# *CONCERNING THE DYNAMICS OF A SUPERFLUID FERMI GAS. II. STATIC EQUATIONS AT FINITE TEMPERATURES*

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Submitted to JETP editor December 2, 1965

J. Exptl. Theoret. Phys. (U.S.S.R.) 50, 1369–1376 (May, 1966)

A static nonlinear differential equation for the characteristic parameter (gap) $\Delta$ in a superfluid Fermi gas at finite temperature is derived by taking into account the motion of the normal component and the presence of a magnetic field. The behavior of the gap $\Delta(\mathbf{r})$ at large distances from the vortex line is determined for arbitrary temperatures.

IN an earlier paper by the author and Pitaevskiĭ [1], a nonlinear differential equation was obtained for the characteristic parameter $\Delta$ in a superfluid Fermi gas at absolute zero temperature. It was assumed that $\Delta$ varies little in space and in time. The purpose of the present paper is to obtain a similar equation for finite temperatures. We shall consider here only the static case, when $\Delta$ does not depend on the time.

Such an equation was obtained for temperatures close to the transition temperature (the Ginzburg-Landau equation) from the microscopic theory first by Gor'kov [2], and in the general case by Tewort [3] and Werthamer [4]. All these authors have assumed, however, that the nominal part of the liquid is at rest. Yet the form of the equations for the case when $\mathbf{v}_n \neq 0$ is of considerable interest in connection with the question of the Galilean invariance of the equations. As seen from [1], this question is very important for a description of the properties of the system. In particular, without introducing $\mathbf{v}_n$, we cannot ascertain whether the change of $\Delta$ is connected with the motion of the superfluid part relative to the normal part or relative to the lattice. We note also that there are discrepancies between the results of Tewort and Werthamer, so that the question remains unclear even when $\mathbf{v}_n = 0$.

The method which we use is an extension of the method of the earliest papers [1] to finite temperatures. The most essential difference consists in the fact that now, in addition to the superfluid velocity $\mathbf{v}_s$, there is a velocity $\mathbf{v}_n$ of the normal component. In order to take $\mathbf{v}_n$ into account, it is necessary to add in the Hamiltonian of the system a term $-\mathbf{p}\cdot\mathbf{v}_n$, where $\mathbf{p}$ is the operator of the total momentum of the system [5,6].

It is obvious that in the static case the equation for $\Delta^*$ can be obtained by minimization with respect to $\Delta$ of the thermodynamic potential $\Omega$ expressed in terms of $\Delta$ and its derivatives with respect to the coordinates. The expansion of $\Omega$ with respect to $\mathbf{v}_n$ and the derivatives of $\Delta$, accurate to second-order terms, has in the general case the form

$$\Omega = \int \{\bar{\Omega}(|\Delta|^2) + \mathbf{l}^2 a_4(|\Delta|^2) + a_1(|\Delta|^2)[(\nabla\varphi)^2 + (\nabla\varphi^*)^2] + a_2(|\Delta|^2)(\nabla\varphi)(\nabla\varphi^*) + a_3(|\Delta|^2) i\mathbf{l}(\nabla\varphi - \nabla\varphi^*)\}\, d^3r. \quad (1)$$

Here $\bar{\Omega}$ is the thermodynamic potential of the homogeneous system when $\mathbf{v}_n = 0$, $\mathbf{l} = 2m\mathbf{v}_n$ and $\varphi = \ln\Delta$, while the coefficients are unknown functions of $\Delta/T$, which we determine from the microscopic theory. As in [1], in writing out (1) we took account of the fact that the potential $\Omega$ should be invariant with respect to the transformation

$$\Delta \to \Delta e^{i\alpha}, \quad \Delta^* \to \Delta^* e^{-i\alpha}.$$

Therefore expression (1) contains derivatives of $\ln\Delta$ only.

Equating to zero the functional derivative of $\Omega$ with respect to $\Delta$ at a constant velocity $\mathbf{v}_n$, we obtain an expression for $\Delta^*$:

$$[\bar{\Omega}' + \mathbf{l}^2 a_4']|\Delta|^2 - 2a_1\nabla^2\varphi - a_2\nabla^2\varphi^* + a_1'|\Delta|^2[(\nabla\varphi^*)^2 - (\nabla\varphi)^2 - 2(\nabla\varphi)(\nabla\varphi^*)] - a_2'|\Delta|^2(\nabla\varphi^*)^2 - 2a_3'|\Delta|^2 i\mathbf{l}\nabla\varphi^* = 0. \quad (2)$$

The primes denote here derivatives with respect to $|\Delta|^2$. We put, just as in [1],

$$\Delta(\mathbf{r}) = [\tilde{\Delta}_0 + \tilde{\Delta}_1(\mathbf{r})]\, e^{i\mathbf{qr}} \quad (3)$$

and assume that $\tilde{\Delta}_1 \ll \tilde{\Delta}_0$. Going over now in the usual manner to Fourier components

$$\Delta(\mathbf{r}) = \int \Delta(\mathbf{k}) e^{i\mathbf{kr}} \frac{d^3k}{(2\pi)^3}, \quad \Delta^*(\mathbf{r}) = \int \Delta^*(\mathbf{k}) e^{i\mathbf{kr}} \frac{d^3k}{(2\pi)^3} \quad (4)$$

and substituting (3) in (2), we obtain after linearization an equation for $\widetilde{\Delta}_0$:

$$\{\overline{\Omega}' + l^2 a_4' + q^2[-2a_1' + a_2'] - 2a_3'(\mathbf{lq})\}\widetilde{\Delta}_0^* = 0 \quad (5)$$

and an equation for $\widetilde{\Delta}_1(\mathbf{k})$:

$$\begin{aligned}&\widetilde{\Delta}_0\widetilde{\Delta}_1^*(\mathbf{k})\{\overline{\Omega}' + |\Delta_0|^2\overline{\Omega}'' + \mathbf{l}^2[a_4' + |\Delta_0|^2 a_4'']\\ &\quad + a_2|\Delta_0|^{-2}k^2 + [-2a_1' + a_2' - 2a_1''|\Delta_0|^2 + a_2''|\Delta_0|^2]q^2\\ &\quad + [4a_1' - 2a_2']\mathbf{kq} + 2a_3'\mathbf{lk} + [-2a_3' - 2a_3''|\Delta_0|^2]\mathbf{lq}\}\\ &\quad + \widetilde{\Delta}_0^*\widetilde{\Delta}_1(\mathbf{k})\{\overline{\Omega}''|\Delta_0|^2 + l^2|\Delta_0|^2 a_4'' - 2a_1|\Delta_0|^{-2}k^2\\ &\quad - [2|\Delta_0|^2 a_1'' - |\Delta_0|^2 a_2'']q^2 - 2|\Delta_0|^2 a_3''(\mathbf{lq})\} = 0. \quad (6)\end{aligned}$$

The coefficients $\overline{\Omega}$, $a_1$, $a_2$, ... are determined from a comparison of (5) and (6) with similar equations obtained from the microscopic theory on the basis of Gor'kov's equations.

Gor'kov's equations in the temperature approach, in the presence of $\mathbf{v}_n$, are of the form

$$\begin{aligned}&\left(-\frac{\partial}{\partial\tau} + \frac{\nabla^2}{2m} + \hat{\mathbf{p}}\mathbf{v}_n + \mu\right)G(x, x') + \Delta(x)F^+(x, x')\\ &\quad = \delta(x - x'),\\ &\left(\frac{\partial}{\partial\tau} + \frac{\nabla^2}{2m} - \hat{\mathbf{p}}\mathbf{v}_n + \mu\right)F^+(x, x') - \Delta^*(x)G(x, x') = 0, \quad (7)\end{aligned}$$

where $\tau$ is the Matsubara "time" and $x = (\mathbf{r}, \tau)$. Eliminating from (7) the function $G(x, x')$, we obtain

$$\begin{aligned}&\left[-\frac{\partial}{\partial\tau} - i\mathbf{v}_n\nabla + \frac{\nabla^2}{2m} + \mu\right]\frac{1}{\Delta^*(x)}\left[\frac{\partial}{\partial\tau} + i\mathbf{v}_n\nabla +\right.\\ &\quad \left. + \frac{\nabla^2}{2m} + \mu\right]F^+(x, x') + \Delta(x)F^+(x, x') = \delta(x - x').\end{aligned}$$

Making now the substitution

$$F^+(x, x') = \widetilde{F}^+(x, x') \exp[-i(\mathbf{q}, \mathbf{r} + \mathbf{r}')], \quad (3')$$

we obtain

$$\begin{aligned}&\left[-\frac{\partial}{\partial\tau} - i\left(\mathbf{v}_n, \nabla + \frac{i\mathbf{q}}{2}\right) + \frac{(\nabla + i\mathbf{q}/2)^2}{2m}\right.\\ &\quad \left. + \mu\right]\frac{1}{\widetilde{\Delta}^*(x)}\left[\frac{\partial}{\partial\tau} + i\left(\mathbf{v}_n, \nabla - \frac{i\mathbf{q}}{2}\right)\right.\\ &\quad \left. + \frac{(\nabla - i\mathbf{q}/2)^2}{2m} + \mu\right]\widetilde{F}^+(x, x') + \widetilde{\Delta}(x)\widetilde{F}^+(x, x')\\ &\quad = \delta(x - x'), \quad (8)\end{aligned}$$

with

$$\widetilde{\Delta}^*(x) = |g|\widetilde{F}^+(x, x). \quad (9)$$

We put

$$\widetilde{F}^+(x, x') = \widetilde{F}_0^+(x - x') + \widetilde{F}_1^+(x, x')$$

and linearize (8) with respect to $\widetilde{\Delta}_1^*$ and $\widetilde{F}_1^+$. As a result we obtain

$$\begin{aligned}&\left\{\left[-\frac{\partial}{\partial\tau} - i\left(\mathbf{v}_n, \nabla + \frac{i\mathbf{q}}{2}\right) + \frac{(\nabla + i\mathbf{q}/2)^2}{2m} + \mu\right]\frac{1}{\widetilde{\Delta}_0^*}\left[\frac{\partial}{\partial\tau}\right.\right.\\ &\quad \left. + i\left(\mathbf{v}_n, \nabla - \frac{i\mathbf{q}}{2}\right) + \frac{(\nabla - i\mathbf{q}/2)^2}{2m} + \mu\right]\\ &\quad \left. + \widetilde{\Delta}_0\right\}\widetilde{F}_1^+(x, x') = \left\{\left[-\frac{\partial}{\partial\tau} - i\left(\mathbf{v}_n, \nabla + \frac{i\mathbf{q}}{2}\right)\right.\right.\\ &\quad \left. + \frac{(\nabla + i\mathbf{q}/2)^2}{2m} + \mu\right]\frac{\widetilde{\Delta}_1^*(x)}{\widetilde{\Delta}_0^{*2}}\left[\frac{\partial}{\partial\tau}\right.\\ &\quad \left. + i\left(\mathbf{v}_n, \nabla - \frac{i\mathbf{q}}{2}\right) + \frac{(\nabla - i\mathbf{q}/2)^2}{2m} + \mu\right]\\ &\quad \left. - \widetilde{\Delta}_1(x)\right\}\widetilde{F}_0^+(x - x'). \quad (10)\end{aligned}$$

$F_0^+(x - x')$ satisfies equation (8) with $\widetilde{\Delta}_0$ in lieu of $\widetilde{\Delta}(x)$. We go over in (8) to the Fourier components with respect to $\mathbf{r}$, $\mathbf{r}'$ and $\tau - \tau'$. Equation (9) in terms of Fourier components becomes

$$\widetilde{\Delta}^*(\mathbf{r}) = |g| \sum_n \widetilde{F}^+_{\omega_n}(\mathbf{r}, \mathbf{r}), \quad (9')$$

where $\omega_n = (2n + 1)\pi T$ is the discrete frequency which arises when $F^+$ is expanded in a Fourier series in $\tau - \tau'$. Expanding further in powers of $\mathbf{k}$, $\mathbf{q}$, and $\mathbf{l}$ up to second-order terms, we get after calculations similar to those given in[1] an equation for $\widetilde{\Delta}_1^*(\mathbf{k})$:

$$\begin{aligned}&\left[\frac{1}{|g|} - \frac{mp_0}{2\pi^2}\int_{-\infty}^{\infty} d\xi\left(\frac{1}{2\varepsilon} - \frac{n}{\varepsilon}\right)\right]\widetilde{\Delta}_1^*(\mathbf{k})\\ &= \frac{mp_0}{2\pi^2}\left\{-\frac{1}{2} - |\Delta_0|^2 f' + \frac{v_F^2k^2}{3}\left[-\frac{1}{2\cdot 3}\frac{1}{|\Delta_0|^2}\right.\right.\\ &\quad + \frac{1}{4}\frac{\rho_n'}{\rho} + \frac{|\Delta_0|^2}{12}\frac{\rho_n''}{\rho} + \frac{2}{3}|\Delta_0|^2 f''\\ &\quad \left. + \frac{1}{6}|\Delta_0|^4 f'''\right] + \frac{v_F^2q^2}{3}\left[\frac{1}{4}\frac{\rho_n'}{\rho} + \frac{|\Delta_0|^2}{4}\frac{\rho_n''}{\rho}\right]\\ &\quad \left. - \frac{v_F^2(\mathbf{kq})}{6}\frac{\rho_n'}{\rho} + \frac{v_F^2l^2}{3}\left[\frac{1}{4}\frac{\rho_n'}{\rho} + \frac{|\Delta_0|^2}{4}\frac{\rho_n''}{\rho}\right]\right\}\widetilde{\Delta}_1^*(k)\\ &\quad + \frac{mp_0}{2\pi^2}\left\{-\frac{1}{2} - |\Delta_0|^2 f' + \frac{v_F^2k^2}{3}\left[\frac{1}{12}\frac{1}{|\Delta_0|^2} + \frac{|\Delta_0|^2}{12}\frac{\rho_n''}{\rho}\right.\right.\\ &\quad \left. + \frac{|\Delta_0|^2}{6}f'' + \frac{|\Delta_0|^4}{6}f'''\right] + \frac{v_F^2q^2}{3}\frac{|\Delta_0|^2}{4}\frac{\rho_n''}{\rho} -\\ &\quad \left. - \frac{v_F^2\mathbf{lq}}{3}\frac{|\Delta_0|^2}{2}\frac{\rho_n''}{\rho} + \frac{v_F^2l^2}{3}\frac{|\Delta_0|^2}{4}\frac{\rho_n''}{\rho}\right\}\widetilde{\Delta}_1(\mathbf{k})\frac{\widetilde{\Delta}_0^*}{\widetilde{\Delta}_0}. \quad (11)\end{aligned}$$

Here $\rho = mp_0^3/3\pi^2$ is the total density of the gas,

$$\rho_n = -\frac{2}{(2\pi)^3}\int \mathbf{p}^2 \frac{\partial n}{\partial \varepsilon} d^3p = -\rho \int_{-\infty}^{\infty} \frac{\partial n}{\partial \varepsilon} d\xi$$

is the density of the normal component;

$$f = \int_{-\infty}^{\infty} \frac{n}{\varepsilon} d\xi, \quad n = \frac{1}{e^{\varepsilon/T}+1},$$

where n is the Fermi distribution function; $p_0$ is the Fermi momentum. Comparing now (11) with (5) we obtain equations for the coefficients $\overline{\Omega}$, $a_1$, $a_2$, ...:

$$\overline{\Omega}' = -\frac{1}{|g|} + \frac{mp_0}{2\pi^2}\int_{-\infty}^{\infty} d\xi\left(\frac{1}{2\varepsilon} - \frac{n}{\varepsilon}\right),$$

$$-2a_1' + a_2' = \frac{v_F^2}{3}\frac{\rho_n'}{4\rho}, \quad a_3' = \frac{v_F^2}{3}\frac{\rho_n'}{4\rho},$$

$$a_4' = \frac{v_F^2}{3}\frac{\rho_n'}{4\rho}, \quad \frac{2a_1}{|\Delta_0|^2} = \frac{v_F^2}{3}\left[\frac{1}{12}\frac{1}{|\Delta_0|^2} + \frac{\rho_n''}{4\rho}\frac{|\Delta_0|^2}{3}\right.$$

$$\left. + \frac{1}{6}|\Delta_0|^2 f'' + \frac{1}{6}|\Delta_0|^4 f'''\right],$$

$$\frac{a_2}{|\Delta_0|^2} = \frac{v_F^2}{3}\left[-\frac{1}{6}\frac{1}{|\Delta_0|^2} + \frac{\rho_n'}{4\rho}\right.$$

$$\left. + \frac{|\Delta_0|^2}{3}\frac{\rho_n''}{4\rho} + \frac{2}{3}|\Delta_0|^2 f'' + \frac{1}{6}|\Delta_0|^4 f'''\right]. \quad (12)$$

The equation for $\widetilde{\Delta}_0$ yields nothing new compared with (12). Substituting in the second equation of (12) the expression for $a_1$ and $a_2$, we obtain the condition

$$\frac{1}{4}\frac{\rho_n''}{\rho} + f'' + \frac{1}{2}|\Delta_0|^2 f''' = 0; \quad (13)$$

It is easy to show that this condition is an identity.[1)]

We must stipulate that the form of the functional $\Omega$ can be established from the equation for $\Delta$ only accurate to a factor which does not depend on $\Delta$. This factor was already chosen by us in (12) in such a way as to make $\overline{\Omega}$ coincide with $\Delta$-dependent part of the thermodynamic potential of the Fermi gas. Indeed, according to the first condition (12) we can write

$$\overline{\Omega} = \int d|\Delta|^2\left[-\frac{1}{|g|} + \frac{mp_0}{2\pi^2}\int_{-\infty}^{\infty} d\xi\left(\frac{1}{2\varepsilon} - \frac{n}{\varepsilon}\right)\right].$$

Charging in the last integral the order of integration and taking into account the fact that $d|\Delta|^2 = 2\epsilon d\epsilon$, we obtain

$$\overline{\Omega} = -\left\{\int d|\Delta|^2\left[-\frac{1}{|g|} + \frac{mp_0}{2\pi^2}\int_{-\infty}^{\infty}\frac{d\xi}{2\varepsilon}\right]\right.$$

$$\left. + \frac{mp_0}{2\pi^2}2T\int_{-\infty}^{\infty}\ln(1+e^{-\varepsilon/T})d\xi\right\}.$$

The last term in this formula coincides, as it should, with the usual formula for the thermodynamic potential of a gas of elementary excitations. The coefficients $a_3$ and $a_4$ are determined from (11) only accurate to a term that does not depend on $\Delta$. It can be determined by noting that as $T \to 0$ we have $a_3$, $a_4 \to 0$, for when $\rho_n \to 0$ the potential $\Omega$ cannot depend on $\mathbf{v}_n$. As a result we obtain

$$a_3 = \frac{v_F^2}{3}\frac{\rho_n}{4\rho}, \quad a_4 = \frac{v_F^2}{3}\frac{\rho_n}{4\rho}.$$

The final expression for $\Omega$ assumes after certain transformations the form

$$\Omega = \overline{\Omega} + \frac{mp_0}{2\pi^2}\frac{v_F^2}{3}\int\left\{\frac{1}{8\cdot 3}[4(\nabla\varphi)(\nabla\varphi^*) - (\nabla\varphi)^2 - (\nabla\varphi^*)^2]\right.$$

$$-\frac{1}{8\cdot 3}\frac{\rho_n}{\rho}[4(\nabla\varphi - 2im\mathbf{v}_n)(\nabla\varphi^* + 2im\mathbf{v}_n)$$

$$-(\nabla\varphi - 2im\mathbf{v}_n)^2 - (\nabla\varphi^* + 2im\mathbf{v}_n)^2] - \frac{1}{8\cdot 3}|\Delta|^2\frac{\rho_n'}{\rho}$$

$$\left.\times(\nabla\varphi^* + \nabla\varphi)^2\right\}d^3r. \quad (14)$$

The nonlinear static equation for $\Delta$ is now[2)]:

$$\left[\ln\frac{\Delta_0(T)}{\Delta} + f(\Delta) - f(\Delta_0(T))\right]|\Delta|^2$$

$$= \frac{v_F^2}{3}\left\{-\frac{1}{4}\frac{\rho_n''}{\rho}|\Delta|^2(\nabla\varphi^* + 2im\mathbf{v}_n)^2\right.$$

$$+ \frac{1}{8\cdot 3}|\Delta|^4\frac{\rho_n''}{\rho}(\nabla\varphi^* + \nabla\varphi)^2 + \frac{1}{6}\frac{\rho_s}{\rho}\left[\nabla^2\varphi^* - \frac{1}{2}\nabla^2\varphi\right]$$

$$\left. - \frac{1}{12}|\Delta|^2\frac{\rho_n'}{\rho}(\nabla^2\varphi^* + \nabla^2\varphi)\right\}. \quad (15)$$

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[1)]Indeed

$$f = 2\int_0^{\infty}\frac{n}{\varepsilon}d\xi = 2\int_{|\Delta|}^{\infty}\frac{n}{\varepsilon}\frac{\partial\xi}{\partial\varepsilon}d\varepsilon = -2\int_{|\Delta|}^{\infty}\frac{\partial}{\partial\varepsilon}\left(\frac{n}{\varepsilon}\right)\xi d\varepsilon.$$

Taking now account of the fact that $d\epsilon = [(\epsilon^2 - |\Delta|^2)/\epsilon]d\xi$, we obtain

$$\int_{-\infty}^{\infty}\frac{n}{\varepsilon}d\xi = -\int_{-\infty}^{\infty}\left[\frac{dn}{\partial\varepsilon} - \frac{n}{\varepsilon} - 2|\Delta|^2\frac{\partial}{\partial|\Delta|^2}\left(\frac{n}{\varepsilon}\right)\right]d\xi,$$

that is, $\rho_n/\rho = -2|\Delta|^2 f'$. Differentiating the obtained identity with respect to $|\Delta|^2$ twice, we obtain (13).

[2)]Equation (15) with $\mathbf{v}_n = 0$ does not coincide with the equation obtained in [7]. This, as already indicated in [1], is probably connected with the insufficient accuracy of the approximations made in [7].

In (15) the quantity $\Delta_0(T)$ is the equilibrium value of $\Delta$ for a given temperature in the spatially-homogeneous case, in the absence of a velocity difference $\mathbf{v}_n - \mathbf{v}_s$.

Equation (15) is, of course, valid only when $\Delta$ varies little over distances of the order of $\xi_0 = v_F/\Delta_0(0)$. On the other hand, the velocity $\mathbf{v}_n$ should be considered in the equation as a constant. Indeed, by definition $\mathbf{v}_n$ should vary little over distances of the order of the mean free path of the elementary excitations $l$. Yet in a superfluid Fermi gas the inequality $l \gg \xi_0$ is satisfied practically in the entire range of temperatures. ($l$ is comparable with $\xi_0$ only when $(T_c - T)/T_c \sim (T_c/\mu)^6$, that is, in the temperature region where the usual theory of superconductivity ceases to be valid because of the increase of the fluctuations at the transition point (see[5], page 308).)

Equation (15), as well as the initial equation (6), has Galilean invariance, admitting the transformation group

$$\Delta(\mathbf{r}) \to \Delta(\mathbf{r})\, e^{2im(\mathbf{vr})}$$

with simultaneous substitution $\mathbf{v}_n \to \mathbf{v}_n - \mathbf{v}$.

Let us now consider on the basis of (15) the asymptotic behavior of $\Delta$ at large distances from the axis of a vortex filament. Substituting in (15) the quantities $\Delta = |\Delta| e^{i\vartheta}$ and $\mathbf{v}_n = 0$, and neglecting the derivatives of $|\Delta|$, we obtain

$$\Delta(\mathbf{r}) = \Delta_0\left(1 - \frac{v_F^2}{3}\frac{\rho_n'}{\rho_s}\frac{1}{r^2}\right) \tag{16}$$

(we recall that (15) is valid only over distances much larger than $\xi_0$). As $T \to T_c$, (16) goes over into the formula obtained first by Abrikosov[8]. As already noted in[1], when $T \to 0$ terms of order $1/r^2$ vanish, so that $|\Delta|^2$ changes only in a higher order.

The presence of an external magnetic field can be taken into account in the usual manner, by making in (14) the substitution

$$\nabla\varphi \to \nabla\varphi - 2ie\mathbf{A}, \qquad \nabla\varphi^* \to \nabla\varphi^* + 2ie\mathbf{A},$$

in accordance with the requirements of gauge invariance (here e is the charge of the gas particle). As a result we obtain

$$\Omega = \bar{\Omega} + \frac{mp_0}{2\pi^2}\frac{v_F^2}{3}\int\Big\{\frac{1}{8\cdot 3}[4(\nabla\varphi - 2ie\mathbf{A})(\nabla\varphi^* + 2ie\mathbf{A}) - (\nabla\varphi - 2ie\mathbf{A})^2 - (\nabla\varphi^* + 2ie\mathbf{A})^2] - \frac{1}{8\cdot 3}\frac{\rho_n}{\rho}[4(\nabla\varphi - 2im\mathbf{v}_n - 2ie\mathbf{A})(\nabla\varphi^* + 2im\mathbf{v}_n + 2ie\mathbf{A}) - (\nabla\varphi - 2im\mathbf{v}_n - 2ie\mathbf{A})^2 - (\nabla\varphi^* + 2im\mathbf{v}_n + 2ie\mathbf{A})^2] - \frac{1}{8\cdot 3}|\Delta|^2\frac{\rho_n'}{\rho}(\nabla\varphi + \nabla\varphi^*)^2\Big\}d^3r. \tag{14a}$$

Then the equation for $\Delta$ for a transverse potential gauge ($\operatorname{div}\mathbf{A} = 0$) will take the form

$$|\Delta|^2\left[\ln\frac{\Delta_0(T)}{\Delta} + f(\Delta) - f(\Delta_0(T))\right]|\Delta|^2 = \frac{v_F^2}{3}\Big\{-\frac{1}{4}\frac{\rho_n'}{\rho}|\Delta|^2(\nabla\varphi^* + 2im\mathbf{v}_n + 2ie\mathbf{A})^2 - \frac{1}{3\cdot 3}|\Delta|^4\frac{\rho_n''}{\rho}(\nabla\varphi^* + \nabla\varphi)^2 + \frac{1}{6}\frac{\rho_s}{\rho}\left[\nabla^2\varphi^* - \frac{1}{2}\nabla^2\varphi\right] - \frac{1}{4\cdot 3}|\Delta|^2\frac{\rho_n'}{\rho}(\nabla^2\varphi^* + \nabla^2\varphi)\Big\}. \tag{17}$$

Of course, we could arrive at the same result by starting directly from Gor'kov's equations with account of the magnetic field.

Varying now (14a) with respect to $\mathbf{A}$, we can use the well known formula

$$\delta\Omega = -\int \mathbf{j}\,\delta\mathbf{A}\,d^3r$$

to obtain the current density $\mathbf{j}$:

$$\mathbf{j} = \frac{ie}{4m^2}[\rho_s(\nabla\varphi^* - \nabla\varphi + 4ie\mathbf{A}) - \rho_n\cdot 4im\mathbf{v}_n], \tag{18}$$

which when $\mathbf{v}_n = 0$ coincides, to an appropriate degree of approximation, with the result of Suhl and Stephen[9]. In the spatially-homogeneous case, when $\mathbf{v}_n = 0$ equation (18) coincides with the London equation. Subtracting from (17) the complex-conjugate expression, we obtain the current conservation law

$$\operatorname{div}\mathbf{j} = 0.$$

Let us see now what form will be taken by the equation for $\Delta$ at temperatures close to absolute zero, and at temperatures close to critical. When $T \ll T_c$, so that $\rho_n \ll \rho$, we have from (17)

$$\left[\frac{1}{|g|} - \frac{mp_0}{2\pi^2}\ln\frac{2\tilde{\omega}}{|\Delta|}\right]|\Delta|^2 = \frac{mp_0}{12\pi^2}\frac{v_F^2}{3}\left[\nabla^2\varphi^* - \frac{1}{2}\nabla^2\varphi\right].$$

This equation coincides with the static part of the equation obtained in[1]. We see that when $T = 0$ in our approximation the magnetic field does not enter in the equation explicitly. However, the expression for the current does change and consequently also the connection between $\Delta$ and the superfluid velocity $\mathbf{v}_s$. In a magnetic field, a superfluid current appears in the superconducting system. At finite temperatures, the motion of the superfluid part relative to the normal part leads to a change in $\Delta$. This explains the explicit presence of A in (17). At absolute zero there is no such effect, since the motion of the superfluid part as a whole cannot change $\Delta$, and the magnetic field appears in the equations only in the higher approximations.

At temperatures close to critical we obtain an equation of the Ginzburg-Landau type. Indeed, for $|T - T_c| \ll T_c$

$$\frac{\rho_n}{\rho} = 1 - \frac{3}{2}|\Delta|^2 \frac{7\zeta(3)}{6\pi^2 T_c^2}, \qquad (19)$$

where $\zeta(x)$ is the Riemann $\zeta$-function. The left side of (17) is equal to

$$\frac{mp_0}{2\pi^2}\left\{\frac{7\zeta(3)}{8\pi^2 T_c^2}|\Delta|^2 - \frac{T_c - T}{T_c}\right\}|\Delta|^2.$$

Thus, (17) takes the form

$$\left\{\frac{1}{4m}[\nabla + 2im\mathbf{v}_n + 2ie\mathbf{A}]^2 + \frac{6\pi^2 T_c^2}{\varepsilon_F 7\zeta(3)}\left[\frac{T - T_c}{T_c} - \frac{7\zeta(3)}{8\pi^2 T_c^2}|\Delta|^2\right\}\Delta^*(r) = 0. \qquad (20)$$

When $\mathbf{v}_n = 0$ it coincides exactly with the Ginzburg-Landau equation.

In the absence of a magnetic field (18) goes over into the equation obtained by Pitaevskiĭ from phenomenological considerations[10]. Expression (14) was obtained, from the nature of the derivation, for $\mathbf{v}_n$ = const. It is natural, however, to assume that it is valid also for variable $\mathbf{v}_n$, accurate to first-order derivatives of $\mathbf{v}_n$ with respect to the coordinates. In this case terms containing div $\mathbf{v}_n$ appear in the equations. In the right side of (15) and (17) there is added the expression

$$-i\frac{mv_F^2}{6}\frac{\rho_n}{\rho}\operatorname{div}\mathbf{v}_n,$$

and in the left side of (20) there appears a term

$$-\frac{i}{2}\frac{\rho}{\rho_s}\tilde{\Delta}^*\operatorname{div}\mathbf{v}_n.$$

These terms ensure the correct form of the continuity equation for variable $\mathbf{v}_n$. We note that there is no such term in the equations in[10].

In the quasiclassical approximation

$$\Delta = |\Delta| \exp\{2i(m\mathbf{v}_s + e\mathbf{A}, \mathbf{r})\},$$

and the thermodynamic potential takes the form

$$\Omega = \bar{\Omega} + {}^1/_2\rho v_s^2 - {}^1/_2\rho_n(\mathbf{v}_n - \mathbf{v}_s)^2.$$

The equation for $\Delta$ reduces to the form

$$\frac{mp_0}{2\pi^2}\left[\ln\frac{\Delta_0(T)}{\Delta} + f(\Delta) - f(\Delta_0(T))\right] = \frac{1}{2}\rho_n'(\mathbf{v}_n - \mathbf{v}_s)^2,$$

and the current density has the usual appearance

$$\mathbf{j} = \frac{e}{m}(\rho_s\mathbf{v}_s + \rho_n\mathbf{v}_n).$$

The method proposed here can be generalized to the nonstationary case, to obtain time-dependent equations. It is necessary, however, to carry out here analytic continuation from discrete frequencies to continuous ones.

The author thanks L. P. Pitaevskiĭ for great help in the work and A. A. Abrikosov, A. F. Andreev, and L. P. Gor'kov for useful discussions.

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Translated by J. G. Adashko
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