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SOLUTION AND QUANTIZATION OF A NONLINEAR TWO-DIMENSIONAL MODEL FOR A BORN-INFELD TYPE FIELD

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The Cauchy problem for the nonlinear equation for a scalar field of the Born-Infeld type[1] is solved in the two-dimensional case. The solution is obtained in parametric form. It is shown that the solution defines an extremal surface in three-dimensional pseudo-euclidean space. Such a geometric interpretation suggests the possibility of solving a more complex problem, viz., that of several nonlinear interacting fields. The solution of the generalized model is interpreted geometrically as a two-dimensional extremal surface in multi-dimensional pseudo-euclidean space. In the solutions the time t and the coordinate x together with the field functions φ_i are interpreted as components of a new multi-component field. On the basis of this interpretation a procedure is developed for quantizing such nonlinear systems without recourse to perturbation theory. As a result of the quantization the time t and the coordinate x turn out to be operators along with the field functions. The result is a theory in quantized space-time.

1. INTRODUCTION

SINCE the appearance of the paper by Born and Infeld[1] on a nonlinear electrodynamics of the free field it has become clear that the nonlinear theory leads to qualitatively new and much richer physical concepts than the linear theory. Therefore the solution and the study of nonlinear models of field theory are of fundamental interest.

Mathematicians know exact solutions only for a very limited class of nonlinear partial differential equations. One of these is the equation of minimal surfaces:[2]

$$(1+\varphi_y^2)\varphi_{x,x} - 2\varphi_x\varphi_y\varphi_{xy} + (1+\varphi_x^2)\varphi_{y,y} = 0. \quad (1)$$

This equation is always of the elliptic type and can therefore, only describe phenomena which do not vary in time as, for example, the shape of a soap film bounded by a fixed contour, or certain stationary motions of a liquid or a gas. [The importance of the analogy between Eq. (1) and the gravitational equations of Einstein was pointed out by Wheeler.[3]] It is easy to write down an equation analogous to (1) which is also exactly soluble:

$$(1-\varphi_t^2)\varphi_{x,x} + 2\varphi_x\varphi_t\varphi_{x,t} - (1+\varphi_x^2)\varphi_{t,t} = 0, \quad (2)$$

this equation can describe processes which develop in time, since it is of the hyperbolic type if $1+\varphi_x^2-\varphi_t^2>0$. Equation (2) is obtained by setting y = it. It is remarkable that this equation is the two-dimensional scalar analog of the nonlinear equations of the Born-Infeld electrodynamics. Similar equations have been considered by Blokhintsev[4] and Heisenberg.[5]

Our aim is to solve the Cauchy problem for Eq. (2). Besides its intrinsic interest it affords the possibility of quantizing this nonlinear field.

From a geometric point of view the solution $z=\varphi(x,y)$ of (1) defines a surface of minimal area in euclidean space which is bounded by a given contour. In the coordinates x, y, z the area of the surface is equal to

$$S = -\iint (1+\varphi_x^2+\varphi_y^2)^{1/2}\,dx\,dy \quad (3)$$

and Eq. (1) is the extremal condition for the integral (3). Analogously, Eq. (2) is the extremal condition for the integral

$$S = -\iint (1+\varphi_x^2-\varphi_t^2)^{1/2}\,dx\,dt, \quad (4)$$

which expresses the area of the surface $z=\varphi(x,t)$ in pseudo-euclidean space with the metric $ds^2 = dt^2 - dx^2 - dz^2$. In connection with this, one can regard Eq. (2) as an equation for the extremal surfaces in pseudo-euclidean space. It follows from (4) that the scalar field satisfying (2) has a Lagrangian of the Born-Infeld type

$$L = -(1+\varphi_x^2-\varphi_t^2)^{1/2}. \quad (5)$$

This geometric interpretation allows us to generalize the problem to a more general soluble case, viz., to consider the problem of the two-dimensional extremal surface in N-dimensional pseudo-euclidean space. From a physical point of view this problem is the generalization of the nonlinear Born-Infeld field to the case of fields interacting in a definite way. This problem is solved in Sec. 3. In Sec. 4 we consider the quantization procedure on the basis of the solution of the Cauchy problem.

2. SOLUTION OF THE CAUCHY PROBLEM FOR EQUATION (2)

Let us find the solution to (2) which satisfies the initial conditions

$$\varphi|_{t=0} = a(x), \quad \varphi_t|_{t=0} = b(x). \tag{6}$$

The hyperbolic condition for (2) implies for the initial conditions that

$$1 + a'^2(x) - b^2(x) > 0. \tag{7}$$

Let us simplify (2) by introducing the new variables α, β:

$$x = x(\alpha, \beta), \quad t = t(\alpha, \beta), \quad z = \varphi(x(\alpha, \beta), t(\alpha, \beta)) = z(\alpha, \beta).$$

Thus we seek a solution of (2) in parametric form: $\mathbf{r} = \mathbf{r}(\alpha, \beta)$, where $\mathbf{r}$ is a vector with components t, x, and z.

If we denote the scalar product of the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ by $(\mathbf{r}_1 \cdot \mathbf{r}_2)$,

$$(\mathbf{r}_1\mathbf{r}_2) = t_1t_2 - x_1x_2 - z_1z_2,$$

then Eq. (2) is written in the following form:[2, 6]

$$\mathbf{r}_\alpha^2 D_{\beta\beta} - 2(\mathbf{r}_\alpha\mathbf{r}_\beta)D_{\alpha\beta} + \mathbf{r}_\beta^2 D_{\alpha\alpha} = 0, \tag{8}$$

where

$$\mathbf{r}_\alpha = \frac{\partial \mathbf{r}}{\partial \alpha}, \quad \mathbf{r}_\beta = \frac{\partial \mathbf{r}}{\partial \beta}, \quad \mathbf{r}_{\alpha;\alpha} = \frac{\partial^2 \mathbf{r}}{\partial \alpha^2}, \quad \mathbf{r}_{\alpha,\beta} = \frac{\partial^2 \mathbf{r}}{\partial \alpha\, \partial \beta}$$

and

$$D_{ik} = \begin{vmatrix} t_{ik} & x_{ik} & z_{ik} \\ t_\alpha & x_\alpha & z_\alpha \\ t_\beta & x_\beta & z_\beta \end{vmatrix}.$$

Let us construct the hyperbolic solution of (8) which satisfies the initial conditions; by a known theorem[5] this solution is unique. The hyperbolic nature of (8) implies that

$$(\mathbf{r}_\alpha\mathbf{r}_\beta)^2 - \mathbf{r}_\alpha^2\mathbf{r}_\beta^2 > 0. \tag{9}$$

Equation (8) has the following simple equations for the characteristics:

$$\mathbf{r}_\alpha^2 = 0, \quad \mathbf{r}_\beta^2 = 0. \tag{10}$$

These "characteristic" equations together with the basic equation (8) can be regarded as a system of three equations for the three functions $t(\alpha, \beta)$, $x(\alpha, \beta)$, and $z(\alpha, \beta)$.

It follows from (8), (9), and (10) that

$$D_{\alpha,\beta} = 0. \tag{11}$$

Equation (11) describes in the general case a linear dependence between the rows of the determinant $D_{\alpha,\beta}$, i.e.,

$$\mathbf{r}_{\alpha,\beta} = A(\alpha, \beta)\mathbf{r}_\alpha + B(\alpha, \beta)\mathbf{r}_\beta. \tag{12}$$

Taking further into consideration that (10) and (12) are valid for all α, β, we have

$$\begin{aligned}(\mathbf{r}_{\alpha,\beta}\ \mathbf{r}_\alpha) &= (\mathbf{r}_\alpha\ \mathbf{r}_\beta)B = \frac{1}{2}\frac{\partial}{\partial\beta}\mathbf{r}_\alpha^2 = 0,\\ (\mathbf{r}_{\alpha\beta}\ \mathbf{r}_\beta) &= (\mathbf{r}_\alpha\ \mathbf{r}_\beta)A = \frac{1}{2}\frac{\partial}{\partial\alpha}\mathbf{r}_\beta^2 = 0.\end{aligned} \tag{13}$$

With $(\mathbf{r}_\alpha \cdot \mathbf{r}_\beta) \neq 0$ we conclude from (13) that $A = B = 0$. We obtain finally the system of equations

$$\mathbf{r}_\alpha^2 = 0, \quad \mathbf{r}_\beta^2 = 0, \quad \mathbf{r}_{\alpha,\beta} = 0 \tag{14}$$

or

$$t_\alpha^2 - x_\alpha^2 - z_\alpha^2 = 0, \quad t_\beta^2 - x_\beta^2 - z_\beta^2 = 0,$$
$$t_{\alpha,\beta} = 0, \quad x_{\alpha,\beta} = 0, \quad z_{\alpha,\beta} = 0.$$

The general solution of the last equation (14) is

$$\mathbf{r}(\alpha, \beta) = \mathbf{r}_1(\alpha) + \mathbf{r}_2(\beta), \tag{15}$$

where $\mathbf{r}_1(\alpha)$ and $\mathbf{r}_2(\beta)$ are arbitrary vector functions. The first two equations (14) have now the form

$$\mathbf{r}_1'^2(\alpha) = 0, \quad \mathbf{r}_2'^2(\beta) = 0. \tag{16}$$

Thus we have six unknown functions x_1, t_1, z_1, x_2, t_2, z_2, which must be determined from (16) and the initial conditions (6) written in the variables α and β.

The following circumstance allows us to simplify radically the parametric representation of the initial conditions (6). Namely, we easily note that the choice of the parameters α and β is determined by (14) up to a transformation $\alpha = A(\alpha')$, $\beta = B(\beta')$, where A and B are arbitrary functions with the only restriction $(\partial A/\partial\alpha)(\partial B/\partial\beta) \neq 0$. Therefore, if we express α and β as functions of x and t,

$$\alpha = f_1(x, t), \quad \beta = f_2(x, t)$$

and set

$$A(x) = f_1(x, 0), \quad B(x) = f_2(x, 0),$$

then the condition $t = 0$ is in the new variables α' and β' written as $\alpha' = \beta' = x$. Omitting the primes, we can without loss of generality assume that

$\alpha = \beta = x$ for $t = 0$. Thus the initial conditions (6) are written in the following form:

$$t(\alpha, \beta)|_{\alpha=\beta} = t_1(\alpha) + t_2(\alpha) = 0,$$
$$x(\alpha, \beta)|_{\alpha=\beta} = x_1(\alpha) + x_2(\alpha) = \alpha,$$
$$z(\alpha, \beta)|_{\alpha=\beta} = z_1(\alpha) + z_2(\alpha) = a(\alpha). \qquad (17)$$

Moreover, expressing φ_t through the derivatives of t, x, and z with respect to α and β, we have for $\alpha = \beta$

$$\left.\frac{\partial\varphi(x, t)}{\partial t}\right|_{\alpha=\beta} = \left.\frac{D(x, z;\ \alpha, \beta)}{D(x, t;\ \alpha, \beta)}\right|_{\alpha=\beta}$$

$$= \frac{\tilde{D}(x, z;\ \alpha)}{\tilde{D}(x, t;\ \alpha)} = b(\alpha), \qquad (18)$$

where

$$D(x, y;\ \alpha, \beta) = \begin{vmatrix} x_\alpha & y_\alpha \\ x_\beta & y_\beta \end{vmatrix},$$

$$\tilde{D}(x, y;\ \alpha) = \begin{vmatrix} x_1'(\alpha) & y_1'(\alpha) \\ x_2'(\alpha) & y_2'(\alpha) \end{vmatrix}.$$

Finally, using (14) for $\alpha = \beta$,

$$t_1'^2(\alpha) - x_1'^2(\alpha) - z_1'^2(\alpha) = 0,$$
$$t_2'^2(\alpha) - x_2'^2(\alpha) - z_2'^2(\alpha) = 0, \qquad (19)$$

we obtain the six equations (17), (18), and (19) for six functions.

If, using (15), we write the desired solution in the form

$$\mathbf{r}(\alpha, \beta) = {}^1/_2[\boldsymbol{\rho}(\alpha) + \boldsymbol{\rho}(\beta)] + {}^1/_2 \int_\alpha^\beta \boldsymbol{\pi}(\lambda)\, d\lambda, \qquad (20)$$

then the vector $\boldsymbol{\rho}$ is easily determined from (17):

$$\boldsymbol{\rho}(\alpha) = \{0, \alpha, a(\alpha)\}.$$

The vector $\pi(\lambda) = \{\pi_t(\lambda), \pi_x(\lambda), \pi_z(\lambda)\}$ is determined from (18) and (19):

$$\pi_t(\lambda) = \frac{1 + a'^2(\lambda)}{[1 + a'^2(\lambda) - b^2(\lambda)]^{1/2}},$$

$$\pi_x = \frac{-a'(\lambda)\, b(\lambda)}{[1 + a'^2(\lambda) - b^2(\lambda)]^{1/2}},$$

$$\pi_z = \frac{-b(\lambda)}{[1 + a'^2(\lambda) - b^2(\lambda)]^{1/2}}.$$

Thus the solution of the Cauchy problem for Eq. (2) has the following form:

$$t(\alpha, \beta) = \frac{\beta - \alpha}{2} + \frac{1}{2}\int_\alpha^\beta \left[\frac{1 + a'^2(\lambda)}{[1 + a'^2(\lambda) - b^2(\lambda)]^{1/2}} - 1\right] d\lambda,$$

$$x(\alpha, \beta) = \frac{\alpha + \beta}{2} + \frac{1}{2}\int_\alpha^\beta \frac{-a'(\lambda)\, b(\lambda)}{[1 + a'^2(\lambda) - b^2(\lambda)]^{1/2}}\, d\lambda,$$

$$z(\alpha, \beta) = \frac{a(\alpha) + a(\beta)}{2}$$
$$+ \frac{1}{2}\int_\alpha^\beta \frac{b(\lambda)}{[1 + a'^2(\lambda) - b^2(\lambda)]^{1/2}}\, d\lambda. \qquad (21)$$

It turns out that the functions π_t, π_x, and π_z have an interesting physical meaning. π_z is the canonical momentum of the field $\varphi(x, t)$ for $t = 0$. Indeed, it is easy to verify that

$$\pi_z = \pi(x, 0) = \left.\frac{\partial L}{\partial \varphi_t}\right|_{t=0} = \frac{b(x)}{[1 + a'^2(x) - b^2(x)]^{1/2}}$$

$$L = -\sqrt{1 + \varphi_x^2 - \varphi_t^2}. \qquad (22)$$

π_x describes the momentum density of the field $\varphi(x, t)$ for $t = 0$:

$$\pi_x = G(x, 0) = -\pi(x, 0)\, a'(x)$$
$$= \frac{-a'(x)\, b(x)}{[1 + a'^2(x) - b^2(x)]^{1/2}} \qquad (23)$$

Finally, π_t is the Hamiltonian density of the field $\varphi(x, t)$ for $t = 0$:

$$\pi_t = H(x, 0) = (\pi\varphi_t - L)_{t=0}$$

$$= \frac{1 + a'^2(x)}{[1 + a'^2(x) - b^2(x)]^{1/2}} = [(1 + a'^2)(1 + \pi^2)]^{1/2}. \quad (24)$$

The solution (21) leads to two particular solutions: $\varphi = u(x+t)$ and $\varphi = v(x-t)$, i.e., two traveling waves of arbitrary form. They are obtained from the general solution (21) by setting

$$b(x) = a'(x), \quad a(x) = u(x)$$

$$\text{or} \quad b(x) = -a'(x), \quad a(x) = v(x).$$

3. GENERALIZATION OF THE PROBLEM TO A SYSTEM OF n INTERACTING FIELDS

On the basis of the geometric interpretation of Eq. (2) and the action function (3), we can generalize our problem to a system of n nonlinear interacting fields allowing for an exact solution. To this end we consider the $(n + 2)$-dimensional pseudo-euclidean space $(t, x, z_1, z_2, \ldots, z_n)$ with the metric

$$ds^2 = dt^2 - dx^2 - \sum_{i=1}^n dz_i^2.$$

In this space, let a two-dimensional surface be given by the n equations

$$z_1 = \varphi_1(x, t);\quad z_2 = \varphi_2(x, t);\ \ldots;\ z_n = \varphi_n(x, t). \quad (25)$$

The area of this surface is given by the integral [cf. (4)]

$$S = -\iint \Big[\Big(1+\sum_{i=1} \varphi_{i,x}^2\Big)\Big(1-\sum_{i=1}^{n} \varphi_{i,t}^2\Big) + \Big(\sum_{i=1}^{n} \varphi_{i,x}\varphi_{i,t}\Big)^2\Big]^{1/2} dx\,dt, \quad (26)$$

where $\varphi_{i,x} = \partial\varphi_i/\partial x$ and $\varphi_{i,t} = \partial\varphi_i/\partial t$.

The quantity S can be interpreted as an action function for the system of n fields (25) with the Lagrangian density

$$L = -\Big[\Big(1+\sum_{i=1}^{n} \varphi_{i,x}^2\Big)\Big(1-\sum_{i=1}^{n} \varphi_{i,t}^2\Big) + \Big(\sum_{i=1}^{n} \varphi_{i,x}\varphi_{i,t}\Big)^2\Big]^{1/2}, \quad (27)$$

which goes over into the Lagrangian density (5) for n = 1.

The Euler equations for this system have the form

$$\Big(1-\sum_{i=1}^{n} \varphi_{i,t}^2\Big)\varphi_{j,xx} + 2\sum_{i=1}^{n} \varphi_{i,x}\varphi_{i,t}\varphi_{j,x,t} - \Big(1+\sum_{i=1}^{n} \varphi_{i,x}^2\Big)\varphi_{j,t,t} = 0. \quad (28)$$

The Cauchy problem for Eq. (28) can be solved exactly.

Let the initial conditions for t = 0 be

$$\varphi_j(x,t)\,|_{t=0} = a_j(x); \quad \varphi_{j,t}(x,t)\,|_{t=0} = b_j(x). \quad (29)$$

The system (28) is of hyperbolic type if[2]

$$\Big(1+\sum_{i=1}^{n} \varphi_{i,x}^2\Big)\Big(1-\sum_{i=1}^{n} \varphi_{i,t}^2\Big) + \Big(\sum \varphi_{i,x}\varphi_{i,t}\Big)^2 > 0.$$

The initial conditions (29) must satisfy the same relation.

As in the case of a single field, we go over to a parametric representation by introducing the new parameters α and β; then

$$t = t(\alpha,\beta), \quad x = x(\alpha,\beta), \quad z_i = z_i(\alpha,\beta).$$

The integral (26) which determines the area of the surface in the (n + 2)-dimensional space, or the action integral for the system of n fields, is written as

$$S = -\iint [(\mathbf{r}_\alpha\mathbf{r}_\beta)^2 - \mathbf{r}_\alpha^2\mathbf{r}_\beta^2]^{1/2}\, d\alpha\, d\beta, \quad (30)$$

where we have introduced the (n + 2)-dimensional vectors

$$\mathbf{r}(\alpha,\beta) = \{t(\alpha,\beta),\ x(\alpha,\beta),\ z_1(\alpha,\beta),\ z_2(\alpha,\beta),\ \ldots,\ z_n(\alpha,\beta)\},$$

$$\mathbf{r}_\alpha(\alpha,\beta) = \{t_\alpha(\alpha,\beta), x_\alpha(\alpha,\beta), z_{1,\alpha}(\alpha,\beta), \ldots, z_{n,\alpha}(\alpha,\beta)\}$$

etc. The scalar product in the pseudo-euclidean space is given in correspondence with the metric form. Equation (30) leads to the Lagrangian $L = -[(\mathbf{r}_\alpha\mathbf{r}_\beta)^2 - \mathbf{r}_\alpha^2\mathbf{r}_\beta^2]^{1/2}$ and the Euler equations in the new variables,

$$\mathbf{r}_\alpha^2\mathbf{r}_{\beta,\beta} - 2(\mathbf{r}_\alpha\mathbf{r}_\beta)\mathbf{r}_{\alpha,\beta} + \mathbf{r}_\beta^2\mathbf{r}_{\alpha,\alpha} - N\mathbf{L}_\alpha - M\mathbf{L}_\beta = 0, \quad (31)$$

where

$$N = L^{-2}(\mathbf{D}\mathbf{r}_\alpha), \quad M = L^{-2}(\mathbf{D}\mathbf{r}_\beta),$$
$$\mathbf{D} = \mathbf{r}_\alpha^2\mathbf{r}_{\beta\beta} - 2(\mathbf{r}_\alpha\mathbf{r}_\beta)\mathbf{r}_{\alpha,\beta} + \mathbf{r}_\beta^2\mathbf{r}_{\alpha,\alpha},$$
$$\mathbf{L}_\alpha = (\mathbf{r}_\alpha\mathbf{r}_\beta)\mathbf{r}_\beta - \mathbf{r}_\beta^2\mathbf{r}_\alpha, \quad \mathbf{L}_\beta = (\mathbf{r}_\alpha\mathbf{r}_\beta)\mathbf{r}_\alpha - \mathbf{r}_\alpha^2\mathbf{r}_\beta$$

The system (31) contains n + 2 equations of which only n are linearly independent. This is easy to see by projecting (31) on the vectors $\mathbf{r}_\alpha$ and $\mathbf{r}_\beta$. Using

$$(\mathbf{r}_\alpha\mathbf{L}_\alpha) = (\mathbf{r}_\beta\mathbf{L}_\beta) = L^2, \quad (\mathbf{r}_\alpha\mathbf{L}_\beta) = (\mathbf{r}_\beta\mathbf{L}_\beta) = 0,$$

we verify that these projections are identically equal to zero. The indefiniteness of the system (31) is connected with the arbitrariness in the choice of the parameters α and β. If we set $\alpha = t$ and $\beta = x$, we obtain the original equation (28).

Let us choose the parameters α and β such that the equations of the characteristics on the surface (25) are satisfied. These equations have the same form as in the three-dimensional case (10):

$$\mathbf{r}_\alpha^2(\alpha,\beta) = 0, \quad \mathbf{r}_\beta^2(\alpha,\beta) = 0. \quad (32)$$

With (32) the system (31) reduces to the form [for details see [7]]

$$\mathbf{r}_{\alpha,\beta} = 0. \quad (33)$$

As a result we will have a system of equations (32) and (33) which agrees in form with (14), except that the vectors $\mathbf{r}_\alpha$ and $\mathbf{r}_\beta$ are n-dimensional. The general solution of (33) will again be

$$\mathbf{r}(\alpha,\beta) = \mathbf{r}_1(\alpha) + \mathbf{r}_2(\beta). \quad (34)$$

We have 2(n + 2) unknown functions $r_{1,i}(\alpha)$, $r_{2,i}(\beta)$ which are to be determined from the two equations (32) and the initial conditions (29) formulated in the variables α and β. As was already shown, the condition t = 0 can be expressed in the new variables as $\alpha = \beta = x$, which gives another two equations for the determination of the components of the vectors $\mathbf{r}_1(\alpha)$ and $\mathbf{r}_2(\beta)$. The initial conditions (29) are written as

$$t(\alpha,\beta)\,|_{\alpha=\beta} = 0, \quad x(\alpha,\beta)\,|_{\alpha=\beta} = \alpha,$$

$$z_i(\alpha,\beta)\,|_{\alpha=\beta} = a_i(\alpha),$$

$$\begin{vmatrix} z_{i,\alpha} & x_\alpha \\ z_{i,\beta} & x_\beta \end{vmatrix} = b_i(\alpha)\begin{vmatrix} t_\alpha & x_\alpha \\ t_\beta & x_\beta \end{vmatrix}. \quad (35)$$

If the desired solution is again written in the form,

$$\mathbf{r}(\alpha, \beta)=\frac{1}{2}[\rho(\alpha)+\rho(\beta)]+\frac{1}{2}\int_{\alpha}^{\beta}\pi(\lambda)\,d\lambda,$$

then the vectors ρ and π are easily determined from (32) and (35). As a result we have

$$\rho(\alpha)=\{0, \alpha, a_1(\alpha), a_2(\alpha), \dots, a_n(\alpha)\},$$

$$\pi_t=-\left[1+\sum_{i=1}^{n} a_i'^2(\alpha)\right]L^{-1}=H(\alpha),$$

$$\pi_x=\sum_{i=1}^{n} a_i'(\alpha)\,b_i(\alpha)L^{-1}=G(\alpha),$$

$$\pi_j=-\left[b_j(\alpha)\left(1+\sum_{i=1}^{n} a_i'^2(\alpha)\right)\right.$$

$$\left.-a_j'(\alpha)-\left(\sum_{i=1}^{n} a_i'(\alpha)\,b_i(\alpha)\right)\right]L^{-1}. \qquad (36)$$

As in the case of a single field, the functions π_t, π_x, π_j have the meaning of the Hamiltonian density H of the system (25) at the initial moment, the momentum density G of the fields, and the canonical momenta of this system at the initial moment, respectively. Therefore the solution of the Cauchy problem for the equation (28) can be expressed in the form

$$t(\alpha, \beta)=\frac{\beta-\alpha}{2}+\frac{1}{2}\int_{\alpha}^{\beta}[H(\lambda)-1]\,d\lambda,$$

$$x(\alpha, \beta)=\frac{\beta+\alpha}{2}+\frac{1}{2}\int_{\alpha} G(\lambda)\,d\lambda,$$

$$z_i(R, \beta)=\frac{a_i(\alpha)+a_i(\beta)}{2}+\frac{1}{2}\int_{\alpha}^{\beta}\pi_i(\lambda)\,d\lambda. \qquad (37)$$

As above, we have particular solutions in the form of waves traveling in one direction if we set $b_i(x) = a_i'(x) = u_i'(x)$ or $b_i(x) = -a_i'(x) = -v'(x)$. In the first case we have

$$t(\alpha, \beta)=\frac{\beta-\alpha}{2}+\frac{1}{2}\int_{\alpha}^{\beta}\sum_{i=1}^{n} u_i'^2(\lambda)\,d\lambda,$$

$$x(\alpha, \beta)=\frac{\beta+\alpha}{2}-\frac{1}{2}\int_{\alpha}^{\beta}\sum_{i=1}^{n} u_i'^2(\lambda)\,d\lambda,$$

$$z_i(\alpha, \beta)=\frac{1}{2}[u_i(\alpha)+u_i(\beta)]+\frac{1}{2}\int_{\alpha}^{\beta}u_i'(\lambda)\,d\lambda=u_i(\beta). \qquad (38)$$

From the first two equations we obtain $t + x = \beta$, therefore $z_i = u_i(x+t)$. The second case is analogous.

For the equations considered here the general Cauchy problem can be solved by standard methods, viz., given an arbitrary space-like curve $t = f(\lambda)$, $x = g(\lambda)$, $z_i = h_i(\lambda)$ in parametric form, together with a tangent plane at each point which intersects the isotropic cone $dt^2 - dx^2 - \sum_{i=1}^{n} dz_i^2$ along two straight lines; find the integral surface of our equation. This problem was solved by the authors in [6].

In conclusion we indicate the limiting transition from our solution (21) to the linear case. To this end we recall that in (2) the derivatives of the field φ are expressed in absolute units, i.e., relative to some characteristic constant k (absolute scale of the field).

We introduce the constant k in the Lagrangian in the following way: $L = [1 + k^{-2}\varphi_x^2 - k^{-2}\varphi_t^2]^{1/2}$ and assume that φ_x^2, $\varphi_t^2 \ll k^2$. Then we have approximately $L \approx 1 + (\varphi_x^2 - \varphi_t^2)/2k^2$, and (2) goes over into the linear equation $\varphi_{xx} - \varphi_{tt} = 0$. In order to obtain our solution (21) in the limiting case of the linear theory, we must set a, b, z $\rightarrow$ a/k, b/k, z/k in (21) and take $k \rightarrow \infty$; then

$$t(\alpha, \beta)=\frac{\beta-\alpha}{2}$$

$$+\frac{1}{2}\int_{\alpha}^{\beta}\left[\frac{1+k^{-2}a'^2(\lambda)}{[1+k^{-2}(a'^2-b^2)]^{1/2}}-1\right]d\lambda \underset{k\to\infty}{\rightarrow} \frac{\beta-\alpha}{2},$$

$$x(\alpha, \beta)=\frac{\beta+\alpha}{2}$$

$$-\frac{1}{2}\frac{1}{k^2}\int_{\alpha}^{\beta}\frac{a'b}{[1+k^{-2}(a'^2-b^2)]^{1/2}}\,d\lambda \underset{k\to\infty}{\rightarrow} \frac{\beta+\alpha}{2},$$

$$\frac{1}{k}z(\alpha, \beta)=\frac{a(\alpha)+a(\beta)}{2k}$$

$$+\frac{1}{2k}\int_{\alpha}^{\beta}\frac{b}{[1+k^{-2}(a'^2-b^2)]^{1/2}}\,d\lambda \underset{k\to\infty}{\rightarrow} \frac{a(\alpha)+a(\beta)}{2k}$$

$$+\frac{1}{2k}\int_{\alpha}^{\beta} b\,d\lambda. \qquad (39)$$

It is seen from (39) that the transition from the variables x, t to the variables α and β means in the linear theory simply the transition to the isotropic coordinates α and β. This is also seen from (10), which are the conditions for the choice of the variables α and β; for $k \rightarrow \infty$ the z component drops out from the vectors $\mathbf{r}_\alpha$ and $\mathbf{r}_\beta$, and (10) goes over into

$$t_\alpha^2(\alpha,\beta) - x_\alpha^2(\alpha,\beta) = 0,$$
$$t_\beta^2(\alpha,\beta) - x_\beta^2(\alpha,\beta) = 0, \qquad (40)$$

which connects only the old and new independent variables. The general solution to (40) is $x - t = A(\alpha)$, $x + t = B(\beta)$. Thus (40) defines a transformation from t and x to arbitrary isotropic coordinates α and β. In our case $A(\alpha) = \alpha$, $B(\beta) = \beta$.

4. QUANTIZATION OF THE NONLINEAR FIELD WITH THE LAGRANGIAN $L = -[1 + \varphi_x^2 - \varphi_t^2]^{1/2}$

In the preceding section we have solved the Cauchy problem for the nonlinear Born-Infeld Lagrangian. Having this solution, i.e., having an expression for the field functions $\varphi_i(x, t)$ in terms of the momenta $\pi_i(x, 0)$ taken at the initial moment, we can give as usual the commutation relations between $\varphi_i(x, 0)$ and $\pi_i(x, 0)$ at $t = 0$ and thus determine the field operators $\varphi_i(x, t) = z_i(\alpha, \beta)$ for an arbitrary time, i.e., determine the operators in the Heisenberg representation. However, in this method of quantization we encounter an unexpected difficulty, which does not exist in the case of the linear equations.

Indeed, let us consider the solution (21) for the case of a single field, introducing for convenience the new variables $\xi = (\alpha + \beta)/2$, $\tau = (\beta - \alpha)/2$; we have

$$t(\xi,\tau) = \tau + \frac{1}{2}\int_{\xi-\tau}^{\xi+\tau} [H(\lambda) - 1]\, d\lambda,$$

$$x(\xi,\tau) = \xi + \frac{1}{2}\int_{\xi-\tau}^{\xi+\tau} G(\lambda)\, d\lambda,$$

$$z(\xi,\tau) = \frac{1}{2}[a(\xi+\tau) + a(\xi-\tau)] + \frac{1}{2}\int_{\xi-\tau}^{\xi+\tau} \pi(\lambda)\, d\lambda \qquad (41)$$

and the linear approximation (39), which is valid for field gradients small compared to the absolute scale of the field k. If now at the initial moment $t = 0$, i.e., in the variables α, β, for $\alpha = \beta$, $\tau = 0$, we prescribe the commutation relations

$$[z(\xi, 0), \pi(\xi', 0)] = \delta(\xi - \xi'), \qquad (42)$$

then it follows immediately from the solution (41) that the time $t(\xi, \tau)$ and the coordinate $x(\xi, \tau)$ become operators which for $\tau \neq 0$ do not commute among themselves and with the field $z(\xi, \tau)$, since $G(\xi)$ and $H(\xi)$ are expressed through the field functions according to (23) and (24). This does not happen in the linear case (39), since there t and x are expressed through ξ and τ without participation of the field functions. Therefore, by postulating the commutation relation (42) for the initial time $\tau = 0$ we obtain inescapably that $t(\xi, \tau)$, $x(\xi, \tau)$ and $z(\xi, \tau)$ become Heisenberg operators; we thus are faced with the problem of formulating the theory in quantized space-time.

In the quantization of our nonlinear system we take the point of view developed in Secs. 2 and 3. That is, instead of one field $\varphi(x, t)$ with the Lagrangian $L = -\sqrt{1 + \varphi_x^2 - \varphi_t^2}$ we consider the system of three fields $t(\xi, \tau)$, $x(\xi, \tau)$, $z(\xi, \tau)$ with the Lagrangian

$$L = {}^1\!/_2 (r_\tau^2 - r_\xi^2);$$

$$\mathbf{r}_\tau = \{t_\tau, x_\tau, z_\tau\}, \quad \mathbf{r}_\xi = \{t_\xi, x_\xi, z_\xi\},$$

which leads to the linear equations

$$\mathbf{r}_{\tau\tau} - \mathbf{r}_{\xi\xi} = 0 \qquad (43)$$

with the auxiliary nonlinear conditions

$$\mathbf{r}_\alpha^2 = 0, \quad \mathbf{r}_\beta^2 = 0 \quad \text{or} \quad \mathbf{r}_\xi^2 + \mathbf{r}_\tau^2 = 0, \quad (\mathbf{r}_\xi \mathbf{r}_\tau) = 0. \qquad (44)$$

Condition (44) can be re-expressed as the vanishing of the energy-momentum tensor of the system of three fields:

$$T_{11} = T_{22} = H = r_\xi^2 + r_\tau^2 = 0,$$

$$T_{21} = T_{12} = G = (\mathbf{r}_\xi \mathbf{r}_\tau) = 0. \qquad (45)$$

It was shown in Secs. 2 and 3 that such a linear system of three fields with the nonlinear auxiliary conditions (45) is equivalent to the original nonlinear field $\varphi(x, t)$. It is easy to see (cf., for example, [6]) that if conditions (45) are satisfied at the initial time $t = 0$, $\tau = 0$, then they will also be satisfied at an arbitrary time τ.

We shall quantize the linear system (43) in the usual fashion by requiring, on the basis of the correspondence principle, that the auxiliary conditions (45) be satisfied on the average. Thus we obtain restrictions on the admissible state vectors of our system in analogy to the requirement in quantum electrodynamics that the Lorentz condition be satisfied on the average with respect to the admissible state vectors.[8] The difference consists in that here the auxiliary conditions are nonlinear.

Since our three-component field fulfils the d'Alembert equation (43), the Fourier expansion of this field has the form

$$\mathbf{r}(\xi,\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\omega}} [\mathbf{a}^{(+)}(k)\, e^{-i(k\xi - \omega\tau)} + \mathbf{a}^{(-)}(k)\, e^{i(k\xi - \omega\tau)}], \qquad (46)$$

where $\omega = |k|$. For the following it is convenient again to go over to the isotropic coordinates $\alpha = \xi - \tau$, $\beta = \xi + \tau$ and to write, according to (20), the vector $\mathbf{r}$ in the form

$$\mathbf{r}(\alpha, \beta) = \mathbf{r}_1(\alpha) + \mathbf{r}_2(\beta),$$

where

$$\mathbf{r}_1(\alpha) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dk}{\sqrt{2k}} [\mathbf{a}^{(+)}(k) e^{-ik\alpha} + \mathbf{a}^{(-)}(k) e^{ik\alpha}],$$

$$\mathbf{r}_2(\beta) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dk}{\sqrt{2k}} [\mathbf{a}^{(+)}(k) e^{ik\beta} + \mathbf{a}^{(-)}(k) e^{-ik\beta}]. \quad (47)$$

As usual, we postulate

$$i\partial \mathbf{r} / \partial \tau = [\mathbf{r}, H];$$

since

$$H = H_t - H_x - H_z,$$

we have

$$i\partial t / \partial \tau = [t, H_t], \quad i\partial x / \partial \tau = -[x, H_x],$$
$$i\partial z / \partial \tau = -[z, H_z]. \quad (48)$$

Using (46), we can satisfy (48) by setting

$$[a_t^{(-)}(k), a_t^{(+)}(k')] = \delta(k - k'),$$
$$[a_x^{(-)}(k), a_x^{(+)}(k')] = -\delta(k - k'),$$
$$[a_z^{(-)}(k), a_z^{(+)}(k')] = -\delta(k - k'). \quad (49)$$

Here we meet a situation which also occurs in the quantization of the electromagnetic field, i.e., the commutator for the field component $t(\xi, \tau)$ has the opposite sign of the commutators of the other components $x(\xi, \tau)$ and $z(\xi, \tau)$. The reason for this lies in the indefiniteness of the metric in the space of the fields t, x, z. In treating the operators $a^{(+)}$ and $a^{(-)}$ such as to avoid the known contradiction,[5] we must regard $a_t^{(+)}$ and $a_t^{(-)}$ as creation and annihilation operators for the field quanta t, and the operators $a_x^{(+)}$, $a_z^{(+)}$ and $a_x^{(-)}$, $a_z^{(-)}$ as annihilation and creation operators for the field quanta x and z.

We must further find the admissible state vectors of our system, with respect to which the auxiliary conditions (45) are satisfied on the average at any point of the space-time ξ, τ. It is sufficient to require that these conditions be satisfied at the initial time $\tau = 0$, i.e.,

$$\langle A' | : T_{\mu,\nu}(\xi, 0) : | A \rangle = 0, \quad (50)$$

where the colons denote the normal operator product. This condition is equivalent to the following two conditions:

$$\langle A' | : \mathbf{r}_1'^2(\xi) : | A \rangle = 0, \quad \langle A' | : \mathbf{r}_2'^2(\xi) : | A \rangle = 0.$$

The latter are conveniently written for the Fourier transforms of the operators $\mathbf{r}_1'^2(\xi)$ and $\mathbf{r}_2'^2(\xi)$:

$$\langle A' | : b_1(p) : | A \rangle = 0, \quad \langle A' | : b_2(p) : | A \rangle = 0, \quad (51)$$

where

$$b_1(p) = \int_{-\infty}^{\infty} \mathbf{r}_1'^2(\xi) e^{ip\xi} d\xi, \quad b_2(p) = \int_{-\infty}^{\infty} \mathbf{r}_2'^2(\xi) e^{ip\xi} d\xi. \quad (52)$$

Let us consider the operators $b_1(p)$ and $b_2(p)$ in more detail. It is easy to see that

$$b_1(-p) = b_1^+(p), \quad b_2(-p) = b_2^+(p). \quad (53)$$

Thus it suffices to calculate the integrals (52) for positive values of p. We have for $p > 0$

$$b_1^+(p) = \frac{1}{4} \int_p^\infty \sqrt{q^2 - p^2}\, \mathbf{a}^{(+)}\left(\frac{q-p}{2}\right) \mathbf{a}^{(-)}\left(\frac{p+q}{2}\right) dq$$
$$- \frac{1}{4} \int_0^p \sqrt{p^2 - q^2}\, \mathbf{a}^{(-)}\left(\frac{p+q}{2}\right) \mathbf{a}^{(-)}\left(\frac{p-q}{2}\right) dq,$$

$$b_2(p) = \frac{1}{4} \int_p^\infty \sqrt{q^2 - p^2}\, \mathbf{a}^{(+)}\left(\frac{p-q}{2}\right) \mathbf{a}^{(-)}\left(-\frac{p+q}{2}\right) dq$$
$$- \frac{1}{4} \int_0^p \sqrt{p^2 - q^2}\, \mathbf{a}^{(-)}\left(-\frac{p+q}{2}\right) \mathbf{a}^{(-)}\left(\frac{q-p}{2}\right) dq. \quad (54)$$

Further, $b_1(p) = :b_1(p)$ and $b_2(p) = :b_2(p):$ if $p \neq 0$. Moreover, we have

$$: b_1(0) : = \int_0^\infty k : \mathbf{a}^{(+)}(k) \mathbf{a}^{(-)}(k) : dk,$$

$$: b_2(0) : = \int_0^\infty k : \mathbf{a}^{(+)}(-k) \mathbf{a}^{(-)}(-k) : dk. \quad (55)$$

We note that the energy and momentum operators are

$$\int_{-\infty}^{\infty} d\xi : H(\xi) : = \int_{-\infty}^{\infty} |k| \mathbf{a}^{(+)}(k) \mathbf{a}^{(-)}(k) dk$$
$$= : b_1(0) : + : b_2(0) : ,$$

$$\int_{-\infty}^{\infty} G(\xi) d\xi = \int_{-\infty}^{\infty} k\, \mathbf{a}^{(+)}(k) \mathbf{a}^{(-)}(k) dk$$
$$= : b_1(0) : - : b_2(0) : . \quad (56)$$

Finally, the operators $b_1(p)$ and $b_2(p)$ satisfy the following commutation relations:

$$[b_1(p), b_1(q)] = (p - q) b_1(p + q),$$
$$[b_2(p), b_2(q)] = (p - q) b_2(p + q),$$
$$[b_1(p), b_2(q)] = 0, \quad (57)$$

which form a Lie algebra.

To satisfy the equations (51) it is sufficient that the admissible state vectors fulfil the condition

$$b_1(0) | A \rangle = 0, \quad b_2(0) | A \rangle = 0. \quad (58)$$

The situation is analogous to quantum electrodynamics, where it suffices for the fulfillment of the Lorentz condition on the average that

$$\frac{\partial A_\mu^{(-)}(x)}{\partial x_\mu}|A\rangle = 0.$$

By the meaning of the operators $b_1(0)$ and $b_2(0)$ [cf. (56)] the equations (58) determine the eigenstates of the total energy-momentum tensor with eigenvalues equal to zero.

Let us determine the vectors $|A\rangle$ as superpositions of orthonormal vectors $|m_t, m_x, m_z\rangle$ with a definite number of photons of all kinds m_t, m_x, m_z and with zero total energy. Since the operators b_1 and b_2 commute, we can write the state vector satisfying (58) as the product of two vectors, $|A\rangle = |A_1\rangle |A_2\rangle$, where $|A_1\rangle$ satisfies the first equation (58) and $|A_2\rangle$ the second. Thus

$$|A_1\rangle = \sum_{m_t, m_x, m_z} \int_0^\infty \int \ldots \int_0^\infty \prod_{i=1}^{m_t} dk_{t,i} \prod_{j=1}^{m_x} dk_{x,j} \prod_{l=1}^{m} dk_{z,l}$$
$$\times \delta\left(\sum_{i=1}^{m_t} k_{t,i} - \sum_{j=0}^{m_x} k_{x,j} - \sum_{l=0}^{m_z} k_{z,l}\right) \prod_{i=1}^{m_t} a_t^{(+)}(k_{t,i})$$
$$\times \prod_{j=0}^{m_x} a_x^{(-)}(k_{x,j}) \prod_{l=0}^{m_z} a_z^{(-)}(k_{z,l}) |0\rangle. \quad (59)$$

The vector $|A_2\rangle$ is constructed analogously, except that the integration over all dk_i is taken from $-\infty$ to zero. It is easy to see that the operator

$$b_1(0) = \int_0^\infty k a^{(+)}(k) a^{(-)}(k)\, dk$$

reduces the vector (59) to zero. Correspondingly, $b_2(0)$ reduces the vector $|A_2\rangle$ to zero.

Let us show now that the equations (51) are fulfilled for these vectors. It is easy to show that for $p > 0$ the operator

$$c(p) = \frac{1}{4}\int_p^\infty \sqrt{q^2 - p^2}\, a^{(+)}\left(\frac{q-p}{2}\right) a^{(-)}\left(\frac{p+q}{2}\right) dq,$$

when acting on the state vector

$$\langle 0| \prod_{s=1}^{l} a^{(-)}(r_s) = \Omega,$$

transforms it to the vector

$$\langle 0| \sum_{j=1}^{l} \sqrt{r_j(p + r_j)} \prod_{s=1}^{l} a(r_s + p\delta_{s,j}) = \Omega'.$$

Thus the sum of the arguments $\sum_{s=1}^{l} r_s$ of the operators $a(r_s)$ in Ω differs from the corresponding sum of the arguments in Ω'. Owing to this circumstance the matrix elements of the operator $c(p)$ between the states (59) vanish. By the same considerations the corresponding matrix elements of the operator

$$d(p) = \frac{1}{4}\int_0^p \sqrt{p^2 - q^2}\, a^{(-)}\left(\frac{p+q}{2}\right) a^{(-)}\left(\frac{p-q}{2}\right) dq.$$

also vanish. Hence the matrix elements (51) are equal to zero, as was to be shown.

The problems connected with the quantum effects in this nonlinear system will be considered in a subsequent paper.

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