbf{k}2,(-)z} = -v_{\mathbf{k}2,(+)z}e^{-2i\varphi}. \tag{4.10}$$

To sum up, for the matrix element corresponding to the process of Fig. 2b, we obtain the expression

$$\frac{-i\pi^2 A\gamma^2 b_2^2 k^2}{M_0\rho\omega_{k\lambda}(\Omega_k^2-\omega^2/4)^2}\,\delta(2\omega_{k\lambda}-\omega)\,(n_{k\lambda}+1)^{1/2}$$

$$\times(n_{-k\lambda}+1)^{1/2}\,g_\lambda e^{i\varphi}\sin 2\theta,$$

$$g_1=-2\Omega_1(\Omega_1+{}^1\!/_2\omega)\cos^2 2\theta,$$
$$g_2=-\omega(\Omega+{}^1\!/_2\omega)\cos^2\theta,$$
$$g_3=2\Omega_1(\Omega_1+{}^1\!/_2\omega)\sin^2 2\theta. \quad (4.11)$$

The probability of the process is determined by the square of the modulus of the matrix element. If it is necessary to consider both processes simultaneously (see Fig. 2a and 2b), then both matrix elements are summed in the calculation of the probability. The square of the δ function is replaced by the expression $t\tau/4\pi^2$ (τ is the relaxation parameter of the phonon), since

$$\delta(2\omega_{k\lambda}-\omega)={}^1\!/_2\delta(\omega_{k\lambda}-{}^1\!/_2\omega)\to\tau/2\pi.$$

The threshold value of the quantity A, which corresponds to the beginning of the unstable process, is determined from the relation $\omega\tau = 1$,[10] where w is the probability, calculated per unit time and referred to a single phonon (after subtraction of the probability of the reverse process, the net effect is proportional to $n_{k\lambda} + n_{-k\lambda}$).

In the special case $\lambda = 3$ and $\varphi = 0$, we obtain from Eqs. (4.7) and (4.11) an expression that is identical with the expression obtained in [11], if we neglect the relaxation of the magnons. In the comparison, it is necessary to take into account the connection of the relaxation time of the phonons and the quality factor Q: $\tau = 2Q/\omega_{k\lambda}$. It should be noted that in the calculations of this section, in contrast with the calculations of Auld,[11] relaxation of magnons was not taken into account. (It was assumed that Ω_k and $^1/_2\,\omega$, which is identical with $\omega_{k\lambda}$, are sufficiently far away from each other.) The problem as to how correct are the different ways of introducing the damping phenomenologically requires special attention.

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