A THEORY OF THE CLEBSCH-GORDAN COEFFICIENTS FOR THE SU_n GROUPS

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The theory of the Clebsch-Gordan coefficients for the SU_n groups is constructed on the basis of a here proposed method of generating invariants. It is shown that the Clebsch-Gordan coefficients of the group SU_n and the corresponding Racah coefficients and other transformation matrices reduce to a set of $n \times n$ symbols introduced in [3]. Expressions are given for the Wigner coefficients of the groups SU_3, SU_4, and for the Racah coefficients of the group SU_3. The method under discussion also introduces simplifications into the theory of the Clebsch-Gordan coefficients of the group SU_2.

THE importance of the Clebsch-Gordan coefficients of the group SU_2 and of their contractions (Racah coefficients, transformation matrices) for atomic and nuclear spectroscopy is well known. With their help a significant time saving is achieved in calculations and a standard calculational scheme has been developed. As regards the groups SU_3, SU_4 ... SU_n, they have been comparatively little used and their theory, as regards physical applications, have been insufficiently developed. In the last few years, however, along with the traditional applications to the theory of fractional parentage coefficients in atomic physics and in nuclear physics, the groups SU_n have become widely used in the physics of elementary particles. One may also point to the possibility of application of the group SU_n to the system of weakly coupled oscillators (for example molecules). Thus there exists at the present time a real need for the development of a corresponding calculational apparatus for these groups. In this paper the basis of a theory for the Clebsch-Gordan coefficients and their contractions for the groups SU_n is constructed. In contrast to the conventionally used infinitesimal approach, where the construction of the Clebsch-Gordan coefficients is carried out with the help of infinitesimal operators (see, for example, [1]), the proposed method may be referred to as algebraic or invariant.

The starting point of our approach consists of the determination of the Wigner coefficients of the group SU_n as the projection of a product of three representations onto a unit invariant space. In other words the Wigner coefficients are the coefficients in the expansion of the invariants of the group in a definite basis. Invariant contractions of the Clebsch-Gordan coefficients (or Wigner coefficients) also are coefficients in the expansion of certain invariants which in what follows will be referred to as generating invariants.

Thus a study of the Clebsch-Gordan coefficients and their contractions for the group SU_n should be preceded by an analysis of all possible invariants.

A complete set of basis invariants for the group SU_n consists of (see [2])

$$(\varepsilon^{ikl\ldots}u_{1i}u_{2k}u_{3l}\ldots)^J, \quad (\delta_i^k u_k \xi^i)^{J'}, \quad (\varepsilon_{ikl\ldots}\xi^{1i}\xi^{2k}\xi^{3l}\ldots)^{J''}. \quad (1)$$

Here u_{1i}, u_{2k}, ... are covariant vectors, ξ^{1i}, ξ^{2k}, ... are contravariant vectors. As will be shown below with the help of the set of vectors u_{ik} it is possible to construct the basis for any representation of the group SU_n. It is therefore useful to expand the invariant determinant in the set of vectors u_{ik}.

The coefficients in such an expansion are the $n \times n$ symbols introduced in [3]:

$$(\varepsilon^{ikl\ldots}u_{1i}u_{2k}u_{3l}\ldots)^J = \begin{vmatrix} u_{11} & \ldots & u_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ u_{n1} & \ldots & u_{nn} \end{vmatrix}^J$$

$$= \sqrt{(J!)^3(J+1)} \sum_{\sum_i R_{ik}=J} \left\| \begin{matrix} R_{11} & \ldots & R_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ R_{n1} & \ldots & R_{nn} \end{matrix} \right\| \frac{\prod_{ik} u_{ik}^{R_{ik}}}{\left(\prod_{ik} R_{ik}!\right)^{1/2}}. \quad (2)$$

Corresponding to the symmetry of the determinant the $n \times n$ symbol $\|R_{ik}^{(n)}\|$ satisfies $n \times n \times 2$ symmetry relations. Its numerical value may be obtained from Eq. (2) [4]

$$\|R_{ik}^{(n)}\| = \left[\frac{\prod_{i,k=1}^{n} R_{ik}!}{(J+1)!}\right]^{1/2} \Phi, \quad \Phi = \sum \frac{(-1)^{\Sigma[l]P_{l_1\ldots l_n}}}{\prod_{l_1\ldots l_n} P_{l_1\ldots l_n}}. \quad (3)$$

Here $l_1 \ldots l_n$ is a set of indices obtained by an arbitrary permutation from 1, 2 ... n. The sum $\Sigma_{[l]}$ denotes summation over even permutations. The overall summation is over all integer non-negative numbers which satisfy a system of n^2 equations of the type

$$R_{ik} = \sum_{l_1 \ldots l_n} P_{l_1 \ldots l_{i-1} k l_{i+1} \ldots l_n}. \tag{4}$$

The number of terms is equal to n!.

Comparing the expansion of the determinant in its minors with the formula

$$(\delta_i^k u_k \xi^i)^J = J! \sum \frac{\prod_i u_i^{R_{1i}} \xi^{iR_{1i}}}{\prod_i R_{1i}}, \tag{5}$$

we see that the role of the constituents of the contravariant vector is played by the minors of the determinant $|u_{ik}|$, obtained by expanding in the row u_{1i}. From Eq. (2) and (5) we have

$$\frac{\sqrt{J!}\prod_i \xi^{iR_{1i}}}{\left(\prod_i R_{1i}!\right)^{1/2}} = \sqrt{(J!)^3(J+1)} \quad \|R_{ik}^{(n)}\| \frac{\prod_i u_{2i}^{R_{2i}} u_{3i}^{R_{3i}}}{\left(\prod_i R_{2i}! R_{3i}!\right)^{1/2}}. \tag{6}$$

The n × n symbol $\|R_{ik}^{(n)}\|$ transforms covariant components into contravariant ones, i.e., serves as the metric tensor in the corresponding space of representations of the group SU_n. This means that with the help of the n × n symbols it is possible to accomplish invariant summation. At that any conjugate vectors are expressed in terms of minors of the determinant $|u_{ik}|$. Thus, the basis of the representation of the group SU_n may be constructed with the help of independent minors of the determinant $|u_{ik}|$ and in the final analysis expressed in terms of the quantities u_{ik}, which emphasizes the universal role of the n × n symbol.

If the independent minors are expressed in terms of the corresponding contravariant quantities whose choice will not be specified, the normalized basis for the representation may be written in the form[4]

$$\left[\frac{P_1! P_2! \ldots P_{n-1}!}{\prod_i p_{1i}! \prod_{ik} p_{2ik}! \ldots \prod_i p_{n-1i}!}\right]^{1/2} x_i^{p_{1i}} a_{ik}^{p_{2ik}} b_{ikl}^{p_{3ikl}} \ldots (\xi^i)^{p_{n-1i}} \tag{7}$$

At that the tensors x_i, a_{ik}, b_{ikl}, ... , α^{ik}, ξ^i should satisfy additional conditions of the type $x_i \xi^i = 0$, $x_i a_{kl} \epsilon^{iklm\ldots} = 0$. The basis defined by Eq. (7) is a generalization of the spinor basis for the group SU_2 and may be referred to as the generalized spinor or symmetric basis.[4] In the special cases of SU_2, SU_3, SU_4 we have

$$\sqrt{\frac{P!}{p_1!\, p_2!}}\, u_1^{p_1} u_2^{p_2},$$

$$\sqrt{\frac{P!}{p_1! p_2! p_3!} \frac{Q!}{q_1! q_2! q_3!}}\, u_1^{p_1} u_2^{p_2} u_3^{p_3} (\xi^1)^{q_1} (\xi^2)^{q_2} (\xi^3)^{q_3},$$

$$\left[\frac{P!}{\prod_i p_i!} \frac{Q!}{\prod_{ik} q_{ik}!} \frac{R!}{\prod_i r_i!}\right]^{1/2} \prod_i x_i^{p_i} \prod_{ik} a_{ik}^{q_{ik}} \prod_i \xi^{ir_i}, \tag{8}$$

where

$$P = \Sigma p, \quad Q = \Sigma q, \quad R = \Sigma r.$$

The basis vectors Eq. (8) are characterized respectively by a set of 2, 6, and 14 numbers. Below we shall define the Wigner coefficients and their contractions for this basis. The general approach is as follows: First one writes the generating invariant, and then with the help of expansions of the type Eq. (2) one finds the corresponding Wigner coefficients and their contractions expressed in terms of n × n symbols.

First of all we shall consider the conventional Clebsch-Gordan coefficients (SU_2 group) and their contractions, i.e., the theory of angular momentum. The generating invariant for the Wigner coefficient is the determinant

$$(\varepsilon_{ikl} u_{1i} u_{2k} u_{3l})^J = \sqrt{(J!)^3 (J+1)} \sum \begin{Vmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{Vmatrix} \frac{\prod_{ik} u_{ik}^{R_{ik}}}{\left(\prod_{ik} R_{ik}!\right)^{1/2}}. \tag{9}$$

The connection of this determinant with the Wigner coefficient was first established by Regge[5] and studied in [3]. The generating invariant for the metric tensor of the group SU_2, i.e., for the 2 × 2 symbol, is the determinant

$$(\varepsilon_{\lambda\mu} u_{1\lambda} u_{2\mu})^J = \sqrt{(J!)^3 (J+1)} \sum \begin{Vmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{Vmatrix} \frac{\prod_{ik} u_{ik}^{R_{ik}}}{\left(\prod_{ik} R_{ik}!\right)^{1/2}}. \tag{10}$$

The generating invariants for any contractions of the Clebsch-Gordan coefficients for the SU_2 group may be constructed with the help of ϵ_{ikl} and $\epsilon_{\lambda\mu}$. Thus, for example, the generating invariant for the product of two Clebsch-Gordan coefficients, summed over the projections m_1 and m_2, has the form

$$(\varepsilon_{lmn} u_{1\mathbf{l}} u_{2\mathbf{m}} u_{3n})^J (\varepsilon_{l'm'n'} u_{1l'} u_{2\mathbf{m}'} u_{3n'})^{J'} (\varepsilon_{\lambda\lambda'})^{B_1} (\varepsilon_{\mu\mu'})^{B_2}. \tag{11}$$

Here the Latin indices l, m, n take on the totality of possible values 1, 2, 3; the bold face Latin indices **l**, **m**, **n** take on only the value 1; the Greek indices λ, μ, ν corresponding to l, m, n take on the remaining values 2, 3. In this notation the generating invariant for the Racah coefficient (6j-symbol) may be written in the form

$$(\varepsilon_{l_1m_1n_1}u_{1l_1}u_{2m_1}u_{3n_1})^{J_1}(\varepsilon_{l_2m_2n_2}u_{1l_2}u_{2m_2}u_{3n_2})^{J_2}(\varepsilon_{l_3m_3n_3}u_{1l_3}u_{2m_3}u_{3n_3})^{J_3}$$
$$\times(\varepsilon_{l_4m_4n_4}u_{1l_4}u_{2m_4}u_{3n_4})^{J_4}(\varepsilon_{\lambda_1\lambda_2})^{B_{12}}(\varepsilon_{\mu_1\mu_3})^{B_{13}}(\varepsilon_{\nu_1\nu_4})^{B_{14}}(\varepsilon_{\lambda_3\lambda_4})^{B_{34}}$$
$$\times(\varepsilon_{\mu_2\mu_4})^{B_{24}}(\varepsilon_{\nu_2\nu_3})^{B_{23}}. \qquad (12)$$

Analogous expressions are written for the generating invariants for any contraction of the Wigner coefficients and any transformation matrix. The coefficients, on the other hand, in the expansion of the generating invariants in powers of u_{ik} are contractions of Wigner coefficients. Thus the product of Wigner coefficients corresponding to the generating invariant (11) is equal to

$$\sum\begin{Vmatrix}R_{11}R_{12}R_{13}\\R_{21}R_{22}R_{23}\\R_{31}R_{32}R_{33}\end{Vmatrix}\cdot\begin{Vmatrix}R_{11}'R_{12}'R_{13}'\\R_{21}'R_{22}'R_{23}'\\R_{31}'R_{32}'R_{33}'\end{Vmatrix}\cdot\begin{Vmatrix}R_{21}R_{21}'\\R_{31}R_{31}'\end{Vmatrix}\cdot\begin{Vmatrix}R_{22}R_{22}'\\R_{32}R_{32}'\end{Vmatrix}. \qquad (13)$$

The summation is to be carried out over repeating R_{ik}. Further, the 6j-symbol corresponding to the generating invariant (12) is equal to

$$\sum\begin{Vmatrix}R_{11}^1R_{12}^1R_{13}^1\\R_{21}^1R_{22}^1R_{23}^1\\R_{31}^1R_{32}^1R_{33}^1\end{Vmatrix}\cdot\begin{Vmatrix}R_{11}^2R_{12}^2R_{13}^2\\R_{21}^2R_{22}^2R_{23}^2\\R_{31}^2R_{32}^2R_{33}^2\end{Vmatrix}\cdot\begin{Vmatrix}R_{11}^3R_{12}^3R_{13}^3\\R_{21}^3R_{22}^3R_{23}^3\\R_{31}^3R_{32}^3R_{32}^3\end{Vmatrix}$$
$$\times\begin{Vmatrix}R_{11}^4R_{12}^4R_{13}^4\\R_{21}^4R_{22}^4R_{23}^4\\R_{31}^4R_{32}^4R_{33}^4\end{Vmatrix}\cdot\begin{Vmatrix}R_{21}^1R_{21}^2\\R_{31}^1R_{31}^2\end{Vmatrix}\cdot\begin{Vmatrix}R_{22}^1R_{22}^3\\R_{32}^1R_{32}^3\end{Vmatrix}\cdot\begin{Vmatrix}R_{23}^1R_{23}^4\\R_{33}^1R_{33}^4\end{Vmatrix}$$
$$\times\begin{Vmatrix}R_{21}^3R_{21}^4\\R_{31}^3R_{31}^4\end{Vmatrix}\cdot\begin{Vmatrix}R_{22}^2R_{22}^4\\R_{32}^2R_{32}^4\end{Vmatrix}\cdot\begin{Vmatrix}R_{23}^2R_{23}^3\\R_{33}^2R_{33}^3\end{Vmatrix}. \qquad (14)$$

The Clebsch-Gordan coefficient is equal to the product of the Wigner coefficient by the metric tensor:

$$\sum\begin{Vmatrix}R_{11}R_{12}R_{13}\\R_{21}R_{22}R_{23}\\R_{31}R_{32}R_{33}\end{Vmatrix}\cdot\begin{Vmatrix}R_{23}R_{23}'\\R_{33}R_{33}'\end{Vmatrix}. \qquad (15)$$

In general all quantities of the theory of angular momentum may be expressed in terms of products of 2×2 and 3×3 symbols.

The generating invariants give considerable information about the corresponding contractions of Wigner coefficients. From them follow for example, all symmetry relations. Thus the symmetries of the 6j-symbol obtained by Regge[6] and studied in [7] follow immediately from Eq. (12):

$$A_{ik}+A_{lk}=A_{im}+A_{lm}\quad(A_{ik}=J_i-B_{ik}=J_i-B_{ki}). \qquad (16)$$

Making use of Eq. (3), which in the special cases of SU_3 and SU_2 has the form

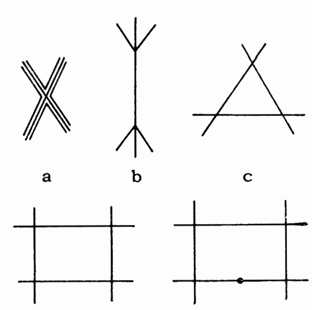
$$\begin{Vmatrix}R_{11}R_{12}R_{13}\\R_{21}R_{22}R_{23}\\R_{31}R_{32}R_{33}\end{Vmatrix}=\left[\frac{\prod\limits_{i,\,k=1}^{3}R_{ik}!}{(J+1)!}\right]^{1/2}\sum\frac{(-1)^{\Sigma_i q_i}}{\prod\limits_{i=1}^{3}p_i!\,q_i!}, \qquad (17)$$

$$\begin{Vmatrix}R_{11}R_{12}\\R_{21}R_{22}\end{Vmatrix}=\left[\frac{\prod\limits_{i,\,k=1}^{2}R_{ik}!}{(J+1)!}\right]^{1/2}\sum\frac{1}{\prod\limits_{i=1}^{2}p_i!}, \qquad (18)$$

it is possible to obtain numerical values for any contractions of the Clebsch-Gordan coefficients.

The method of generating invariants makes possible a consistent construction of the theory of angular momentum. Without stopping to explore the possibilities of the matter with respect to notation and simplification of contractions of Clebsch-Gordan coefficients, or calculations of their numerical values, we shall indicate just one consequence: a new graphical method in the theory of transformation matrices. According to the well known graphical methods,[8] of substantial importance in the theory of angular momentum, the Wigner coefficient is represented in the form of three lines departing from a single point. To each line corresponds one of the angular momenta being added. Contractions of Wigner coefficients are constructed graphically by contracting lines with a common angular momentum. Thus, the 6j-symbol consisting of a sum of products of four Wigner coefficients, is represented in the form of a tetrahedron.

As is seen from Eq. (14), in the construction of transformation matrices instead of a sum over the projections one carries out a summation over the first lines of the 3×3 symbols, corresponding to the triads $j_1 j_2 j_3$. Therefore the Wigner coefficient may be represented in the form of four lines, issuing from a single point; three of them correspond to the projections of the angular momentum and one to the triad (the first line of the 3×3 symbol). In the case of transformation matrices, when the summation is carried out over the projections, only the triad lines remain free. Thus the 6j-symbol may be represented in the form of four triple lines—12 free ends (see figure, picture a), corresponding to the four triads and twelve arguments of the 3×4 symbol.[7] The triad lines



are in general triple, however, the summation, according to Eq. (14) is carried out over the whole triad in its entirety.

An arbitrary transformation matrix may be represented as a sum of products of Racah coefficients, i.e., as a contraction of graphs a. However, in a number of cases it is sufficient to deal with a simplified graph, when the 6j-symbol is represented in the form of four simple lines issuing from a single point. Here to each line corresponds a Wigner coefficient. Transformation matrices are represented as contractions of such four-lines. Thus, the product of two 6j-symbols with a common triad is represented in graph b, the 9j-symbol in graph c. However, in the presence in the graph of a closed cycle two independent ways of contractions of the graph a are topologically possible: (1) Triple lines, connecting the intersections, are parallel to each other; (2) On one triple line there is interlacing.

On simplified graphs the second case may be denoted by a dot on the line. Thus a 12j-symbol of the first kind is represented on the graph b, a 12j-symbol of the second kind on the graph e. When a large number of closed cycles is present one should make use of contractions of graphs a.

The structure and symmetry of an arbitrary transformation matrix is determined by the corresponding graph. It is necessary to note that the theory of Clebsch-Gordan coefficients and their contractions for the SU_2 group is sufficiently well known so that the simplifications introduced here, although substantial, are not of a decisive character.

The situation is very different for groups SU_n, with $n > 2$, where the corresponding theory is practically nonexistent. Taking into account the characteristic peculiarities of the basis, by generalizing the approach here described for the group SU_2 it is possible to write generating invariants for the Wigner coefficients of the group SU_n. The generating invariant for the Wigner coefficients of the SU_3 group has the form

$$(\varepsilon^{iklm}x_i y_k z_l t_m)^J (\varepsilon_{i'k'l'm'}\xi^{i'}\eta^{k'}\varphi^{l'}\tau^{m'})^{J'} (\delta_\mu{}^{\mu'})^B. \qquad (19)$$

Analogous to the case of SU_2 the Latin indices take on all values 1, 2, 3, 4; the bold face Latin indices take on the one value 1, and the Greek indices the remaining values: 2, 3, 4.

The generating invariants for various contractions of Wigner coefficients are constructed analogously to the above considered formula for SU_2. Thus the generating invariant for the Racah coefficient of the SU_3 group may be written in the following manner:

$$(\varepsilon^{k_1l_1m_1n_1}u_{1\mathbf{k}_1}u_{2\mathbf{l}_1}u_{3\mathbf{m}_1}u_{4\mathbf{n}_1})^{J_1} (\varepsilon_{k_1'l_1'm_1'n_1'}\xi^{1\mathbf{k}_1'}\xi^{2\mathbf{l}_1'}\xi^{3\mathbf{m}_1'}\xi^{4\mathbf{n}_1'})^{J_1'}$$
$$\times (\varepsilon^{k_2l_2m_2n_2}u_{1\mathbf{k}_2}u_{2\mathbf{l}_2}u_{3\mathbf{m}_2}u_{4\mathbf{n}_2})^{J_2} (\varepsilon_{k_2'l_2'm_2'n_2'}\xi^{1\mathbf{k}_2'}\xi^{2\mathbf{l}_2'}\xi^{3\mathbf{m}_2'}\xi^{4\mathbf{n}_2'})^{J_2'}$$
$$\times (\varepsilon^{k_3l_3m_3n_3}u_{1\mathbf{k}_3}u_{2\mathbf{l}_3}u_{3\mathbf{m}_3}u_{4\mathbf{n}_3})^{J_3} (\varepsilon_{k_3'l_3'm_3'n_3'}\xi^{1\mathbf{k}_3'}\xi^{2\mathbf{l}_3'}\xi^{3\mathbf{m}_3'}\xi^{4\mathbf{n}_3'})^{J_3'}$$
$$\times (\varepsilon^{k_4l_4m_4n_4}u_{1\mathbf{k}_4}u_{2\mathbf{l}_4}u_{3m_4}u_{4\mathbf{n}_4})^{J_4} (\varepsilon_{k_4'l_4'm_4'n_4'}\xi^{1\mathbf{k}_4'}\xi^{2\mathbf{l}_4'}\xi^{3\mathbf{m}_4'}\xi^{4\mathbf{n}_4'})^{J_4'}$$
$$\times (\delta_{\nu_1}{}^{\nu_1'})^{B_{11}} (\delta_{\nu_2}{}^{\nu_2'})^{B_{22}} (\delta_{\nu_3}{}^{\nu_3'})^{B_{33}} (\delta_{\nu_3}{}^{\nu_4'})^{B_{44}} (\delta_{\mu_1}{}^{\mu_2'})^{B_{12}} (\delta_{\mu_2}{}^{\mu_1'})^{B_{21}}$$
$$\times (\delta_{\mu_3}{}^{\mu_4'})^{B_{34}} (\delta_{\mu_4}{}^{\mu_3'})^{B_{43}} (\delta_{\lambda_1}{}^{\lambda_3'})^{B_{13}} (\delta_{\lambda_3}{}^{\lambda_1'})^{B_{31}} (\delta_{\lambda_2}{}^{\lambda_4'})^{B_{24}} (\delta_{\lambda_4}{}^{\lambda_2'})^{B_{42}}$$
$$\times (\delta_{\varkappa_1}{}^{\varkappa_4'})^{B_{14}} (\delta_{\varkappa_4}{}^{\varkappa_1'})^{B_{41}} (\delta_{\varkappa_2}{}^{\varkappa_3'})^{B_{23}} (\delta_{\varkappa_3}{}^{\varkappa_2'})^{B_{32}}. \qquad (20)$$

The generating invariant for the Wigner coefficient of the SU_4 group has the form

$$(\varepsilon_{i_1k_1l_1m_1n_1}\xi^{1i_1}\xi^{2k_1}\xi^{3l_1}\xi^{4\mathbf{m}_1}\xi^{5\mathbf{n}_1})^{J_1} (\varepsilon^{i_2k_2l_2m_2n_2}u_{1i_2}u_{2k_2}u_{3l_2}u_{4\mathbf{m}_2}u_{5\mathbf{n}_2})^{J_2}$$
$$\times (\varepsilon^{i_3k_3l_3m_3n_3}\varepsilon^{i_4k_4l_4m_4n_4}a_{1i_3i_4}a_{2k_3k_4}a_{3l_3l_4}v_{1m_3}v_{2n_3}v_{3m_4}v_{4n_4})^{J_3} (\delta_{\mu_2}{}^{\mu_1})^{B_1}$$
$$\times (\delta_{\nu_4}{}^{\nu_1})^{B_2} (\varepsilon_{\nu_2\mu_3\nu_3\mu_4})^{B_3}. \qquad (21)$$

The generating invariant for the Wigner coefficient for the SU_5 group will be

$$(\varepsilon^{i_1k_1l_1m_1n_1r_1}x_{1i_1}x_{2k_1}x_{3l_1}x_{4\mathbf{m}_1}x_{5\mathbf{n}_1}x_{6\mathbf{r}_1})^{J_1}$$
$$\times (\varepsilon^{i_2k_2l_2m_2n_2r_2}\varepsilon^{i_3k_3l_3m_3n_3r_3}a_{1i_2i_3}a_{2k_2k_3}a_{3l_2l_3}u_{1\mathbf{m}_2}u_{2\mathbf{n}_2}$$
$$\times u_{3\mathbf{r}_2}u_{4\mathbf{m}_3}u_{5\mathbf{n}_3}u_{6\mathbf{r}_3})^{J_2}$$
$$\times (\varepsilon_{i_4k_4l_4m_4n_4r_4}\varepsilon_{i_5k_5l_5m_5n_5r_5}\alpha^{1i_4i_5}\alpha^{2k_4k_5}\alpha^{3l_4l_5}v^{1\mathbf{m}_4}v^{2\mathbf{n}_4}v^{3\mathbf{r}_4}v^{4\mathbf{m}_5}v^{5\mathbf{n}_5}v^{6\mathbf{r}_5})^{J_3}$$
$$\times (\varepsilon_{i_6k_6l_6m_6n_6r_6}\xi^{1i_6}\xi^{2k_6}\xi^{3l_6}\xi^{4\mathbf{m}_6}\xi^{5\mathbf{n}_6}\xi^{6\mathbf{r}_6})^{J_4} (\delta_{\mu_1}{}^{\mu_6})^{B_1} (\delta_{\nu_2}{}^{\nu_6})^{B_2} (\delta_{\nu_1}{}^{\nu_5})^{B_3}$$
$$\times (\delta_{\rho_2}{}^{\rho_4}\delta_{\rho_3}{}^{\rho_5})^{B_4} (\varepsilon_{\rho_1\mu_2\mu_3\nu_3})^{B_5} (\varepsilon^{\rho_6\mu_4\nu_4\mu_5})^{B_6}. \qquad (22)$$

Completely analogously one may write the generating invariants for the Wigner coefficients of an arbitrary group SU_n and an arbitrary transformation matrix of the group SU_n.

With the help of the generating invariants and the decompositions, equation (2), it is easy to obtain the corresponding expressions for the Wigner coefficients and their contractions for the group SU_n. If the basis for the representations of SU_2, SU_3, SU_4 is chosen according to Eq. (8), then, if one denotes the indices $p_i q_i r_i$, corresponding to the first, second, and third representation respectively by $p_i^1 q_i^1 r_i^1$, $p_i^2 q_i^2 r_i^2$; $p_i^3 q_i^3 r_i^3$, we find the following expression for the Wigner coefficients in terms of $n \times n$ symbols:

$$SU_2: \quad \left\| \begin{matrix} A_1A_2A_3 \\ p_1{}^1p_1{}^2p_1{}^3 \\ p_2{}^1p_2{}^2p_2{}^3 \end{matrix} \right\|, \qquad (23)$$

$$SU_3: \quad \sum_{x_1x_2x_3} \left\| \begin{matrix} A_1A_2A_3A_4 \\ p_1{}^1p_1{}^2p_1{}^3x_1 \\ p_2{}^1p_2{}^2p_2{}^3x_2 \\ p_3{}^1p_3{}^2p_3{}^3x_3 \end{matrix} \right\| \cdot \left\| \begin{matrix} B_1B_2B_3B_4 \\ q_1{}^1q_1{}^2q_1{}^3x_1 \\ q_2{}^1q_2{}^2q_2{}^3x_2 \\ q_3{}^1q_3{}^2q_3{}^3x_3 \end{matrix} \right\|, \qquad (24)$$

$$SU_4: \quad \sum_{x_k{}^i y_k{}^i z_k{}^i} \left\| \begin{matrix} A_1A_2A_3A_4A_5 \\ p_1{}^1p_1{}^2p_1{}^3x_1{}^1x_1{}^2 \\ p_2{}^1p_2{}^2p_2{}^3x_2{}^1x_2{}^2 \\ p_3{}^1p_3{}^2p_3{}^3x_3{}^1x_3{}^2 \\ p_4{}^1p_4{}^2p_4{}^3x_4{}^1x_4{}^2 \end{matrix} \right\| \cdot \left\| \begin{matrix} B_1B_2B_3B_4B_5 \\ y_1{}^1y_1{}^2y_1{}^3x_1{}^3x_1{}^4 \\ y_2{}^1y_2{}^2y_2{}^3x_2{}^3x_2{}^4 \\ y_3{}^1y_3{}^2y_3{}^3x_3{}^3x_3{}^4 \\ y_4{}^1y_4{}^2y_4{}^3x_4{}^3x_4{}^4 \end{matrix} \right\|$$

$$\times \left\| \begin{matrix} C_1C_2C_3C_4C_5 \\ z_1^1z_1^2z_1^3x_1^5x_1^6 \\ z_2^1z_2^2z_2^3x_2^5x_2^6 \\ z_3^1z_3^2z_3^3x_3^5x_3^6 \\ z_4^1z_4^2z_4^3x_4^5x_4^6 \end{matrix} \right\| \cdot \left\| \begin{matrix} D_1D_2D_3D_4D_5 \\ q_1^1q_1^2q_1^3x_1^1x_1^6 \\ q_2^1q_2^2q_2^3x_2^1x_2^6 \\ q_3^1q_3^2q_3^3x_3^1x_3^6 \\ q_4^1q_4^2q_4^3x_4^1x_4^6 \end{matrix} \right\| \cdot \left\| \begin{matrix} x_1^2x_1^3x_1^4x_1^5 \\ x_2^2x_2^3x_2^4x_2^5 \\ x_3^2x_3^3x_3^4x_3^5 \\ x_4^2x_4^3x_4^4x_4^5 \end{matrix} \right\|$$

$$\times \left\| \frac{\begin{matrix} y_1^1y_2^1y_3^1y_4^1 \\ z_1^1z_2^1z_3^1z_4^1 \end{matrix}}{q_{12}^1q_{13}^1q_{14}^1q_{23}^1q_{24}^1q_{34}^1} \right\| \cdot \left\| \frac{\begin{matrix} y_1^2y_2^2y_3^2y_4^2 \\ z_1^2z_2^2z_3^2z_4^2 \end{matrix}}{q_{12}^2q_{13}^2q_{14}^2q_{23}^2q_{24}^2q_{34}^2} \right\|$$

$$\times \left\| \frac{\begin{matrix} y_1^3y_2^3y_3^3y_4^3 \\ z_1^3z_2^3z_3^3z_4^3 \end{matrix}}{q_{12}^3q_{13}^3q_{14}^3q_{23}^3q_{24}^3q_{34}^3} \right\|. \tag{25}$$

Here $A_i B_i C_i D_i$ are definite linear combinations of the indices of the representations. The symbol

$$\left\| \frac{\begin{matrix} y_1y_2y_3y_4 \\ z_1z_2z_3z_4 \end{matrix}}{q_{12}q_{13}q_{14}q_{23}q_{24}q_{34}} \right\|$$

means that summation is to be carried out over all values satisfying the condition $y_i + z_i = y_k + z_k = z_i + z_k = y_i + y_k = q_{ik}$. For higher groups there arise natural generalizations of this symbol, owing to the fact that minors of higher order enter into the basis vector Eq. (7).

It is easy to write equations also for the Racah coefficients for the group SU_3 and the Wigner coefficients of the group SU_5. They are all expressable in terms of $n \times n$ symbols.

Formulae of the type (23), (24), (25) are convenient in that there appear in them directly indices of the representations $p_i^k q_i^k r_i^k$, all symmetry relations are immediately apparent, numerical values may be obtained and convenient graphical methods may be devised analogous to that described above. The main convenience of reducing the theory of Clebsch-Gordan coefficients of the SU_n group to $n \times n$ symbols consists however, in the possibility of carrying out intermediate calculations in a general form and of the possibility for the studying of a number of important applications at once for all SU_n groups. It is clear that the approach discussed in this paper may be generalized also to other groups.

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