QUASILINEAR TRANSFORMATION OF WAVES IN AN INHOMOGENEOUS PLASMA

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Quasilinear equations are obtained for an inhomogeneous plasma in a magnetic field: these equations take account of convection of charge across the magnetic field. The equations are used to treat the quasilinear pumping of energy associated with electron plasma oscillations, excited in a plasma by an inhomogeneous electron beam, into low-frequency oscillations. It is shown that this phenomenon can lead to a substantial reduction of the energy of the plasma oscillations.

1. INTRODUCTION

If a given class of waves is excited in a plasma the energy of these waves can, by virtue of nonlinear interactions (for example decay), be transformed into energy associated with waves of another class. This phenomenon has been studied by many authors, one of the earliest of whom is Sturrock.[1] However, in addition to the interaction between waves, another mechanism for the transformation of wave energy can be associated with quasilinear phenomena. The case of a uniform plasma in a magnetic field has been treated by Andronov and Trakhtengerts.[2] The effect is the following: The plateau formed in velocity space as a result of any one of a number of interactions between particles and waves (Cerenkov ($v_z = \omega/k_z$), or cyclotron ($v_z = (\omega - n\omega_B)/k_z$, $n \neq 0$), is unstable with respect to another mechanism. Thus, the plateau of the initially excited waves does not remain as such; in addition to the wave excitation there is also an absorption effect. This effect is due to the fact that the equilization of the distribution function $f(v_z, v_\perp)$, which leads to the plateau, proceeds along different lines in velocity space depending on the type of interaction. (For example in the Cerenkov interaction the plateau equation is of the form $\partial f/\partial v_z = 0$, while in the cyclotron interaction the equation is of the form $\alpha\partial f/\partial v_z + \beta\partial f/\partial v_\perp = 0$ where $\alpha \neq 0$ and $\beta \neq 0$.) The necessary condition for wave transformation in a uniform plasma is the requirement that both wave classes interact with the same resonance particles:

$$(\omega_1 - n_1\omega_B) / k_{1z} = (\omega_2 - n_2\omega_B) / k_{2z},$$

and $n_1 \neq n_2$.

In the present work we consider quasilinear transformation of waves in an inhomogeneous plasma. The qualitatively new feature is the introduction of convective (drift) effects.[3] In the linear approximation, taking account of particle convection in the direction of the plasma inhomogeneity means that the growth rate (or damping rate) of the oscillations depends on arbitrary distribution functions which describe the distributions in coordinate space as well as velocity space,[4] that is to say,

$$\gamma \sim \alpha\partial f / \partial v_z + \beta\partial f / \partial v_\perp + \sigma\partial f / \partial r,$$

while the plateau for waves in this plasma corresponds to a line in the generalized phase space. In contrast with the case of a uniform plasma, quasilinear transformation in an inhomogeneous plasma is possible only by virtue of a particular kind of interaction, for example the Cerenkov interaction

$$\omega_1 / k_{1z} = \omega_2 / k_{2z}, \quad n_1 = n_2 = 0,$$

which, incidentally, plays the principal role in most of the instabilities of an inhomogeneous plasma.

Energy pumping is investigated for a system consisting of a cold plasma penetrated by a thin electron beam. This example gives results which are clear and also of direct interest for a number of experiments concerned with the interaction of a beam with a plasma, in particular,[5].

We show that high-frequency (plasma) oscillations, the relaxation of which is treated below, are excited by an electron beam with transverse dimension a which moves along the axis of a plasma cylinder of radius R located in a longitudinal magnetic field. Thus, the present work is essentially the nonlinear extension of the linear theory of the interaction between a plasma and a spatially inhomogeneous electron beam presented in an earlier

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paper by the present authors.[4] All of the basic assumptions and, wherever possible, the notation are the same as in the earlier paper.[4] In particular, it is assumed that the ratio of the beam density to the density of the cold plasma is a small quantity ($N \ll n_0$) and that the dimensions of the beam are smaller than the radius of the cylinder ($a < R$); the analysis is restricted to irrotational electron plasma oscillations ($\mathbf{E} = -\nabla\psi$).

In Sec. 2 we derive the basic equations of a quasilinear approximation for this inhomogeneous system and obtain the time integrals. In Sec. 3 we consider the quasilinear pumping of energy of the high-frequency oscillations into the low-frequency region of the spectrum. The original beam distribution function is assumed to be a wide step formed earlier because of the smearing of a δ-function (in velocity) beam; the total initial energy of the high-frequency oscillations is estimated to be of the same order as the kinetic energy of the beam. (In this formulation of the problem we are thus assuming that the usual beam relaxation process, as treated, for example, by Shapiro[6] has already terminated. It is shown in[6] that as a result of this process approximately 2/3 of the energy of the δ function beam remains in the particles of the smeared out beam while 1/3 goes into the high-frequency oscillations.) It is assumed that the radius of the beam remains unchanged during the time in which the low-frequency oscillations grow and it is shown in Sec. 3 that for a sufficiently high initial amplitude of the high-frequency oscillations almost all of the high frequency energy is converted into low frequency energy. During this time the kinetic energy of the beam does not change substantially (it remains of the same order of magnitude) and in this sense the stage of the process considered in Sec. 3 can be called the initial stage.

Section 4 is devoted to the analysis of a later stage in the growth of the low-frequency oscillations and the relaxation of the beam distribution function. In this stage the negative derivative of the velocity distribution function becomes appreciable and this implies damping of the low-frequency oscillations; this effect compensates for the excitation of these oscillations by the spatially inhomogeneous beam. As in Sec. 3, we neglect the effect of the spatial broadening of the beam. Under these assumptions it is shown that the energy acquired by the low-frequency oscillations directly from the beam is a small fraction of the energy obtained from the high frequencies if the frequency exceeds the characteristic drift frequency of the beam: $\omega_2 > \omega_{dr} = v_z^2|m|/a^2\omega_B$ (m is the azimuthal wave number). If $\omega_2 \lesssim \omega_{dr}$ these energies become comparable.

In Sec. 5 we estimate the effect of beam expansion on the growth of the low-frequency oscillations. A criterion is obtained for which this effect can be shown to be unimportant; it is shown that the analysis in Section 3 and 4 is valid with respect to this criterion. The growth of low-frequency oscillations under conditions of beam expansion is also investigated.

2. QUASILINEAR EQUATIONS FOR AN INHOMOGENEOUS PLASMA

In the quasilinear approximation the equation for the slowly varying (in time) part of the distribution function F is of the form:

$$\frac{\partial F}{\partial t} + \mathbf{v}\frac{\partial F}{\partial \mathbf{r}} + [\mathbf{v}\boldsymbol{\omega}_B]\frac{\partial F}{\partial \mathbf{v}} = \frac{e}{M}\left\langle \nabla\psi \frac{\partial f^{(1)}}{\partial \mathbf{v}} \right\rangle = S. \quad (2.1)^*$$

Assuming that F varies slowly over a period of the cyclotron gyration of the particles $\partial F/\partial t \ll \omega_B F$ we simplify this equation by an expansion in powers of ω_B^{-1}, writing

$$F = F^{(0)} + F^{(1)} + F^{(2)} + \ldots$$

We now introduce a cylindrical coordinate system in velocity space $v_\perp$, α, and v_z and assume that the plasma is uniform in the direction $z \parallel \mathbf{B}$. In the zeroth approximation we have from (2.1)

$$\partial F^{(0)} / \partial \alpha = 0, \quad (2.2)$$

whence $F^{(0)} \equiv F^0(v_\perp, v_z, \mathbf{r}, t)$.

In the next approximation, the oscillatory part (with respect to α) of Eq. (2.1) is

$$v\frac{\partial F^{(0)}}{\partial \mathbf{r}} - \omega_B \frac{\partial F^{(1)}}{\partial \alpha} = S - \bar{S}, \quad (2.3)$$

where the bar denotes an average over α. The quantity $F^{(1)}$ can be computed from Eq. (2.3):

$$F^{(1)} = \frac{1}{\omega_B}\left[v_\perp \sin\alpha \frac{\partial F^{(0)}}{\partial x} - v_\perp \cos\alpha \frac{\partial F^{(0)}}{\partial y} - \int (S - \bar{S})\, d\alpha \right]. \quad (2.4)$$

Substituting the expression for $F^{(1)}$ in the equation for the non-oscillatory (with respect to α) part of Eq. (2.1), we obtain the required equation for the time variation of $F^{(0)}$:

$$\frac{\partial F^{(0)}}{\partial t} + \frac{1}{\omega_B}\,\mathrm{rot}_z\, \mathbf{v}\bar{S} = \bar{S}. \quad (2.5)^\dagger$$

Hereinafter we limit our analysis to the beam-plasma system assuming that the plasma is at zero temperature and that the beam particles have only a longitudinal component of velocity v_z. Under

*$[\mathbf{v}\omega_B] \equiv \mathbf{v} \times \boldsymbol{\omega}_B$.

† rot ≡ curl.

these conditions we can integrate Eq. (2.5) over transverse velocities and take the limit $v_\perp \to 0$. Then the following equation is obtained for $F_0 = \int F^{(0)} dv_\perp$:

$$\frac{\partial F_0}{\partial t} = \frac{ie}{M}\int d\mathbf{k}'\, d\omega' \left\{k_z' \frac{\partial}{\partial v_z} + \frac{1}{\omega_B}[\nabla \mathbf{k}']_z\right\} \psi_{k'\omega'} f^{(1)}(\mathbf{r}_\perp, v_z, t) \times e^{i(\mathbf{k}'\mathbf{r}-\omega' t)}, \tag{2.6}$$

where $f^{(1)}$ is the rapidly varying part of the distribution function, an expression for which can be obtained from [4]:

$$f^{(1)}(\mathbf{r}_\perp, v_z, t) = -\frac{e}{M}\left\{\frac{k_z\,\partial F_0/\partial v_z}{\omega - k_z v_z}\psi + \frac{\nabla_\perp(F_0\nabla_\perp\psi)}{2\omega_B} \times\left(\frac{1}{\omega - k_z v_z - \omega_B} - \frac{1}{\omega - k_z v_z + \omega_B}\right) + i\frac{[\nabla F_0, \nabla\psi]_z}{2\omega_B} \times\left(\frac{1}{\omega - k_z v_z - \omega_B} + \frac{1}{\omega - k_z v_z + \omega_B} - \frac{2}{\omega - k_z v_z}\right)\right.. \tag{2.7}$$

For the case of a cylindrical beam which is coaxial with the cold-plasma cylinder it follows, in particular, that

$$\frac{\partial F_0}{\partial t} = \frac{e^2}{2M^2}\int dk_z \sum_{m,n}\left(k_z\frac{\partial}{\partial v_z} + \frac{m}{r\omega_B}\frac{\partial}{\partial r}\right)\psi^*\left\{\psi\left(k_z\frac{\partial F_0}{\partial v_z} + \frac{m}{r\omega_B}\frac{\partial F_0}{\partial r}\right)\frac{\gamma}{(\omega - k_z v_z)^2 + \gamma^2} + \frac{1}{2\omega_B}\left[\nabla_\perp(F_0\nabla_\perp\psi) - \frac{m\psi}{r}\frac{\partial F_c}{\partial r}\right]\frac{\gamma}{(\omega - k_z v_z - \omega_B)^2 + \gamma^2} - \frac{1}{2\omega_B}\left[\nabla_\perp(F_0\nabla_\perp\psi) + \frac{m\psi}{r}\frac{\partial F_0}{\partial r}\right]\frac{\gamma}{(\omega - k_z v_z + \omega_B)^2 + \gamma^2}\right\}. \tag{2.8}$$

Here it is convenient to isolate the part corresponding to the resonance particles of the beam ("hot" particles) and the cold plasma F_c:

$$\frac{\partial F_h}{\partial t} = \frac{\pi e^2}{2M^2}\int dk_z \sum_{m,n}\left(k_z\frac{\partial}{\partial v_z} + \frac{m}{r\omega_B}\frac{\partial}{\partial r}\right)\psi^* \times\left\{\psi\left(k_z\frac{\partial F_h}{\partial v_z} + \frac{m}{r\omega_B}\frac{\partial F_h}{\partial r}\right)\cdot\delta(\omega - k_z v_z) + \frac{1}{2\omega_B}\left[\nabla_\perp(F_h\nabla_\perp\psi) - \frac{m\psi}{r}\frac{\partial F_h}{\partial r}\right] \times \delta(\omega - k_z v_z - \omega_B) - \frac{1}{2\omega_B}\left[\nabla_\perp(F_h\nabla_\perp\psi) + \frac{m\psi}{r}\frac{\partial F_h}{\partial r}\right] \times \delta(\omega - k_z v_z + \omega_B)\right., \tag{2.9}$$

$$\frac{\partial F_c}{\partial t} = \frac{e^2}{2M^2}\int dk_z \sum_{m,n}\left(k_z\frac{\partial}{\partial v_z} + \frac{m}{r\omega_B}\frac{\partial}{\partial r}\right)\psi^*\left\{\frac{\psi\gamma}{(\omega - k_z v_z)^2} \times\left(k_z\frac{\partial F_c}{\partial v_z} + \frac{m}{r\omega_B}\frac{\partial F_c}{\partial r}\right) + \frac{1}{2\omega_B}\left[\nabla_\perp(F_c\nabla_\perp\psi) - \frac{m\psi}{r}\frac{\partial F_0}{\partial r}\right]\frac{\gamma}{(\omega - k_z v_z - \omega_B)^2} - \frac{1}{2\omega_B}\left[\nabla_\perp(F_c\nabla_\perp\psi) + \frac{m\psi}{r}\frac{\partial F_c}{\partial r}\right]\frac{\gamma}{(\omega - k_z v_z + \omega_B)^2}\right.. \tag{2.10}$$

The equation for the variation in the energy of the oscillations can be obtained from Poisson's equation and (2.7) (this has been done in [4]). The result is then

$$\frac{\partial}{\partial t}\int W_{m,n,k_z}(r)\,dr_\perp = \frac{\pi e^2}{2M}\omega\int dv_z\,dr_\perp\left\{|\psi|^2\left(k_z\frac{\partial F_h}{\partial v_z} + \frac{m}{r\omega_B}\frac{\partial F_h}{\partial r}\right)\delta(\omega - k_z v_z) - \frac{1}{2\omega_B}\left(|\nabla_\perp\psi|^2 F_h + \frac{m|\psi|^2}{r}\frac{\partial F_h}{\partial r}\right)\delta(\omega - k_z v_z - \omega_B) + \frac{1}{2\omega_B}\left(|\nabla_\perp\psi|^2 F_h - \frac{m|\psi|^2}{r}\frac{\partial F_h}{\partial r}\right)\delta(\omega - k_z v_z + \omega_B)\right.. \tag{2.11}$$

Here, $W_{m,n,k_z}(r)$ has the meaning of the energy density of the oscillations and is given by the expression

$$W_{m,n,k_z}(r) = \frac{1}{16\pi}\frac{\partial}{\partial\omega}\{\omega[|\nabla_\perp\psi|^2\varepsilon_\perp + k_z^2|\psi|^2\varepsilon_\parallel]\}, \tag{2.12}$$

where

$$\varepsilon_\perp = 1 - \frac{\omega_p^2}{\omega^2 - \omega_B^2}, \quad \varepsilon_\parallel = 1 - \frac{\omega_p^2}{\omega^2}, \quad \omega_p^2 = \frac{4\pi e^2 n_0}{M},$$

while the oscillation frequency $\omega(m, n, k_z)$ is determined by the dispersion equation for the cold plasma (cf. [4]).

Equations (2.9)—(2.11) form a closed system of equations in the quasilinear approximation for an inhomogeneous electron beam and a nearly homogeneous cold dense plasma. The equations for a more complicated system would be of analogous form but these will not be treated in the present paper.

We show that our quasilinear equations, like the equations for uniform plasma, have an integral in time. In order to be convinced of this we integrate Eq. (2.9) over the transverse coordinates and Eq. (2.11) over transverse wave numbers (that is to say, we sum over n and m). Combining these two results we obtain a conservation law in time for a function which contains the parameters of the beam and the energy of the oscillations.

Quasilinear Integral for Cerenkov Interaction

Retaining terms that contain $\delta(\omega - k_z v_z)$ in the right sides of Eqs. (2.9) and (2.11) we have

$$\frac{\partial}{\partial t}\left\{\int F_h\,d\mathbf{r}_\perp - \frac{\partial}{\partial v_z}\frac{1}{Mv_z^3}\sum_{m,n}|\omega_{m,n}|\int W_{m,n}(r)\,d\mathbf{r}_\perp\right\} = 0, \tag{2.13}$$

where, the quantity k_z is to be replaced by $k_z = \omega/v_z$.

Quasilinear Integral for the Normal and Anomalous Doppler Effects

Similarly, if we take account only of the terms containing $\delta(\omega - k_z v_z \pm \omega_B)$ in the right sides of Eqs. (2.9) and (2.11) we have

$$\frac{\partial}{\partial t}\left\{\int F_h d\mathbf{r}_\perp - \frac{\partial}{\partial v_z}\frac{1}{Mv_z^3}\sum_{m,n}\frac{(\omega_{m,n}\pm\omega_B)\,|\omega_{m,n}\pm\omega_B|}{\omega_{m,n}} \times \int W_{m,n}(r)\,d\mathbf{r}_\perp\right\} = 0, \quad (2.14)$$

where $k_z = (\omega_{m,n} \pm \omega_B)/v_z$.

Another interesting consequence of the quasilinear equations is the fact that spatial diffusion of the beam corresponds to axisymmetric waves only ($m \neq 0$). Actually, by integration of Eq. (2.9) over velocity one is easily convinced that $\partial N/\partial t$ if $m = 0$ in the right side of Eq. (2.9). This result is a consequence of the slowness of the relaxation processes that are being considered $\partial \ln F/\partial t \ll \omega_B$.

3. QUASILINEAR TRANSFER OF ENERGY OF HIGH-FREQUENCY WAVES INTO THE LOW-FREQUENCY REGION OF THE SPECTRUM

We now use the equations of the preceding section to analyze the possibility of energy transfer from the high-frequency waves $\omega_1 \approx \omega_p$ to the low frequency waves $\omega_2 \ll \omega_p$. This transfer is a consequence of the fact that the plateau on the beam distribution function for one frequency range (high-frequency range) is not a general plateau for the other (low-frequency).

For reasons of simplicity we assume that as a consequence of the initial conditions there are excited in the plasma two wave classes with frequencies that are appreciably different $\omega_1 \gg \omega_2$. We limit our analysis to the case of Cerenkov interaction and assume that in the high-frequency region it is legitimate to neglect the effect of gradient terms (this requires that $\omega_p \gg \omega_{dr} = mv_z^2/a^2\omega_B$). Let $t = t_0$ be the time at which the high-frequency oscillations are excited by the usual beam mechanism, forming a plateau on the distribution function $\partial F_h/\partial t = 0$. In the absence of convection the formation of this plateau would be the terminal stage of the process. However, the convective effects can be extremely important in the low-frequency region so that it is necessary to trace the further change in the distribution function and the resulting transfer of energy from the high-frequency waves to the low-frequency waves.

As in the linear theory,[4] we choose a beam distribution function of the form

$$F_h = F(v_z, t)\exp(-r^2/a^2)$$

and assume that the beam width a does not change in the course of the entire process. (The validity of this assumption is discussed in Sec. 5.) We also assume that $\partial F/\partial v_z$ terms can be neglected in Eq. (2.9) for the low-frequency oscillations when comparison is made with the corresponding terms with the high-frequency oscillations. This implies

$$W_2(t)\,/\,W_1(t) \ll \omega_1/\omega_2, \quad (3.1)$$

where W_1 and W_2 are the total energies associated with the corresponding wave classes. If Eq. (2.9) is integrated over the transverse coordinate $d\mathbf{r}_\perp$, assuming that the assumptions given above are satisfied, we have

$$\frac{\partial F}{\partial t} = \frac{\pi e^2}{2M^2}\frac{\partial}{\partial v_z}\left\{\left[\sum_1 |\Psi_1|^2\frac{\omega_1^2}{v_z^3}\frac{\partial F}{\partial v_z}\right]_{k_z=\omega_1/v_z} - \frac{2m}{a^2\omega_B}\left[\sum_2 |\Psi_2|^2\frac{\omega^2}{v_z^2}F\right]_{k_z=\omega_2/v_z}\right\}, \quad (3.2)$$

where

$$|\Psi|^2 = \int |\psi|^2 e^{-r^2/a^2} r\,dr \Big/ \int e^{-r^2/a^2} r\,dr.$$

Here, in the summation over m we actually sum up to a value of m which corresponds to the maximum growth rate in the low-frequency region.

Solution of the Kinetic Equation

In order to proceed with the solution of Eq. (3.2) we assume that as a consequence of the initial conditions the region of excitation of the low-frequency oscillations in velocity space is narrower than the region of high-frequency oscillations so that

$$|\Psi_1|^2 \neq 0 \quad \text{for } u_2 > v_z > u_1, \quad (3.3)$$
$$|\Psi_2|^2 \neq 0 \quad \text{for } v_2 > v_z > v_1 \quad (3.3')$$

where the limits of these regions satisfy the inequality $u_1 < v_1 < v_2 < u_2$. Hereinafter we denote the regions in the following manner:

$$\mathrm{I} \equiv (u_1, v_1); \quad \mathrm{II} \equiv (v_1, v_2); \quad \mathrm{III} \equiv (v_2, u_2).$$

From the point of view of energy transfer, the most interesting region is II. However, in order to find the solutions in this region we must solve Eq. (3.2) and determine the boundary conditions at the points v_1 and v_2 at which the potential ψ_2 vanishes. Inasmuch as low-frequency oscillations are not excited in regions I and III, it is legitimate to assume that the relation $\partial F/\partial v_z$ holds in these regions throughout the entire process. Then, using Eq. (3.2) we obtain the following approximate boundary conditions:

$$[\partial F / \partial v_z]_{v_z = v_1, v_2} = 0. \quad (3.4)$$

It should be noted that these conditions would be exact boundary conditions for regions I and III if $|\Psi_2|^2 = 0$ as well in region II. Thus, in choosing the boundary conditions in the form in (3.4) we are neglecting the interaction between regions. In order to obtain the variation in the number of particles in regions I and III we must invoke the conservation of the total number of particles in the beam:

$$N = F_{\mathrm{I}}(t)(v_1 - u_1) + \int_{v_1}^{v_2} F_2(v_z, t)\, dv_z$$
$$+ F_{\mathrm{III}}(t)(u_2 - v_2) = \mathrm{const} \quad (3.5)$$

and the condition of continuity on the distribution function F.

In order to obtain the function $F_{\mathrm{II}}(v_z, t)$ we solve Eq. (3.2) by a method similar to that used in the derivation of the basic quasilinear equations in Sec. 2. For values of the time that are not too large $t > t_0$, for which the term $|\Psi_1|^2$ dominates, we can use the following method of successive approximations: The quantity F_{II} is written in the form of a sum $F_{\mathrm{II}} = F_0 + F_1 + F_2$ in which each term satisfies the boundary condition (3.4). Then, to a first approximation, using the equation

$$\frac{\partial}{\partial v_z}\left[\sum_1 |\Psi_1|^2 \frac{\omega_1^2}{v_z^3} \frac{\partial F_0}{\partial v_z}\right]_{k_z = \omega_1/v_z} = 0 \quad (3.6)$$

and (3.4) we find that

$$F_0 = C_1(t). \quad (3.7)$$

In the next approximation we introduce the effect of the low-frequency oscillations on the distribution function:

$$\frac{\partial}{\partial v_z}\left[\sum_1 |\Psi_1|^2 \frac{\omega_1}{v_z^3} \frac{\partial F_1}{\partial v_z}\right]_{k_z = \omega_1/v_z}$$
$$= \frac{2m}{a^2 \omega_B} \frac{\partial}{\partial v_z}\left[\sum_2 |\Psi_2|^2 \frac{\omega_2}{v_z^2} F_0\right]_{k_z = \omega_2/v_z}. \quad (3.8)$$

In view of the fact that the quantity $|\Psi_2|^2$ vanishes at the boundaries of region II, using (3.4) we can write

$$F_1 = \vartheta(v_z, t) + C_2(t), \quad (3.9)$$

where

$$\vartheta(v_z, t) = \frac{2m}{a^2 \omega_B} F_0 \int_{v_1}^{v_z} \left[\sum_2 |\Psi_2|^2 \omega_2 / v_z'^2\right]_{k_z = \omega_2/v_z'}$$
$$\times \left[\sum_1 |\Psi_1|^2 \omega_1^2 / v_z'^3\right]^{-1}_{k_z = \omega_1/v_z'} dv_z'.$$

The constant $C_2(t)$ is determined from the continuity condition on F at the points v_1 and v_2 and from the conservation of the number of particles (3.5). The corrections to the distribution functions $F_{\mathrm{I}1}(t)$ and $F_{\mathrm{III}1}(t)$, which describe the slow change of the plateau in regions I and III, are given by

$$F_{\mathrm{I}1}(t) = C_2(t); \quad F_{\mathrm{III}1}(t) = \vartheta(v_2, t) + C_2(t), \quad (3.10)$$

so that the complete distribution function in this approximation F is given by

$$F = F_0 + C_2(t) + \vartheta(v_2, t). \quad (3.11)$$

Substituting (3.11) in (3.5) and assuming that the boundary points u_1 and u_2 are independent of time, we have

$$C_2(t) \approx \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} \vartheta(v_z, t)\, dv_z. \quad (3.12)$$

In the next approximation we find

$$\frac{\partial F_0}{\partial t} = \frac{\pi e^2}{2M^2} \frac{\partial}{\partial v_z}\left\{\sum_1 |\Psi_1|^2 \frac{\omega_1^2}{v_z^3} \frac{\partial F_2}{\partial v_z}\right.$$
$$\left. - \frac{2m}{a^2 \omega_B} \sum_2 |\Psi_2|^2 \frac{\omega_2}{v_z^2} F_1\right\}. \quad (3.13)$$

Using (3.4) we find that the orthogonality condition for Eq. (3.13) reduces to the form $\partial F_0/\partial t = 0$, that is to say, $F_0 = N/(u_2 - u_1)$.

We note that the condition for applicability of the method used above to obtain successive approximations $\vartheta(v_2, t) \ll 1$ means, in order-of-magnitude terms, that $W_2(t)/W_1(t) \ll \omega_p/\omega_{\mathrm{dr}}$. Taking account of (3.1) and assuming that the beam radius a remains constant, the condition for applicability of all the results obtained above can be written in the form

$$\frac{W_2(t)}{W_1(t)} \ll \min\left(\frac{\omega_p}{|\omega_{\mathrm{dr}}|}, \frac{\omega_p}{\omega_2}\right), \quad a \approx \mathrm{const}. \quad (3.14)$$

Effect of Energy Transfer

Using the results obtained in the above sections it is easy to compute what fraction of the energy that is converted into low-frequency oscillations in region II comes from the beam and what fraction comes from the high-frequency oscillations. We first compute the variation in the energy of the high frequency oscillation δW_1^{II}. For this purpose, in Eq. (2.11) we replace F by (3.11) and take account of the time dependence $|\Psi_2|^2 \sim \exp(2\gamma_2 t)$ where γ_2 is the maximum growth rate in the low-frequency region, this quantity being of order [4]

$$\gamma_2 \approx \frac{\pi^{3/2} a N}{R n_0} |\omega_{\mathrm{dr}}|. \quad (3.15)$$

As a result we find

$$\delta W_1^{\mathrm{II}} \approx \frac{\pi^2 e^2 m}{2M\omega_B\gamma_2 g} F_0 \int_{v_1}^{v_2} \sum_2 |\Psi_2|^2 \frac{\omega_2}{v_z} dv_z,$$

$$g = \pi a^2 \sum_1 |\Psi_1|^2 \omega_1 \Big/ \int_0^R \sum_1 |\psi_1|^2 \omega_1 \, dr_\perp. \qquad (3.16)$$

The variation in this fraction of kinetic energy of the beam which, in region II, is converted into the energy of low-frequency oscillations, can be written in the form

$$\delta W_{\mathrm{kin}}^{\mathrm{II}} = \int e^{-r^2/a^2} \Big\{ \int_{v_1}^{v_2} (F_{\mathrm{II}}(t) - F_{\mathrm{II}}(t_0)) \frac{1}{2} M v_z^2 \, dv_z - {}^1\!/_6 M (u_2^3 - v_2^3)(F_{\mathrm{III}}(t) - F_{\mathrm{III}}(t_0)) - {}^1\!/_6 M (v_1^3 - u_1^3) \times (F_{\mathrm{I}}(t) - F_{\mathrm{I}}(t_0)) \Big\} dr_\perp. \qquad (3.17)$$

In order to determine the order-of-magnitude of the ratio $\delta W_1^{\mathrm{II}}/\delta W_{\mathrm{kin}}^{\mathrm{II}}$ we must estimate the integrals appearing in Eqs. (3.16) and (3.17). For reasons of simplicity we take $v_1 \approx u_1$ and $v_2 \approx u_2$ so that the order of magnitude of this quantity is found to be

$$\frac{\delta W_1^{\mathrm{II}}(t)}{\delta W_{\mathrm{kin}}^{\mathrm{II}}} \approx \frac{\omega_p}{|\omega_{\mathrm{dr}}|} \frac{R}{a} \frac{W_1(t)}{W_0}, \qquad (3.18)$$

where $W_0 = \pi a^2 N M v_2^2/2$ is the total energy of the δ function beam while $W_1(t)$ is the total energy of the high-frequency oscillations per unit length in the z-direction.

It follows from Eq. (3.18) that when the initial energy of the high-frequency oscillations is not too small, in which case

$$\frac{W_1(t_0)}{W_0} \gg \frac{|\omega_{\mathrm{dr}}|}{\omega_p} \frac{a}{R}, \qquad (3.19)$$

in the initial stage (when $W_1(t)$ is still comparable with $W_1(t_0)$) the energy growth of the low-frequency oscillations is due to the transfer of energy from high-frequency oscillations rather than from the beam. If the high-frequency oscillations are excited by a beam which is not smeared out greatly so that $W_1(t_0) \approx W_0$, then at time t_1, at which point one expects

$$1 \gg \frac{W_1(t_1)}{W_0} \gg \max\left(\frac{|\omega_{\mathrm{dr}}|}{\omega_p}, \frac{\omega_2}{\omega_p} \right), \qquad (3.20)$$

almost all of the energy in the high-frequency oscillations has been transferred to the low-frequency region whereas the change in the kinetic energy of the beam is still negligibly small. It is evident from (3.18) and (3.20) that the first of the conditions (3.14) is not violated when $t \approx t_1$. (The condition $a \approx$ const remains satisfied as before.)

We now estimate the ratio $\delta W_1(\tau)/\delta W_{\mathrm{kin}}(\tau)$ for later times $t \approx \tau$ when the first condition (3.14) is violated, that is to say, when

$$W_2(\tau) \approx \min\left(\frac{\omega_p}{|\omega_{\mathrm{dr}}|}, \frac{\omega_p}{\omega_2} \right) W_1(\tau). \qquad (3.21)$$

Substituting Eq. (3.21) in Eq. (3.18) we find

$$\frac{\delta W_1(\tau)}{\delta W_{\mathrm{kin}}(\tau)} \approx \frac{R}{a} \frac{W_2(\tau)}{W_0} \max\left(1, \frac{\omega_2}{|\omega_{\mathrm{dr}}|} \right). \qquad (3.22)$$

We now consider the following two limiting cases.

1. Let

$$\frac{W_1(t_0)}{W_0} \gtrsim \frac{a}{R} \min\left(1, \frac{|\omega_{\mathrm{dr}}|}{\omega_2} \right).$$

It follows from Eqs. (3.21) and (3.22) that $W_2(\tau) \approx W_1(t_0)$. This means that after time τ almost all the energy of the high-frequency oscillations has been transferred to low-frequency oscillations and that the amount of energy transferred by this means from the latter is not smaller than the energy transferred from the beam.

2. Let

$$W_1(t_0) / W_0 \ll a / R$$

(assuming for simplicity that $\omega_2 < |\omega_{\mathrm{dr}}|$). Then, when $t = \tau$ the amount of energy obtained by the low-frequency oscillations from the beam is greater than the energy obtained from the high-frequency oscillations. If the initial energy of the high-frequency oscillations is so small that the stronger inequality

$$W_1(t_0) / W_0 \ll (a/R) \omega_{\mathrm{dr}}^2 / \omega_p^2,$$

is satisfied then at time $t \approx \tau$ there will be transferred only a small fraction of the high-frequency energy. In the opposite case

$$a/R \gg W_1(t_0) / W_0 \gg (a/R) \omega_{\mathrm{dr}}^2 / \omega_p^2,$$

the energy of the high-frequency oscillations is almost completely transferred to the low frequencies although, as indicated above, its fractional effect in the energy balance is small.

4. ENERGY OF THE LOW-FREQUENCY OSCILLATIONS IN THE STATIONARY STATE

As a consequence of the effect described in the preceding sections the energy of the low-frequency oscillations is increased as the energy of the high-frequency oscillations is reduced so that the condition in (3.14) must ultimately be violated. It is of interest to determine what amount of energy will be transferred to the low-frequency oscillations in the course of the entire process in which a steady state is established. If we neglect effects such as beam expansion, particle collisions, or excitation of other waves which do not appear in our two wave classes, then at long times $t \to \infty$ we have $|\psi_1|^2 \to 0$, $\partial F_h/\partial t \to 0$, using Eq. (2.9) we find that the function F satisfies the following equation:

$$\frac{\partial}{\partial v_z}\left[\sum_2 |\psi_2|^2 \frac{\omega_2^2}{v_z^3}\frac{\partial F}{\partial v_z}\right] = \frac{2m}{a^2\omega_B}\frac{\partial}{\partial v_z}\left[\sum_2 |\psi_2|^2 \frac{\omega_2}{v_z^2}F\right]. \quad (4.1)$$

Integrating Eq. (4.1) with respect to velocity and assuming that when $t \to \infty$ the quantity

$$\frac{\partial}{\partial t}\int W dr_\perp \to 0,$$

we find that in the region in which $|\psi_2|^2 \neq 0$,

$$\frac{\partial F}{\partial v_z} - \frac{2mv_z}{a^2\omega_B\omega_2}F = 0. \quad (4.2)$$

If, for simplicity we take $|\psi_2|^2 \neq 0$ for $v_2 > v_z > 0$ and at $t = t_0$ F is equated to the constant part, F_0, then when $t \to \infty$ we find

$$F = F_0 v_2 \exp(mv_z^2/a^2\omega_B\omega_2) \Big/ \int_0^{v_2} \exp(mv_z^2/a^2\omega_B\omega_2)\, dv_z. \quad (4.3)$$

In order to determine the kinetic energy of the beam $W_{f\,kin}$ when $t \to \infty$ we consider the following two limiting cases.

A. $\omega_2 \gg |\omega_{dr}|$. In this case using Eq. (4.3) we find

$$\frac{W_{i\,kin} - W_{f\,kin}}{W_{i\,kin}} \approx \frac{|\omega_{dr}|}{\omega_2}, \quad (4.4)$$

where $W_{i\,kin}$ is the kinetic energy of the beam for $t = t_0$ so that when $t \to \infty$

$$\frac{\delta W_1}{\delta W_{kin}} \approx \frac{\omega_2}{|\omega_{dr}|}\frac{W_1(t_0)}{W_0}. \quad (4.5)$$

It is then evident that if

$$W_1(t_0)\,/\,W_0 > |\omega_{dr}|\,/\,\omega_2$$

the fraction of the total energy obtained by the low-frequency oscillations from the high-frequency oscillations will be appreciably greater than the fraction of the energy obtained from the beam particles so that the final energy level of the low-frequency noise can be properly estimated only if the quasilinear energy transfer is considered.

B. $\omega_2 \ll |\omega_{dr}|$. In this case, using Eq. (4.3) we find

$$W_{f\,kin}\,/\,W_{i\,kin} \approx \omega_2\,/\,|\omega_{dr}|, \quad (4.6)$$

so that when $t \to \infty$ we have

$$\delta W_1\,/\,\delta W_{kin} \approx W_1(t_0)\,/\,W_0. \quad (4.7)$$

This means that when $\omega_2 < |\omega_{dr}|$ the final fraction of the energy obtained by the low-frequency oscillations from the beam will exceed or be of the same order as the energy obtained from the high-frequency oscillations, in determining the final energy level of the low-frequency noise in this case the effect of energy transfer is not so important. However, this effect is always important in estimating the level of the high-frequency oscillations because of the quasilinear transfer the level can be appreciably smaller than would be expected from the theory of a uniform plasma.

5. EFFECT OF SPATIAL BROADENING OF THE BEAM ON QUASILINEAR ENERGY TRANSFER

We now discuss the validity of our assumption above that the spatial diffusion of the beam can be neglected in the transfer of energy. We also consider the case in which this assumption does not hold. Integrating Eq. (2.9) with respect to dv_z and retaining the term that corresponds to the low-frequency oscillations we have

$$\frac{\partial}{\partial t}Ne^{-r^2/a^2} \approx -\frac{\pi e^2}{2M^2}\int dk_z\, dv_z \sum_2 \frac{2m^2}{a^2\omega_B^2 r}\frac{\partial}{\partial r}(|\psi_2|^2 F_h) \times \delta(\omega - k_z v_z). \quad (5.1)$$

Here, as before, we assume that the particles have a Gaussian distribution in space. Integrating Eq. (5.1) with respect to $dr_\perp$ and keeping in mind the fact that $R \gg a$, we can replace the limits of integration by infinity in the left side of this equation, thus obtaining

$$N(t)\,a^2(t) \approx N_0 a_0^2, \quad (5.2)$$

where N_0 and a_0 are the values of N and a at $t = t_0$.

Estimating the value of the expression on the right side of Eq. (5.1), in order-of-magnitude terms we have

$$\partial N(t)\,/\,\partial t \approx N(t)\,W_2(t)\,m^2 v_2^2\,/\,M n_0 a^4(t) R^2 \omega_B^2 \omega_2. \quad (5.3)$$

Making the substitution $2\gamma_2 W_2(t) \to dW_2(t)/dt$, and using Eqs. (3.15) and (4.2) we can now integrate Eq. (5.3):

$$\left(\frac{N(t)}{N_0}\right)^{-1/2} = 1 + \frac{|\omega_{dr}|}{\omega_2}\frac{a_0}{R}\frac{W_2(t)}{W_0}. \quad (5.4)$$

Then the condition that the number of particles at the axis at time t remains essentially unchanged $N(t) - N_0/N_0 \ll 1$ can be written in the form

$$W_2(t) \ll \frac{\omega_2}{|\omega_{dr}|}\frac{R}{a_0}W_0. \quad (5.5)$$

In particular, it follows from Eq. (5.5) that all the calculations in the earlier sections remain valid so long as

$$\frac{\omega_2}{|\omega_{dr}|}\frac{R}{a_0} > 1. \quad (5.6)$$

If the condition (5.5) is not satisfied when $W_2(t) \approx W_1(t_0)$ (as is possible when $R\omega_2/a_0|\omega_{dr}| < 1$), an important question arises as to whether it is generally possible for the high-frequency oscillations to transfer energy into low-frequency oscillations because the spatial diffusion of the beam can occur so rapidly that all particles will reach the walls of the chamber. At the limit of applicability of Eq. (5.4), where the width of the beam $a(t_2) \approx R$, $W_2(t_2)$ reaches the value

$$W_2(t_2) \approx \frac{\omega_2}{|\omega_{dr}|} \frac{R}{a_0} W_0, \qquad (5.7)$$

which coincides with the right side of Eq. (5.5).

It is not difficult to analyze the continuation of the process of spatial diffusion when $a(t) \approx R$ assuming that the transverse dimensions of the beam are not changed but that the dependence of N(t) on time is determined by loss of particles to the chamber walls. The correction obtained in this way to the energy of the low frequency oscillations W_2 is R/a_0 times smaller than $W_2(t_2)$ so that it can generally be neglected. Thus, if one satisfies the inequality which is the inverse of (5.6) the derivations of Secs. 3 and 4, which were obtained without taking beam expansion into account, no longer hold. In this case the fraction of the energy obtained by the low-frequency oscillations is determined by Eq. (5.7). When $W_1(t_0) \sim W_0$ this means that the energy obtained by the low frequency oscillations from the high-frequency oscillations represents a small fraction of $W_1(t_0)$ so that the total transformation of high-frequency energy into low-frequency energy holds only when the initial energy of the high-frequency oscillations is sufficiently large.

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