SOVIET PHYSICS JETP VOLUME 23, NUMBER 4 OCTOBER, 1966

BOUNDARY CONDITIONS AND SURFACE SUPERCONDUCTIVITY

R. O. ZAĬTSEV

Submitted to JETP editor November 6, 1966

J. Exptl. Theoret. Phys. (U.S:S.R.) 50, 1055-1063 (April, 1966)

The boundary conditions for the Ginzburg-Landau-Gor'kov [4,8,9] equations on the interface between two superconductors, and also between a superconducting and normal metal, are found. In particular, the limiting cases of pure or very impure metals are considered. Boundary conditions for an arbitrary impurity concentration can also be derived by means of a simple generalization. The quasi-classical trajectory method employed in the paper can be used for determining the boundary condition for a plane or a rough surface. The critical field for a system consisting of a superconducting and normal metal is calculated; it slightly exceeds the second critical field H_{c2} of a bulk superconductor.

1. INTRODUCTION

THE boundary conditions on a surface between two superconductors, and also between a superconductor and a normal metal, were obtained in [1] under certain simplifying assumptions. It was assumed there that the metals differ only in the magnitude of the interaction (g) between the electrons, and that the boundary is a plane. For a quantitative comparison between theory and experiment it is necessary to take into account the difference in the electron velocities on the Fermi surface (v_1 and v_2), the difference of the mean free paths l_1 and l_2 if the metals contain impurities, and also the reflection of the electrons from the boundary (specular for a plane boundary and diffuse for a rough boundary). We consider in this paper the case when the metals are not separated at all, that is, there is no potential barrier at all on the boundary (direct contact), and the coefficient of transmission of the electrons through the boundary is arbitrary.

The boundary conditions obtained are expressed in terms of the quantities used in the phenomenological theory of Ginzburg and Landau. At the end of the paper we calculate the third critical field [2,3], the existence of which is directly connected with the boundary conditions.

2. FUNDAMENTAL EQUATION

Let a superconducting metal I occupy the half-space $z > 0$, and metal II the half-space $z < 0$. We assume that the temperature T is close to the critical temperature of the metal I. The critical temperature of the second metal is lower than that of the first, so that the latter can be either superconducting or normal.

As shown by Gor'kov[1)] (see also [1]), to obtain the boundary conditions it is sufficient to confine oneself to an approximation linear in Δ, which is valid at distances $|z| \ll \xi_0/[T_c - T)/T_c]^{1/2}$ from the boundary[2)]:

$$\Delta(z) = -g(z)T\sum_{\omega}\int_{-\infty}^{+\infty} G_\omega{}^0(\mathbf{r}', \mathbf{r})\Delta(z')G_{-\omega}{}^0(\mathbf{r}', \mathbf{r})\,d\mathbf{r}'. \quad (1)$$

Here $G_\omega^0(\mathbf{r}, \mathbf{r}')$ is the Green's function of the system without allowance for the interaction between the electrons; this function satisfies the equation

$$[i\omega + \nabla^2/2m + \mu - \varphi(\mathbf{r})]G_\omega{}^0(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'), \quad (2)$$

where μ is the chemical potential of the system and $\varphi(\mathbf{r})$ is the potential.

Near the boundary, the potential φ varies over distances of the order of the interatomic spacing, with

$$\varphi(\mathbf{r}) \to \mu - mv_1^2/2 \quad \text{as} \quad z \to +\infty,$$
$$\varphi(\mathbf{r}) \to \mu - mv_2^2/2 \quad \text{as} \quad z \to -\infty.$$

It will be shown below that terms of the order $[(T_c - T)/T_c]^{1/2}$, $(T_c - T)/T_c$, and higher arise in Eq. (1), whereas terms of the next approximations in Δ are of the order of $[(T_c - T)/T_c]^{3/2}$. To determine the boundary conditions it is therefore sufficient to solve Eq. (1) and to discard quantities of order higher than $(T_c - T)/T_c$.

1)The author is grateful to L. P. Gor'kov for reporting the results of his unpublished paper.

2)To obtain Eq. (1) from Gor'kov's equations [4] it is necessary to assume that on the boundary the functions $G_\omega(\mathbf{r}, \mathbf{r}')$ and $F_\omega{}^+(\mathbf{r}, \mathbf{r}')$ are continuous together with their normal derivatives. This requirement is certainly satisfied when the metals are in direct contact.

3. BOUNDARY CONDITIONS FOR PURE METALS

To calculate the kernel of the integral equation (1)

$$K_\omega(z, z') = \int_{-\infty}^{+\infty} G_\omega^0(\mathbf{r}', \mathbf{r}) G_{-\omega}^0(\mathbf{r}', \mathbf{r})\, dx'\, dy'$$

we use the method of quasiclassical trajectories[5,6], which is applicable to our problem if there is no potential barrier on the boundary. Carrying out the usual transformations, we obtain for $z > 0$

$$K_\omega(z, z') = 2\pi\zeta_1 \left\langle \int_0^\infty e^{-2|\omega|t}\, \delta[z' - z(t)]\, dt \right\rangle, \quad (3)$$

where ζ_1 is the level density on the Fermi surface of the first metal, $z(t)$ is the equation of the electron trajectory along the z axis with initial conditions $z(0) = z$ and $z'(0) = v_{1z}$; the average is taken over all the directions of the initial velocity of the electron on the Fermi surface of the first metal.

From this we obtain immediately the kernel K_ω for the case of a plane boundary. When $z > 0$ and $z' > 0$ we have

$$K_\omega(z, z') = K_\omega^{11}(z, z') = \left(\frac{m}{2\pi}\right)^2 \int_{-\infty}^{+\infty} \frac{dv_x\, dv_y}{v_{1z}^2} \times \left[\exp\left(-\frac{2|\omega||z - z'|}{v_{1z}}\right) + R(\mathbf{v}) \exp\left(-\frac{2|\omega|(z + z')}{v_{1z}}\right)\right]; \quad (4a)$$

when $z > 0$ and $z' < 0$ we get

$$K_\omega(z, z') = K_\omega^{12}(z, z') = \left(\frac{m}{2\pi}\right)^2 \int_{-\infty}^{\infty} \frac{dv_x\, dv_y}{v_{1z} v_{2z}} \times D(\mathbf{v}) \exp\left(-\frac{2|\omega| z}{v_{1z}} + \frac{2|\omega| z'}{v_{2z}}\right). \quad (4b)$$

Here R and D are the coefficients of reflection and transmission of an electron with initial velocity v_{1z}. The kernels K_ω^{22} and K_ω^{21} are obtained from K_ω^{11} and K_ω^{12} by making the substitution

$$z \to -z, \quad z' \to -z', \quad 1 \rightleftarrows 2.$$

If the boundary is diffuse, then the distribution function for the reflected electrons[5] takes the form

$$R \cos\theta_1 d\mathbf{n}_1 / \pi \quad (5a)$$

and for "refracted" electrons

$$D p_2 |\cos\theta_2|\, d\mathbf{n}_2 / \pi p_1. \quad (5b)$$

Here θ is the angle between the direction of the electron and the normal to the surface; R and D are the average reflection and refraction coefficients; the factor p_2/p_1 can be obtained from the obvious condition $K_{12}(z, z') = K_{21}(z', z)$.

Carrying out the averaging with the aid of the distribution functions (5), we obtain the kernel $K_\omega(z, z')$:

$$K_\omega^{11}(z, z') = \frac{m^2}{2\pi}\left\{\int_0^1 \frac{ds}{s} \exp\left(-\frac{2|\omega||z - z'|}{s v_1}\right) + 2R \int_0^1\!\!\int \exp\left(-\frac{2|\omega| z}{s v_1} - \frac{2|\omega| z'}{t v_1}\right) ds\, dt\right\},$$

$$K_\omega^{12}(z, z') = \frac{m^2 D}{\pi} \int_0^1\!\!\int \exp\left(-\frac{2|\omega| z}{s v_1} + \frac{2|\omega| z'}{t v_2}\right) ds\, dt. \quad (6)$$

We rewrite Eq. (1) in a more convenient form:

$$\Delta(z) = -g_1 T \sum_\omega \int_0^\infty K_\omega^{11}(z, z') \Delta(z')\, dz' - g_1 T \sum_\omega \int_{-\infty}^0 K_\omega^{12}(z, z') \Delta(z')\, dz' \quad (7a)$$

for $z > 0$ and

$$\Delta(z) = -g_2 T \sum_\omega \int_0^\infty K_\omega^{21}(z, z') \Delta(z')\, dz' - g_2 T \sum_\omega \int_{-\infty}^0 K_\omega^{22}(z, z') \Delta(z')\, dz' \quad (7b)$$

for $z < 0$.

We solve the equations in (7) by a variational method[7], choosing as trial functions the asymptotic expansions of the exact solution (7) for $R = 0$ and $K_{21} = K_{12} = K_{11} = K_{22}$ (see[1]).

Let us assume that the second metal has a transition temperature equal to zero, i.e., $g_2 = 0$. We seek the solution of (7) in the form

$$\Delta(z) = \beta\xi_1 + z \quad (z > 0); \quad \Delta(z) = 0 \quad (z < 0). \quad (8)$$

Substituting (8) into the system (7) and integrating with respect to z, we readily obtain the value of β (for diffuse reflection):

$$\beta = 14\zeta(3)(1 + R) / 3\pi^2 D. \quad (9)$$

In the case of a plane boundary it is necessary to make in (9) the substitutions

$$R \to \frac{3}{2\pi v_1^3} \int R(\mathbf{v}) v_{1z}\, dv_x\, dv_y, \quad D \to \int D(\mathbf{v}) \frac{dv_x\, dv_y}{\pi v_1^2}. \quad (10)$$

The quantity β enters in the boundary condition:

$$-\beta\xi_1(\mathbf{n}\nabla\Delta) = \Delta. \quad (11)$$

From (9) and (10) we see that when $D \to 0$ we get $\beta \to \infty$, so that the boundary condition (11) goes over into the "natural" Ginzburg-Landau condition for a superconductor bordering on vacuum or on an insulator. It is important to note that this result does not depend on the character of reflection of the electrons from the boundary (L. P. Gor'kov, private communication).

When $D = 1$ we obtain $\beta \approx 0.6$, which differs insignificantly from the exact value of β for this case ($\beta \approx 0.7$)[1].

Let us assume now that the temperature is close not only to the transition temperature of superconductor I, but also to the transition temperature of the second metal

$$\left|\frac{T_{c2}-T}{T_{c2}}\right| \ll 1, \quad \frac{T_{c1}-T}{T_{c1}} \ll 1.$$

Whether the second metal is normal ($T > T_{C2}$) or superconducting ($T < T_{C2}$) is immaterial in this case. The asymptotic expansions then take the form

$$\Delta(z) = A_1 z + B_1, \quad z > 0; \quad \Delta(z) = -A_2 z + B_2, \quad z < 0. \tag{12}$$

Substituting (12) in the system (7) we obtain the following boundary conditions. For a diffuse boundary

$$\zeta_1\Delta(0+) - \eta_1\zeta_1\xi_1\Delta'(0+) = \zeta_2\Delta(0-) + \eta_2\zeta_2\xi_2\Delta'(0-),$$
$$\frac{N_1}{\zeta_1}\Delta'(0+) = \frac{N_2}{\zeta_2}\Delta'(0-), \qquad \eta = \frac{14\zeta(3)R}{3\pi^2 D}; \tag{13}$$

for a plane boundary

$$\Delta(0+) - \eta_1\xi_1\Delta'(0+) = \Delta(0-) + \eta_2\xi_2\Delta'(0-),$$
$$N_1\Delta'(0+) = N_2\Delta'(0-),$$
$$\eta = 7\zeta(3)\int R(\mathbf{v})v_z\,dv_x\,dv_y \Big/ \pi^2 v_0 \int D(\mathbf{v})\,dv_x\,dv_y. \tag{14}$$

In (13) and (14) N is the electron density, $\Delta(0+)$ is the value of $\Delta(z)$ to the right of the boundary (on the side of metal I), $\Delta(0-)$—to the left (on the side of metal II); the notation for the derivative Δ' is similar.

Inasmuch as the function Δ varies slowly ($\Delta' \sim [(T_C - T)/T_C]^{1/2}\Delta$), for not too small a transmission coefficient

$$D \gg [(T_c - T)/T_c]^{1/2}$$

we can neglect the terms containing η under conditions (13) and (14). Then the boundary conditions simplify: for a plane boundary we have

$$\Delta(0+) = \Delta(0-), \quad N_1\Delta'(0+) = N_2\Delta'(0-); \tag{15}$$

and for a rough one

$$\zeta_1\Delta(0+) = \zeta_2\Delta(0-), \quad N_1\Delta'(0+)/\zeta_1 = N_2\Delta'(0-)/\zeta_2. \tag{16}$$

In the opposite limiting case $D \ll [(T_C - T)/T_C]^{1/2}$ we obtain the 'natural' boundary conditions:

$$\Delta'(0+) = \Delta'(0-) = 0.$$

4. BOUNDARY CONDITIONS FOR CONTAMINATED METALS

If the metals contain impurities, then Eq. (1) must be averaged over the positions of the impurity atoms[8,9]. For strongly contaminated metals ($l \ll \xi_0$) and for not too small a transmission coefficient $D \gg l(/\xi_0)$ the result does not depend on the magnitude of the latter, but depends on the form of the boundary.

It is shown in the Appendix that for a plane boundary the kernel K_ω coincides with the expression obtained by deGennes[10]:

$$K_{11}(z, z') = \frac{\pi\zeta_1\beta_1}{2|\omega|}\Big[\exp(-\beta_1|z-z'|) + \frac{\zeta_1\beta_2 - \zeta_2\beta_1}{\zeta_1\beta_2 + \zeta_2\beta_1}\exp(-\beta_1(z+z'))\Big],$$
$$K_{12}(z, z') = \frac{\pi\zeta_1\zeta_2\beta_1\beta_2}{|\omega|(\zeta_1\beta_2 + \zeta_2\beta_1)}\exp(-\beta_1 z + \beta_2 z'), \tag{17}$$

where $\beta(6|\omega|/lv_0)^{1/2}$ and l is the mean free path.

For a diffuse boundary, the result is somewhat different:

$$K_{11}(z, z') = \frac{\pi\zeta_1\beta_1}{2|\omega|}\Big[\exp(-\beta_1|z-z'|) + \frac{\zeta_2\beta_2 - \zeta_1\beta_1}{\zeta_2\beta_2 + \zeta_1\beta_1} \times \exp(-\beta_1(z+z'))\Big],$$
$$K_{12}(z, z') = \frac{\pi\zeta_1\zeta_2\beta_1\beta_2}{|\omega|(\zeta_1\beta_1 + \zeta_2\beta_2)}\exp(-\beta_1 z + \beta_2 z'). \tag{18}$$

The functions K_{11} and K_{12} coincide in form with the corresponding expressions for pure metals, so that as a result of analogous operations we obtain the following boundary conditions.

For $T_{C2} = 0$ the function Δ satisfies the condition

$$-\beta D_1(\mathbf{n}\nabla\Delta) = \Delta, \tag{19}$$

where $L_1^2 = l_1 v_2/6\pi T$, and the value of β for the plane boundary is equal to

$$\beta = \frac{\pi^2\zeta_1 D_1}{2\zeta(3/2)(4-\sqrt{2})\zeta_2 D_2} \tag{20}$$

and for a diffuse boundary

$$\beta = \frac{\pi^2\zeta_2 D_1}{2\zeta(3/2)(4-\sqrt{2})\zeta_1 D_2}. \tag{21}$$

When $T \sim T_{C1} \sim T_{C2}$ the continuity conditions are again valid

$$\Delta(0+) = \Delta(0-), \quad \zeta_1 D_1^2\Delta'(0+) = \zeta_2 D_2^2\Delta'(0-) \tag{22}$$

for a plane boundary and

$$\zeta_1\Delta(0+) = \zeta_2\Delta(0-), \quad D_1^2\Delta'(0+) = D_2^2\Delta'(0-) \quad (23)$$

for a diffuse boundary.

5. GENERALIZATION OF RESULTS

Comparing (15) with (22) and (16) with (23), we observe that at not too small a transmission coefficient, for a plane boundary, the function Δ turns out to be continuous regardless of the impurity concentration. If the boundary is diffuse, then the quantity $\zeta\Delta$ is continuous.

The condition for the derivative can be easily obtained from the requirement of continuity of the normal component of the current density [8]:

$$j_n = \left\{\frac{ie}{m}\left(\Delta\frac{\partial\Delta^*}{\partial z} - \Delta^*\frac{\partial\Delta}{\partial z}\right) - \frac{4e^2}{mc}A_n|\Delta|^2\right\}\chi(\rho)\frac{7\zeta(3)N}{16\pi^2T^2},$$

where

$$\chi(\rho) = \frac{8}{7\zeta(3)}\left(\frac{1}{\rho}\right)\left\{\frac{\pi^2}{8} + \frac{1}{2\rho}\left[\psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{2}+\rho\right)\right]\right\};$$
$$\rho = {}^1/_{2\pi T\tau_{tr}}, \quad \psi(x) = \Gamma'(x)/\Gamma(x).$$

($\chi(\rho) \to 1$ when $\rho \to 0$ and $\chi(\rho) \approx \pi^2/7\zeta(3)\rho$ when $\rho \to \infty$.) From this we find that for a plane boundary the quantities

$$\Delta, \quad N\chi(\rho)\left(\mathbf{n}\left(\nabla - \frac{2ie}{c}\mathbf{A}\right)\right)\Delta,$$

should be continuous and for a rough one,

$$\zeta\Delta, \quad \frac{N\chi(\rho)}{\zeta}\left(\mathbf{n}\left(\nabla - \frac{2ie}{c}\mathbf{A}\right)\right)\Delta.$$

Let us rewrite these conditions in terms of the variables of the phenomenological theory:

$$[N_1\chi(\rho_1)]^{-1/2}\psi(0+) = [N_2\chi(\rho_2)]^{-1/2}\psi(0-),$$

$$[N_1\chi(\rho_1)]^{1/2}\left(\mathbf{n}\left(\nabla - \frac{2ie}{c}\mathbf{A}\right)\right)\psi(0+)$$
$$= [N_2\chi(\rho_2)]^{1/2}\left(\mathbf{n}\left(\nabla - \frac{2ie}{c}\mathbf{A}\right)\right)\chi(0-), \quad (24)$$

if the boundary is plane, and

$$\zeta_1[N_1\chi(\rho_1)]^{-1/2}\psi(0+) = \zeta_2[N_2\chi(\rho_2)]^{-1/2}\psi(0-),$$

$$\frac{[N_1\chi(\rho_1)]^{1/2}}{\zeta_1}\left(\mathbf{n}, \nabla - \frac{2ie}{c}\mathbf{A}\right)\psi(0+)$$
$$= \frac{[N_2\chi(\rho_2)]^{1/2}}{\zeta_2}\left(\mathbf{n}\left(\nabla - \frac{2ie}{c}\mathbf{A}\right)\right)\psi(0-), \quad (25)$$

if the boundary is diffuse.

When the transmission coefficient is small, $D \ll [(T_c - T)/T_c]^{1/2}$, the Ginzburg-Landau conditions hold (see [3,11]):

$$\left(\mathbf{n}\left(\nabla - \frac{2ie}{c}\mathbf{A}\right)\right)\psi(0+) = \left(\mathbf{n}\left(\nabla - \frac{2ie}{c}\mathbf{A}\right)\right)\psi(0-) = 0.$$

When normal metals have a very low transition temperature ($T_{c2} \ll T \lesssim T_{c1}$), it is easy to write an expression generalizing (11) and (19):

$$-\tilde{\beta}\left(\frac{\lambda_\tau}{2m}\right)^{1/2}\left(\mathbf{n}\left(\nabla - \frac{2ie}{c}\mathbf{A}\right)\right)\Delta(0+) = \Delta(0+), \quad (26)$$

where

$$\lambda_\tau = \lambda_0\chi(\rho), \quad \lambda_0 = 7\zeta(3)p_0^2/24\pi^2T^2m.$$

The quantity $\tilde{\beta}$ is proportional to β: $\tilde{\beta} = C_l\beta$, where

$$C_l = \begin{cases}[12/7\zeta(3)]^{1/2} & \text{for} \quad l \to +\infty,\\ 2/\pi & \text{for} \quad l \ll \xi_0.\end{cases}$$

Let us rewrite conditions (26) in terms of the variables of the Ginzburg-Landau theory:

$$-\tilde{\beta}\frac{\delta(T)}{\varkappa}\left(\frac{T_c - T}{T_c}\right)^{1/2}\left(\mathbf{n}\left(\nabla - \frac{2ie}{c}\mathbf{A}\right)\right)\psi = \psi; \quad (27)$$

$\delta(T)$ is the depth of penetration at the given temperature, and is a constant of the phenomenological theory.

6. CRITICAL FIELD

It is well known that an increase of the critical field, for superconductors of finite dimensions (surface superconductivity), is connected with the fulfillment of the condition $\psi' = 0$ on the boundary. It is therefore of interest to calculate the critical field for the case when the superconductor borders on another metal, so that a condition similar to (27) is satisfied.

To calculate the critical field it is sufficient to solve the linear equation [2,3]

$$[(\mathbf{p} - 2iec^{-1}\mathbf{A})^2 + (\varkappa/\delta(T)^2]\psi = 0. \quad (28)$$

We take the potential $\mathbf{A}$ in the form $A_y = -H(z - z_0)$. The constant z_0 is chosen such that the function ψ is real and depends only on z. After making the substitution

$$z - z_0 \to (c/2eH)^{1/2}z$$

we obtain the following equation

$$\psi'' + (E - z^2)\psi = 0 \quad (29)$$

($E = H_{c2}/H$, H_{c2} is the second critical field). The boundary condition takes the form

$$(1 - T/T_c)^{1/2}E^{-1/2}\tilde{\beta}\psi'(-z_0) = \psi(-z_0). \quad (30)$$

It turns out that the minimum of the "energy" E, as a function of z_0, is attained when

$$z_0 \sim (1 - T/T_c)^{-1/2} \gg 1,$$

so that we can substitute in the condition (30) the asymptotic expansion of the solution of (29), which attenuates as $z \to +\infty$:

$$\psi = \exp(-z_0^2/2) - \pi^{1/2}\nu z_0^{-1}\exp(z_0^2/2),\quad \nu = (E-1)/2.$$

As a result we obtain the minimum value of E:

$$E_{min} = 1 - \frac{(1-T/T_c)^{1/2}\tilde\beta}{2\sqrt{\pi}}\exp\left[-1/\tilde\beta^2\left(1-\frac{T}{T_c}\right)\right].$$

Thus, under boundary conditions (27), the third critical field practically coincides with the second:

$$H_{c3} = H_{c2}\left\{1 + \frac{(1-T/T_c)^{1/2}\tilde\beta}{2\sqrt{\pi}}\exp\left[-1/\tilde\beta^2\left(1-\frac{T}{T_c}\right)\right]\right\}. \quad (31)$$

It is important that this result is valid if $\tilde\beta(1 - T/P_c)^{1/2} \ll 1$. In the opposite case we arrive at the condition $\psi' = 0$, so that, in accordance with the original paper of Saint-James and deGennes [2], $H_{c3} \approx 1.7 H_{c2}$.

In conclusion I am grateful to Professor B. T. Geĭlikman and A. I. Larkin for fruitful discussions and help with the work, and to E. G. Maksimov and A. I. Rusinov for a discussion of the results.

APPENDIX

Let us find the equation that must be satisfied by the quantity $K_\omega(z, z')$, averaged over the positions of the impurity atoms. We assume that the interaction with the impurities has a δ-function character, and after averaging in the coordinate representation we obtain the following integral equation [9,11]:

$$K_\omega(z,z') = K_\omega{}^0(z,z') + \int_{-\infty}^{+\infty} K_\omega{}^0(z,z'')n(z'')u^2(z'') \times K_\omega(z'',z')\,dz'', \quad \text{(A1)}$$

where $n(z)$ is the number of impurity atoms per unit volume, $u(z)$ is the "amplitude" of scattering by a δ-function potential; the quantity $K_\omega^0(z, z')$ is the product of the mean values of two Green's functions

$$K_\omega{}^0(z,z') = \int_{-\infty}^{+\infty} \langle G_\omega{}^0(\mathbf{r}',\mathbf{r})\rangle\langle G_{-\omega}{}^0(\mathbf{r}',\mathbf{r})\rangle\,dx'\,dy'.$$

We note further that the mean value of the Green's function $\langle G_\omega^0(\mathbf{r}, \mathbf{r}')\rangle$ satisfies the equation

$$[i\omega\eta(\mathbf{r}) + \nabla^2/2m + \mu - \varphi(\mathbf{r})]\langle G_\omega{}^0(\mathbf{r},\mathbf{r}')\rangle = \delta(\mathbf{r}-\mathbf{r}'),$$
$$\eta(\mathbf{r}) = \begin{cases} 1 + 1/2|\omega|\tau_1 & \text{for } z \gg 1/p_0 \\ 1 + 1/2|\omega|\tau_2 & \text{for } |z| \gg 1/p_0 \end{cases}. \quad \text{(A.2)}$$

Using this equation and carrying out the usual transformations [5,6] we find that for $z > 0$

$$K_\omega{}^0(z,z') = 2\pi\zeta_1\left\langle \int_0^\infty e^{-f(t)}\delta(z'-z(t))\,dt \right\rangle, \quad \text{(A.3)}$$

where

$$f(t) = \int_0^t 2|\omega|\eta[\mathbf{r}(\tau)]\,d\tau,$$

$z(t)$ is the equation of motion of electron without account of the interaction with the impurities, and the averaging is carried out over all the directions of the initial velocity on the Fermi surface of the first metal.

It follows therefore that the kernel $K_\omega^0(z, z')$ can be obtained from the corresponding expressions (4) and (6) for the pure metal with the aid of the substitutions

$$2|\omega|v_{1z} \to 2|\omega|\eta_1/v_{1z}, \quad 2|\omega|/v_{2z} \to 2|\omega|\eta_2/v_{2z}.$$

If the metals are highly contaminated, $l \ll \xi_0$, then far from the boundary equation (A.1) reduces to a differential equation (see [10]):

$$\left[2|\omega| - \frac{lv_0}{3}\frac{d^2}{dz^2}\right]K_\omega(z,z') = 2\pi\zeta\delta(z-z'), \quad \text{(A.4)}$$

since the kernel K_ω^0 attenuates over distances of the order of l, while the kernel K_ω varies at large distances like $\sim(l\xi_0)^{1/2}$.

Thus, the integral equation (A.1) also has the same relation to (A.4), as the Gor'kov equation has to the Ginzburg-Landau equations. It is therefore clear that we can use the method of Sec. 3 to find the boundary conditions in (A.4) (see also [7]). Using as the trial function the solution of the equation (A.4), replacing the inhomogeneous part in (A.1) by a δ-function, and integrating this equation with respect to z', we obtain the solution (17) and (18).

This procedure enables us to find the boundary conditions for the equation (A.4). These conditions are

$$\frac{U(0+,z')}{\zeta_1} = \frac{K(0-,z')}{\zeta_2}, \quad v_1l_1K'(0+,z') = v_2l_2K'(0-,z') \quad \text{(A.5)}$$

for a flat boundary [10] and

$$K(0+,z') = K(0-,z'), \quad l_1K'(0+,z') = l_2K'(0-,z') \quad \text{(A.6)}$$

for a rough boundary.

It is necessary, of course, to recall that the obtained conditions are effective and give the solution accurate to terms of order l/ξ_0.

Note added in proof (5 March 1966). We present without proof the boundary conditions which can be readily obtained if account is taken of the difference between the effective electron masses. For metals with nearly equal transition temperatures, the following quantities should be continuous: for a plane boundary.

$$\Delta \text{ and } \frac{N\chi(\rho)}{m}\left(\mathbf{n}\nabla - \frac{2ie}{c}\mathbf{A}\right)\Delta;$$

and for a rough boundary

$$p\Delta \text{ and } \frac{N\chi(\rho)}{\zeta}\left(\mathbf{n}\nabla - \frac{2ie}{c}\mathbf{A}\right)\Delta.$$

If the normal metal has a low transition temperature, then condition (26) is again satisfied, with m the effective electron mass of the superconductor.

The values of the constant β (9), (10), and (20) do not change. For contaminated metals with rough boundary, however,

$$\beta = \pi^2 v_2 D_1 / 2\zeta(3/2)(4 - \sqrt{2}) v_1 D_2.$$

[1] R. O. Zaĭtsev, JETP 48, 644, 1759 (1965), Soviet Phys. JETP 21, 1178 (1965).

[2] D. Saint-James and P. G. deGennes, Phys. Let. 7, 306 (1963).

[3] A. A. Abrikosov, JETP 47, 720 (1964), Soviet Phys. JETP 20, 480 (1965).

[4] L. P. Gor'kov JETP 36, 1918 (1959), Soviet Phys. JETP 9, 1364 (1959).

[5] E. A. Shapoval, JETP 47, 1007 (1964), Soviet Phys. JETP 20, 675 (1965).

[6] P. G. deGennes and M. Tinkham, Physics 1, 107 (1964).

[7] P. M. Morse and H. Feshbach, Methods of Theoretical Physics, McGraw-Hill, Chapter 12.

[8] L. P. Gor'kov, JETP 37, 1407 (1959), Soviet Phys. JETP 10, 998 (1960).

[9] A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, Metody kvantovoĭ teorii polya v statisticheskoĭ fizike (Methods of Quantum Field Theory in Statistical Physics), Fizmatgiz.

[10] P. G. deGennes, Revs. Modern Phys. 36, 225 (1964).

[11] M. R. Werthamer, Phys. Rev. 132, 2440 (1963).

Translated by J. G. Adashko
131