

THEORY OF A BOSE GAS PRODUCED BY BOUND STATES OF FERMI PARTICLES

V. N. POPOV

Leningrad Division, V. A. Steklov Mathematics Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor December 17, 1965

J. Exptl. Theoret. Phys. (U.S.S.R.) 50, 1550-1558 (June, 1966)

Temperature Green's functions that are correct in the small-density limit, expressed in terms of many-particle scattering amplitudes, are found for a system of Fermi particles that can form pair bound states in vacuum. It is shown that the system behaves as a Bose gas whose particles (bound states) form a condensate at low temperatures.

1. THE purpose of this investigation was to obtain, within the limit of low densities, an exact solution of the following model problem of many-particle theory:

We consider a system of Fermi particles that interact via a short-range pair potential. It is assumed that a particle pair with opposite spins has one bound s-state (in vacuum). In order for the density of such a system to be low, it is necessary, as will be shown later, to assume that the chemical potential is negative. In this case there is no Fermi sphere (the radius  $k_F$  of which is usually defined by the equation  $k_F^2/2m = \lambda$ ).

These singularities distinguish such a system from the model of Fermi gas of repelling particles, investigated in detail by Galitskiĭ<sup>[1]</sup> by quantum field theory methods, as well as from different superconductivity models,<sup>[2]</sup> where the principal role is played by attraction between particles near the Fermi surface. It is natural to expect the Fermi system under consideration to behave like a Bose gas of molecules-bound states, and, especially at sufficiently low temperature, the bound pairs may form a Bose condensate.

The calculations presented below confirm these assumptions. In Sec. 2 we present a preliminary calculation for  $T = 0$ , using a well known variational principle<sup>[2]</sup> based on the Bogolyubov transformation. In Sec. 3 we calculate for  $T = 0$  expressions for the single-particle Green's functions (normal and anomalous); these expressions are accurate within the limit of low densities. The answer is expressed in terms of functions describing the scattering of 2, 3, and 4 particles in vacuum. In Sec. 4, finally, we obtain the Green's functions for the region of temperatures that are comparable with the phase-transition temperature.

2. The Hamiltonian of the system is<sup>1)</sup>

$$\begin{aligned}
 H' = H - \lambda N = & \sum_{\mathbf{k}, s} (k^2 - \lambda) c_{\mathbf{k}s}^+ c_{\mathbf{k}s} \\
 & + V^{-1} \sum_{\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4} U_{\mathbf{k}_1 \mathbf{k}_4} c_{\mathbf{k}_1}^+ c_{\mathbf{k}_2}^+ - c_{\mathbf{k}_3} c_{\mathbf{k}_4} \\
 & + (4V)^{-1} \sum_{s, \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4} (W_{\mathbf{k}_1 \mathbf{k}_4} - W_{\mathbf{k}_1 \mathbf{k}_3}) c_{\mathbf{k}_1 s}^+ c_{\mathbf{k}_2 s}^+ c_{\mathbf{k}_3 s} c_{\mathbf{k}_4 s}.
 \end{aligned} \tag{2.1}$$

Here  $s = \pm$  is the spin index, and  $U_{\mathbf{k}}$  and  $W_{\mathbf{k}}$  are the short-range potentials of interaction between particles with opposite and identical spins. We assume that the potential  $U_{\mathbf{k}}$  has one bound s-state in vacuum, that is, there exists a normalized solution of the Schrödinger equation

$$(2k^2 - E_0) \varphi_{\mathbf{k}} + V^{-1} \sum_{\mathbf{k}_1} U_{\mathbf{k} \mathbf{k}_1} \varphi_{\mathbf{k}_1} = 0, \tag{2.2}$$

$$V^{-1} \sum_{\mathbf{k}} \varphi_{\mathbf{k}}^2 = 1. \tag{2.2a}$$

In (2.2),  $E_0$  ( $E_0 < 0$ ) is the bound-state energy and  $\varphi_{\mathbf{k}}$  is a spherically-symmetrical eigenfunction with the normalization indicated in (2.2a). We also assume that the Schrödinger equation with potential  $W_{\mathbf{k}}$  does not have any eigenfunctions (antisymmetrical because of the statistics requirement).

The question arises: What should be the chemical potential  $\lambda$  in order for the density  $\rho$  to be small, that is, for the interaction radius  $r_0$  to be small compared with the mean distance between particles  $\rho^{-1/3}$ ? An answer to this question is obtained with the aid of a variational calculation<sup>[2]</sup>

<sup>1)</sup>We use a system of units with  $\hbar = k = 2m = 1$ , where  $\hbar$  and  $k$  are Planck's and Boltzmann's constants and  $m$  is the particle mass.

based on the well known Bogolyubov transformation

$$c_{ks} = u_k b_{ks} + s v_k b_{-k-s}^+, \quad u_k^2 + v_k^2 = 1. \quad (2.3)$$

Let us consider the mean value  $\langle H' \rangle_{b_0} \equiv \Omega_0$  of the operator  $H'$  over the vacuum for the operators  $b_{ks}$ . The requirement that  $\Omega_0$  be extremal with respect to the parameters  $u_k$  and  $v_k$  of the transformation (2.3) leads, as is well known,<sup>[2]</sup> to the equation

$$\begin{aligned} & 2u_k v_k \left[ k^2 - \lambda + (2V)^{-1} \sum_{k_1} (W_0 - W_{kk_1}) v_{k_1}^2 \right] \\ & + (u_k^2 - v_k^2) V^{-1} \sum_{k_1} U_{kk_1} u_{k_1} v_{k_1} \equiv 2u_k v_k F_k \\ & + (u_k^2 - v_k^2) G_k = 0. \end{aligned} \quad (2.4)$$

This equation reduces in first approximation (assuming that  $|v_k| \ll 1$ ) to the Schrödinger equation for the function  $u_k v_k$ , so that

$$u_k v_k \approx C_0 \varphi_k, \quad 2\lambda \approx E_0 < 0. \quad (2.5)$$

We now rewrite (2.4) in the form

$$\begin{aligned} & (2k^2 - E_0) u_k v_k + V^{-1} \sum_{k_1} U_{kk_1} u_{k_1} v_{k_1} \equiv (H - E_0) u_k v_k \\ & = (2\lambda - E_0) u_k v_k - u_k v_k V^{-1} \sum_{k_1} (W_0 - W_{kk_1}) v_{k_1}^2 \\ & + 2v_k^2 V^{-1} \sum_{k_1} U_{kk_1} u_{k_1} v_{k_1}. \end{aligned} \quad (2.6)$$

We first replace  $v_k \approx u_k v_k$  in (2.6) by its first approximation  $C_0 \varphi_k$ . Multiplying (2.6) by  $\varphi_k$  and summing over  $k$ , we obtain an equation for the determination of  $C_0$ :

$$\begin{aligned} 0 & = (2\lambda - E_0) C_0 - C_0^3 V^{-2} \sum_{kk_1} (W_0 - W_{kk_1}) \varphi_k^2 \varphi_{k_1}^2 \\ & + 2C_0^3 V^{-2} \sum_{kk_1} \varphi_k^3 U_{kk_1} \varphi_{k_1}, \end{aligned} \quad (2.7)$$

which has, besides the solution  $C_0 = 0$ , also the nontrivial solution

$$\begin{aligned} C_0^2 & = a_0 \Delta \lambda \equiv (\lambda - E_0/2) \left[ V^{-1} \sum_k (2k^2 - E_0) \varphi_k^4 \right. \\ & \left. + (2V^2)^{-1} \sum_{kk_1} (W_0 - W_{kk_1}) \varphi_k^2 \varphi_{k_1}^2 \right]^{-1}, \end{aligned} \quad (2.8)$$

where on going from (2.7) to (2.8) the potential  $U_{kk_1}$  is eliminated with the aid of the Schrödinger equation for  $\varphi_k$ .

The formulas for the density and for the pressure

$$\begin{aligned} \rho & = 2V^{-1} \sum_k v_k^2 = 2C_0^2 V^{-1} \sum_k \varphi_k^2 = 2C_0^2 = 2a_0 \Delta \lambda, \\ p & = -\Omega_0/V = \int_{E_0/2}^{\lambda} \rho d\lambda = a_0 (\Delta \lambda)^2, \end{aligned} \quad (2.9)$$

show that the stability condition ( $p_{\lambda\lambda} \gg 0$ ) requires that  $a_0$  be positive, and a nontrivial solution exists only when  $\Delta \lambda > 0$ . This solution describes a state of the system, in which the macroscopic number of bound pairs with zero momentum forms a Bose condensate.

The low-density requirement ( $r_0 \ll \rho^{-1/3}$ ) leads to the inequality

$$|\Delta \lambda / E_0| \equiv \theta^2 \ll 1 \quad (2.10)$$

by virtue of (2.9) and to the relations  $r_0 \sim |E_0|^{-1/2}$  and  $a_0 \sim |E_0|^{1/2}$ .

The condition (2.10) is necessary also for good convergence of the successive approximations when solving (2.6). Thus, the chemical potential  $\lambda$ , as already noted, should be negative (but larger than  $E_0/2$  if a nontrivial solution exists). The parameter  $\theta$  defined in (2.10) will be the parameter with respect to which the expansion is carried out in the perturbation theory developed below.

3. An exact solution can be obtained in low-density limit by considering the temperature Green's functions and using diagram perturbation theory<sup>[3]</sup> in their calculation. It must be taken into account that, besides the normal Green's function (3.1), there can also appear the anomalous function (3.1a).

$$G(k, s, \tau - \tau_1) = -\langle T c_{ks}(\tau) c_{ks}^+(\tau_1) \rangle, \quad (3.1)$$

$$G_1(k, s, \tau - \tau_1) = -\langle T c_{ks}(\tau) c_{-k-s}(\tau_1) \rangle, \quad (3.1a)$$

where  $c_{ks}(\tau) = e^{H'\tau} c_{ks} e^{-H'\tau}$ , and  $\langle \dots \rangle$  denotes averaging over the grand ensemble. The possibility of appearance of anomalous functions is deduced from the result of the preliminary calculation in Sec. 2, where (nonvanishing) averages of the type  $\langle c_{ks} c_{-k-s} \rangle$  actually did arise. As always, it is more convenient to deal with the Fourier coefficients  $G(p)$  and  $G_1(p)$  of the functions (3.1) and (3.1a) ( $p = (k, \omega_n)$ ,  $\omega_n = (2n+1)\pi T$ ). We present the required expressions (3.2) for the functions  $G(p)$  and  $G_1(p)$  in terms of irreducible self-energy parts<sup>[4]</sup> satisfying the Dyson equations (Fig. 1):

$$\begin{aligned} G(p) & = [i\omega_n + k^2 - \lambda + A(-p)] Z^{-1}(p), \\ G_1(p) & = B(p) Z^{-1}(p), \\ Z(p) & = (i\omega_n + k^2 - \lambda + A(-p)) \\ & \quad \times (i\omega_n - k^2 + \lambda - A(p)) - B^2(p). \end{aligned} \quad (3.2)$$

In Fig. 1 each heavy line terminating in arrows corresponds to a Green's function (normal if the arrows have the same direction and anomalous if opposite). When there are no arrows in the diagrams it is understood that summation takes

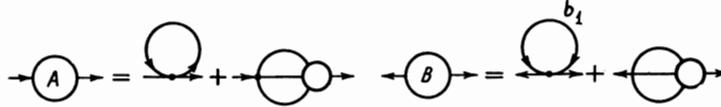


FIG. 1

place over all possible arrangements of the arrows at the beginning and end of each line. The vertices in the diagrams of Fig. 1 correspond to the factors  $V^{-1}U_{\mathbf{k}_1\mathbf{k}_4}$  and  $V^{-1}(W_{\mathbf{k}_1\mathbf{k}_4} - W_{\mathbf{k}_1\mathbf{k}_3})$ , while circles with four arrows correspond to irreducible vertex parts.

Assume now (and to the end of this section) that  $T = 0$ . In this case we must replace  $i\omega_n$  by  $E$ , and the sum over the frequencies  $\sum_n$  must be understood as the integral along the imaginary axis  $(2\pi i)^{-1} \int dE$ .

To estimate the order of the Green's functions let us consider the functions obtained by the variational method (if after substituting (2.3) and reducing to the normal form with respect to operators  $b_{\mathbf{k}S}$  and  $b_{\mathbf{k}S}^+$  we retain in  $H'$  only the quadratic form in the operators  $b$  and  $b^+$ )

$$\begin{aligned} G_{(0)}(p) &= (i\omega_n + F_{\mathbf{k}}) [(i\omega_n)^2 - \varepsilon_{\mathbf{k}}^2]^{-1} \\ &= u_{\mathbf{k}}^2 (i\omega_n - \varepsilon_{\mathbf{k}})^{-1} + v_{\mathbf{k}}^2 (i\omega_n + \varepsilon_{\mathbf{k}})^{-1}, \\ G_{(1)}(p) &= G_{\mathbf{k}} [(i\omega_n)^2 - \varepsilon_{\mathbf{k}}^2]^{-1} = u_{\mathbf{k}} v_{\mathbf{k}} [(i\omega_n - \varepsilon_{\mathbf{k}})^{-1} \\ &\quad - (i\omega_n + \varepsilon_{\mathbf{k}})^{-1}], \quad \varepsilon_{\mathbf{k}}^2 = F_{\mathbf{k}}^2 + G_{\mathbf{k}}^2. \end{aligned} \quad (3.3)$$

Assuming that the complete Green's functions are of the same order of magnitude as the functions (3.3), we arrive at the conclusion that the anomalous functions  $G_1(p)$  introduce a smallness order  $\theta$  (since  $u_{\mathbf{k}} v_{\mathbf{k}} \sim \theta$ ), and closed loops made up of normal lines introduce a smallness of the order  $\theta^2$ . The latter is the result of the fact that, when integrating over the energy in the closed loop, the contribution of the principal term in the product of the Green's functions, that is,  $\prod_i [u_i^2 \times (E_i - \varepsilon_i)^{-1}]$  vanishes, since all the poles of this expression are located on one side of the imaginary axis, and the contribution of the remaining terms contains at least one function  $v^2 \sim \theta^2$ . It is clear from these estimates that  $A(p) \sim \theta^2$  and  $B(p) \sim \theta$ , the main contribution to  $B(p)$  coming from diagram  $b_1$  of Fig. 1, while the remaining diagrams have an order  $\lesssim \theta^3$ . Therefore the approximate equation for  $B(p)$  is

$$\begin{aligned} B(p) &\approx b_1(\mathbf{k}) = TV^{-1} \sum_{\mathbf{k}_1 n_1} U_{\mathbf{k}\mathbf{k}_1} G_1(p_1) \\ &\approx TV^{-1} \sum_{\mathbf{k}_1, n_1} U_{\mathbf{k}\mathbf{k}_1} b_1(\mathbf{k}_1) [(i\omega_{n_1})^2 - (k_1^2 - \lambda)^2]^{-1} \\ &= V^{-1} \sum_{\mathbf{k}_1} U_{\mathbf{k}\mathbf{k}_1} b_1(\mathbf{k}_1) (2k_1^2 - 2\lambda)^{-1}, \end{aligned} \quad (3.4)$$

and its solution and the anomalous function  $G_1(p)$  are expressed in first approximation in terms of the eigenfunction of the bound state

$$\begin{aligned} \lambda &\approx E_0/2, \quad b_1(\mathbf{k}) \approx -C(2k^2 - E_0)\varphi_{\mathbf{k}} \equiv -\tilde{C}\tilde{\varphi}_{\mathbf{k}}, \\ G_1(p) &\approx -C\tilde{\varphi}_{\mathbf{k}} Z_0^{-1}(p) \equiv -C\tilde{\varphi}_{\mathbf{k}} [(i\omega_n)^2 - (k^2 - E_0/2)^2]^{-1}. \end{aligned} \quad (3.5)$$

Expression (2.9) for the density and for the pressure remain in force if  $C_0$  is replaced by the constant  $C$  which enters in (3.5), and  $a_0$  is replaced by the constant  $a$  defined below. In fact, substituting the Green's function on the imaginary axis

$$\begin{aligned} G(\mathbf{k}, E) &= (E + k^2 - \lambda + A(-p))Z^{-1}(p) \\ &\approx (E - k^2 + \lambda - A(p))^{-1} + B^2(p) \\ &\quad \times (E - k^2 + \lambda - A(p))^{-2} (E + k^2 - \lambda \\ &\quad + A(-p))^{-1} \approx (E - k^2 + \lambda - A(p))^{-1} \\ &\quad + C^2\tilde{\varphi}_{\mathbf{k}}^2 (E - k^2 + E_0/2)^{-2} (E + k^2 - E_0/2)^{-1} \end{aligned} \quad (3.6)$$

into the formula for the density ( $\epsilon \rightarrow +0$ ) we obtain

$$\begin{aligned} \rho &= 2V^{-1} \sum_{\mathbf{k}} (2\pi i)^{-1} \int e^{\varepsilon E} G(\mathbf{k}, E) dE \\ &\approx 2C^2 V^{-1} \sum_{\mathbf{k}} \tilde{\varphi}_{\mathbf{k}}^2 (2\pi i)^{-1} \int \left( E - k^2 + \frac{E_0}{2} \right)^{-2} \\ &\quad \times \left( E + k^2 - \frac{E_0}{2} \right)^{-1} dE = 2C^2 V^{-1} \sum_{\mathbf{k}} \varphi_{\mathbf{k}}^2 = 2C^2. \end{aligned} \quad (3.7)$$

Since, as will be shown later,  $A(p) = A(\mathbf{k}, E)$  has in first approximation no singularities in the left  $E$  half-plane, the first term in (3.6) makes no contribution to the integral with respect to  $E$ .

Taking into account the foregoing remarks concerning the estimate of the normal and anomalous Green's functions, we pick out the diagrams that make the main contribution to  $A(p)$  and  $B(p)$  (Fig. 2). For  $A(p)$  these will be all the diagrams with one loop or two anomalous lines, and to calculate  $B(p)$ —the addition to  $b_1(\mathbf{k})$ —it is necessary to take into account all the diagrams with three anomalous lines or one anomalous line and one loop.

In Fig. 2 the circles with  $s$  incoming and  $s$  outgoing arrows denote the functions  $t_s(\mathbf{k}_j, E_j)$  ( $s = 2, 3, 4$ ), which represent the contributions from all the connected diagrams of the indicated structure without the loops and anomalous ele-

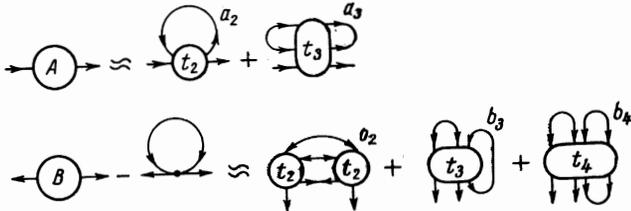


FIG. 2

ments. In calculating these functions, the Green's functions represented by their internal lines can be replaced by the zeroth-approximation functions  $(E - k^2 + \lambda)^{-1} \approx (E - k^2 + E_0/2)^{-1} \equiv G_0(p)$ .

The functions  $t_s$  are the amplitudes for scattering of  $s$  particles in vacuum, in terms of which the self-energy parts of  $A(p)$  and  $B(p)$  are expressed in the low-density limit (see (3.8)). Two types of functions  $t_2$  are encountered:  $t_{2U}$ , describing the scattering of particles with opposite spin and expressed in terms of a paired  $t$ -matrix (see (4.1)) with potential  $U_k$ , and  $t_{2W}$ —antisymmetrical combination of  $t$  matrices with potential  $W_k$  for the case of particles with identical spin. The function  $t_3$  describes the scattering of three particles, not all of which have identical spin, and  $t_4$  the scattering of four particles of which two have the same and two have opposite spins.

If we close the incoming and outgoing arrows of the diagrams for  $t_s$  by means of normal and anomalous lines, as was done in Fig. 2, then each anomalous function gives a factor  $C$  by virtue of (3.5), and the normal one gives a factor  $C^2$ . The reason for the latter is that in the expansion (3.6) the first term, assuming that  $A(k, E)$  has no singularities in the left  $E$  half-plane, makes no contribution when the contour is closed in the left half-plane, while the second contains just  $C^2$ .

As a result we obtain for  $A(p)$  and  $B(p) = B(p) - b_1(k)$  the expressions

$$\begin{aligned}
 A(p) &\equiv C^2 a(p) = -C^2 \left[ TV^{-1} \sum_{p_1} (t_{2U}(p, p_1; p, p_1) \right. \\
 &\quad + t_{2W}(p, p_1; p, p_1)) G_0^2(p_1) G_0(-p_1) \tilde{\varphi}_{k_1}^2 \\
 &\quad + T^2 V^{-2} \sum_{p_1, p_2} t_3(p, p_1, -p_1; p_1, p_2, -p_2) \\
 &\quad \times \prod_{i=1}^2 G_0(p_i) G_0(-p_i) \tilde{\varphi}_{k_i} \left. \right], \\
 \tilde{B}(p) &\equiv C^3 b(p) = -C^3 \left[ T^2 V^{-2} \sum_{p_1, p_2} (|t_{2U}(p_1, p_2; p, p_3)|^2 \right. \\
 &\quad + |t_{2W}(p_1, p_2; p, p_3)|^2) \prod_{i=1}^3 G_0(p_i) G_0(-p_i) \tilde{\varphi}_{k_i} \\
 &\quad + 2T^2 V^{-2} \sum_{p_1, p_2} t_3(p_1, p_2, -p_2; p_1, p, -p) G_0^2(p_1) \\
 &\quad \times G_0(-p_1) G_0(p_2) G_0(-p_2) \tilde{\varphi}_{k_1}^2 \tilde{\varphi}_{k_2}^2 + (T^3 / 2V^3)
 \end{aligned}$$

$$\begin{aligned}
 &\times \sum_{p_1, p_2, p_3} t_4(p_1, -p_1, p_2, -p_2; p_3, -p_3, p, -p) \\
 &\times \prod_{i=1}^3 G_0(p_i) G_0(-p_i) \tilde{\varphi}_{k_i} \left. \right]. \quad (3.8)
 \end{aligned}$$

The first approximation (3.8) for  $A(k, E)$ , as can be readily verified, has indeed no singularities in the left  $E$  half-plane, as assumed above.

To determine  $C$  we write out the equation

$$b_1(k) = TV^{-1} \sum_{p_1} U_{kk_1} (b_1(k_1) + C^3 b(p_1)) Z^{-1}(p_1), \quad (3.9)$$

and add to its both sides the same term, in the form

$$\begin{aligned}
 b_1(k) + V^{-1} \sum_{k_1} U_{kk_1} (2k_1^2 - E_0)^{-1} b_1(k_1) \\
 = TV^{-1} \sum_{p_1} U_{kk_1} b_1(k_1) [Z^{-1}(p_1) - Z_0^{-1}(p_1)] \\
 + C^3 TV^{-1} \sum_{p_1} U_{kk_1} b(p_1) Z^{-1}(p_1). \quad (3.10)
 \end{aligned}$$

We replace  $b_1(k_1)$  on the right side of (3.10) by its first approximation, multiply by  $\varphi_k$ , and sum over  $k$ , after which the left side vanishes. Recognizing that

$$\begin{aligned}
 Z^{-1}(p_1) - Z_0^{-1}(p_1) &\approx C^2 \tilde{\varphi}_{k_1}^2 Z_0^{-2}(p_1) \\
 &+ (A(p_1) - \Delta\lambda) Z_0^{-1}(p_1) G_0(p_1) - (A(-p_1) - \Delta\lambda) \\
 &\times Z_0^{-1}(p_1) G_0(-p_1), \quad (3.11)
 \end{aligned}$$

we obtain an equation for the determination of  $C$ :

$$\begin{aligned}
 C \left[ -2(\Delta\lambda) TV^{-1} \sum_p \tilde{\varphi}_k^2 Z_0^{-1}(p) G_0(p) \right. \\
 + C^2 \left( TV^{-1} \sum_p \tilde{\varphi}_k^4 Z_0^{-2}(p) + 2TV^{-1} \sum_p \tilde{\varphi}_k^2 a(p) Z_0^{-1}(p) \right. \\
 \left. \left. \times G_0(p) - TV^{-1} \sum_p \tilde{\varphi}_k b(p) Z_0^{-1}(p) \right) \right] = 0. \quad (3.12)
 \end{aligned}$$

Using the equalities

$$\begin{aligned}
 TV^{-1} \sum_p \tilde{\varphi}_k^2 Z_0^{-1}(p) G_0(p) &= 1, \\
 TV^{-1} \sum_p \tilde{\varphi}_k^4 Z_0^{-2}(p) &= V^{-1} \sum_k (2k^2 - E_0) \varphi_k^4, \quad (3.13)
 \end{aligned}$$

we write the nontrivial solution (3.12) in the form

$$\begin{aligned}
 C^2 = a\Delta\lambda \equiv \Delta\lambda \left[ V^{-1} \sum_k (2k^2 - E_0) \varphi_k^4 \right. \\
 + TV^{-1} \sum_p \tilde{\varphi}_k^2 a(p) Z_0^{-1}(p) G_0(p) \\
 \left. - \frac{T}{2V} \sum_p \tilde{\varphi}_k b(p) Z_0^{-1}(p) \right]^{-1}. \quad (3.14)
 \end{aligned}$$

We note that the first terms in the expression (2.8)

for  $a_0^{-1}$ , obtained by the variational method, coincide with those in (3.14) for  $a^{-1}$ .

In the Appendix we obtain a different formula for  $a^{-1}$ :

$$a^{-1} = \langle AH_0^{-1}AH_0^{-1}A^+H_0^{-1}A^+ \rangle_0 \equiv \langle BH_0^{-1}B^+ \rangle_0,$$

$$A = V^{-1/4} \sum_{\mathbf{k}} \tilde{\varphi}_{\mathbf{k}} c_{\mathbf{k}+} c_{-\mathbf{k}-}, \quad B = AH_0^{-1}A,$$

$$H_0 = H - (E_0/2)N. \tag{3.15}$$

Here  $\langle \dots \rangle_0$  denotes the average over the vacuum  $|0\rangle$  ( $c_{\mathbf{k}S}|0\rangle = 0$ ). From (3.15), as shown in the Appendix, it follows that  $a^{-1}$  is positive if there are no three- or four-particle bound states.

4. We now consider the system at temperatures of the order of the temperature of the phase transition connected with the vanishing of the condensate. In this region, in addition to the condition for the applicability of the gas approximation  $\rho^{1/3} \ll 1$ , we should have  $\rho \sim T^{3/2}$ , that is, the thermal length is of the order of the mean distance between particles.

When  $T \neq 0$  the integrals with respect to  $E$  are replaced by integrals around the imaginary axis with weight  $(e^{\beta E} + 1)^{-1} = n_F(E)$ . Temperature corrections are due essentially to functions having singularities near the imaginary axis at  $\text{Re } E \sim T$ . Such a function is  $t_2U$ , which in first approximation is equal to the  $t$ -matrix

$$t\left(\frac{\mathbf{k}_1}{2} - \frac{\mathbf{k}_2}{2}, \frac{\mathbf{k}_4}{2} - \frac{\mathbf{k}_3}{2}; E + E_0 - \frac{k^2}{2}\right)$$

$$(k = \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4, \quad E = E_1 + E_2 = E_3 + E_4),$$

defined by the equation

$$t(\mathbf{k}_1, \mathbf{k}_2; z) = U_{\mathbf{k}_1\mathbf{k}_2} - V^{-1} \sum_{\mathbf{k}_3} U_{\mathbf{k}_1\mathbf{k}_3} (2k_3^2 - z)^{-1} t(\mathbf{k}_3, \mathbf{k}_2; z). \tag{4.1}$$

Since  $t(\mathbf{k}_1, \mathbf{k}_2; z)$  has a pole at  $z = E_0$  (with residue  $\tilde{\varphi}_{\mathbf{k}_1} \tilde{\varphi}_{\mathbf{k}_2}$ ),  $t_2U$  has a pole at  $E = k^2/2$ ; this pole makes a contribution to the temperature corrections at  $k \sim T^{1/2}$ . It is clear from the foregoing that when  $T \neq 0$  it is necessary to add diagrams (Fig. 3) which make a contribution from the poles of  $t_2U$  near the imaginary axis, and find the temperature corrections to the diagrams which have already been taken into account for  $T = 0$ . The corrections to the diagrams  $a_i$  and  $b_i$  for the self-energy parts will be denoted by  $\delta a_i$  and  $\delta b_i$  (and the new diagrams of Fig. 3 by  $\Delta a_i$  and  $\Delta b_i$ ).

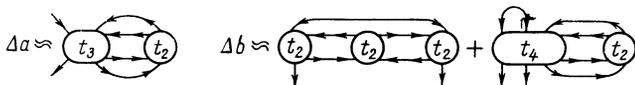


FIG. 3

For example, a contribution to  $\delta a_2$  is made by the pole of the function  $t_2U$ , and also by the fact that the residue of  $G(\mathbf{k}, E)$  in the left  $E$  half-plane differs from the residue at  $T = 0$  by a certain factor (which we denote by  $(1 + \alpha)$ ).

The expressions for  $\Delta a_i$ ,  $\Delta b_i$ ,  $\delta a_i$ , and  $\delta b_i$  are given by the formulas

$$\delta a_2 = \alpha a_2 - D^2 \tilde{\varphi}_{\mathbf{k}}^2 G_0(-p), \quad \Delta a_3 = (D/C)^2 a_3,$$

$$\Delta b_2 = 2(D/C)^2 b_2, \quad \delta b_3 = (\alpha + (D/C)^2) b_3$$

$$- (D^2/C) \tilde{\varphi}_{\mathbf{k}} [(a_2(p) + a_3(p)) G_0(p)$$

$$+ (a_2(-p) + a_3(-p)) G_0(-p)], \quad \Delta b_4 = 2(D/C)^2 b_4,$$

$$D^2 = V^{-1} \sum_{\mathbf{k}} [\exp(\beta k^2/2) - 1]^{-1} = (2\pi)^{-3/2} \zeta(3/2) T^{3/2}. \tag{4.2}$$

The expression for the residue of the function  $G(p)$  with corrections (4.2) differs from the corresponding expression at  $T = 0$  by a factor  $(1 + (D/C)^2)$ , so that  $\alpha = (D/C)^2$ .

When determining  $C$  from (3.9)–(3.12), the role of the temperature corrections reduces, as can be readily shown, to a replacement of  $C^2$  by  $C^2(1 + 2\alpha) = C^2 + 2D^2$  in the left side of (3.14). As a result we obtain for the density  $\rho_0 = 2C^2$  of the condensate and for the total density  $\rho$ , which differs from  $2C^2$  by a factor  $(1 + \alpha)$ , the expressions

$$\rho_0 = 2C^2 = 2(a\Delta\lambda - 2(2\pi)^{-3/2} \zeta(3/2) T^{3/2}) \equiv 2a\Lambda,$$

$$\rho = 2C^2(1 + \alpha) = 2(a\Delta\lambda - (2\pi)^{-3/2} \zeta(3/2) T^{3/2}). \tag{4.3}$$

It is clear from (4.3) that the condensate exists when  $\Lambda > 0$ , and the equation  $\Lambda = 0$  gives the line of the phase transition in the  $(\lambda, T)$  plane.

In the absence of condensate ( $\Lambda < 0$ ) we obtain for the self-energy part of  $A(p)$  ( $B(p) \equiv 0$ ) and for the density  $\rho$  the expressions

$$A(p) = D^2(a(p) - \tilde{\varphi}_{\mathbf{k}}^2 G_0(-p)),$$

$$\rho = 2D^2 = 2(2\pi)^{-3/2} \zeta(3/2) T^{3/2}, \tag{4.4}$$

where  $a(p)$  is defined in (3.8).

The results show that the Fermi system under consideration actually behaves like a Bose gas whose particles (bound states) form a condensate at sufficiently low temperatures. The thermodynamic functions (in particular, expressions (4.3) and (4.4) for the density) and the equation of the phase-transition curve coincide with the corresponding function of a Bose gas,<sup>[5]</sup> and the quantity (3.5) then plays the role of the parameter  $t_0$  ( $t$ -matrix at zero energy), describing the collision of two Bose particles.

The author is grateful to L. D. Faddeev for valuable advice and continuous interest in the work.

APPENDIX

To prove (3.15) let us consider the vacuum mean of the four operators  $A(t_j) = \exp(iH_0 t_j) \times A \exp(-iH_0 t_j)$

$$G(t_i) = 4^{-1}(-i)^3 \langle TA(t_1)A(t_2)A^+(t_3)A^+(t_4) \rangle_0. \quad (A.1)$$

We can obtain a diagram expansion for this function by setting in correspondence to the lines in vertices of the two types the usual expressions for  $G_0(p)$ ,  $V^{-1}U_{\mathbf{k}_1\mathbf{k}_4}$  and  $V^{-1}(W_{\mathbf{k}_1\mathbf{k}_4} - W_{\mathbf{k}_1\mathbf{k}_3})$ , with  $p = (\mathbf{k}, E)$ , and by setting in correspondence to the operators  $A^+(A)$  the vertices with two outgoing (incoming) lines. If in  $G(t_j)$  we confine ourselves to the contribution of the connected diagrams (we denoted by  $G_c(t_j)$ ), then we obtain for the quantity

$$g = \int G_c(t_i) \prod_{i=1}^3 d(t_i - t_4) \quad (A.2)$$

the diagrams of Fig. 4.

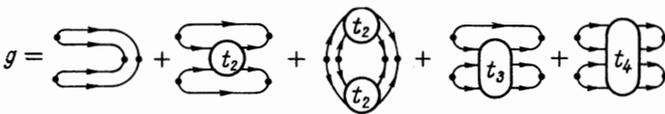


FIG. 4

Comparing with expression (3.14) for  $a^{-1}$  (in which we must substitute  $a(p)$  and  $b(p)$  from (3.8)), we verify that  $a^{-1} = g$ . On the other hand, a contribution to  $g$  is made only by the integral over the region  $(t_1, t_2 > t_3, t_4)$  (or the integral over  $(t_1 > t_2 > t_3 > t_4)$  multiplied by 4). Integrating, we obtain the required result

$$a^{-1} = g = (-i)^3 \int_0^\infty \prod_{i=1}^3 d(t_i - t_{i+1}) \times \langle e^{iH_0 t_1} A e^{iH_0(t_2-t_1)} A e^{iH_0(t_3-t_2)} A^+ e^{iH_0(t_4-t_3)} A^+ e^{-iH_0 t_4} \rangle_0 = \langle AH_0^{-1}AH_0^{-1}A^+H_0^{-1}A^+ \rangle_0. \quad (A.3)$$

The vector  $B^+ \rangle_0 = A^+H_0^{-1}A^+ \rangle_0$  in (A.3) is a four-particle vector, since  $A^+$  produces a pair of particles, and  $H_0$  conserves the number of particles. The central operator  $H_0^{-1}$  in Eq. (A.3) acts on the four-particle sector, where  $H_0$  is equal to  $H - (E_0/2)N = H - 2E_0$ . If there are no three or four-particle bound states, then the spectrum of  $H$  begins from  $2E_0$  (double the energy of the two-particle state), and therefore  $H_0$ , together with  $H_0^{-1}$ , is positive definite, and the expression for  $a^{-1}$  in the right side of (A.3) is positive.

<sup>1</sup>V. M. Galitskiĭ, JETP 34, 151 (1958), Soviet Phys. JETP 7, 104 (1958).

<sup>2</sup>N. N. Bogolyubov, V. V. Tolmachev, and D. V. Shirkov, *Novyĭ method v teorii sverkhprovodimosti (New Method in the Theory of Superconductivity)*, AN SSSR, 1958.

<sup>3</sup>A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, *Metody kvantovoi teorii polya v statisticheskoi fizike (Methods of Quantum Field Theory in Statistical Physics)*, Fizmatgiz, 1962.

<sup>4</sup>S. T. Belyaev, JETP 34, 417 (1958), Soviet Phys. JETP 7, 289 (1958).

<sup>5</sup>V. N. Popov, JETP 47, 1759 (1964), Soviet Phys. JETP 20, 1185 (1965).

Translated by J. G. Adashko  
186