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CONTRIBUTION TO THE THEORY OF DECAY OF ELECTROMAGNETIC WAVES IN A MAGNETOACTIVE PLASMA

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Cerenkov and cyclotron decay of electromagnetic waves in an homogeneous magnetoactive plasma is considered in the quasilinear approximation, with account taken of collisions between resonant particles responsible for absorption of the waves and the remaining particles of the plasma. The velocity of the resonant particles along the magnetic field may be commensurate with or less than the thermal velocity. For three-dimensional wave packets it is shown that in the absence of collisions the system finally reaches a steady state in which either the oscillation energy vanishes or the distribution function has a plateau. In the case of Cerenkov resonance, diffusion of particles on the waves in velocity space occurs only along the magnetic field, whereas in the case of cyclotron resonance diffusion takes place along as well as across the magnetic field. The distribution function is determined for the quasistationary state when particle diffusion on the waves is balanced by collisions; the nonlinear decay decrement is determined.

1. INTRODUCTION

As is well known,[1] the reaction exerted by an electromagnetic field on the resonant plasma particles responsible for the attenuation of the field leads to deformation of the distribution function and to a decrease in the damping decrement. In the absence of collisions, a "plateau" is formed on the distribution function, and the absorption of the energy by the plasma stops. The collisions Maxwellize the distribution function and this leads to a certain quasi-stationary state of the distribution function, for which the damping decrement is smaller than the linear damping decrement.

Cerenkov damping of Langmuir oscillations and cyclotron damping of the extraordinary wave propagating along an external magnetic field were considered in the quasi-linear approximation, with allowance for the Coulomb collisions of the resonant electrons with the remaining plasma particles, by Vedenov, Velikhov, and Sagdeev.[2] The Cerenkov and cyclotron damping of the electromagnetic waves propagating at an arbitrary angle to the magnetic field were investigated by Yakimenko.[3] These investigations concerned exponentially small damping, due to resonant frequencies with velocity $v_{\parallel}$ along the magnetic field much larger than the thermal velocity. In these investigations, the collision integral of the resonant particles with respect to the velocity component $v_{\perp}$ perpendicular to the magnetic field was averaged, and in addition it was assumed that the distribution function $f(v_{\perp}, v_{\parallel})$ breaks up into a product $f(v_{\perp})f(v_{\parallel})$. These assumptions, however, are incorrect and therefore the expressions obtained in these papers for the distribution functions of the resonant particles and for the damping decrement are only of the correct order of magnitude.

In this paper we consider the damping of electromagnetic waves and the quasi-linear approximation without the foregoing limitations.

2. FUNDAMENTAL EQUATIONS

The background distribution function of the resonant particles of species α is determined in the quasi-linear approximation from the kinetic equation

$$\partial f^{\alpha} / \partial t = L^{\alpha}(f^{\alpha}) + \mathrm{St}^{\alpha}\{f^{\beta}\}. \tag{2.1}$$

The term $L^{\alpha}(f^{\alpha})$, which takes into account "diffusion on the waves," is of the form[3]

$$L^{\alpha}(f^{\alpha}) = \frac{\pi e_{\alpha}^2}{|\omega_{\alpha}| m_{\alpha}^2} \sum_{\mathbf{k}} \sum_{n=-\infty}^{\infty} \frac{1}{v_{\perp}} R \Big\{ v_{\perp} \delta(b+n) \Big| E_1 n \frac{J_n(a)}{a} - iE_2 J_n'(a) + E_3 \frac{v_{\parallel}}{v_{\perp}} J_n(a) \Big|^2 R f^{\alpha} \Big\}, \tag{2.2}$$

where*

$$R \equiv \frac{n\omega_\alpha}{\omega}\frac{\partial}{\partial v_\perp} + \frac{k_\parallel v_\perp}{\omega}\frac{\partial}{\partial v_\parallel}, \quad a = \frac{k_\perp v_\perp}{\omega_\alpha}, \quad b = \frac{k_\parallel v_\parallel - \omega}{\omega_\alpha},$$

$$E_1 = \frac{\mathbf{k}_\perp \mathbf{E}_\mathbf{k}}{k_\perp}, \quad E_2 = \frac{[\mathbf{k}_\perp \mathbf{E}_\mathbf{k}]\,\mathbf{B}_0}{k_\perp B_0}, \quad E_3 = E_\parallel = \frac{\mathbf{E}_\mathbf{k}\mathbf{B}_0}{B_0},$$

$\omega_\alpha = e_\alpha B_0/m_\alpha c$ is the gyrofrequency of particles with charge e_α and mass m_α in an external magnetic field $\mathbf{B}_0$, $J_n(a)$ and $J_n'(a)$ are a Bessel function and its derivative, $k_\perp$ and $k_\parallel$ are the components of the wave vector $\mathbf{k}$ perpendicular and parallel to $\mathbf{B}_0$ respectively, $\omega = \omega(\mathbf{k})$ is the frequency of the oscillations, determined from the dispersion equation of the linear theory, and $\mathbf{E}_\mathbf{k}$ is the Fourier component of the electric field intensity. Equation (21) was derived for rather broad wave packets such that there are no captured particles[1] and the damping decrement is sufficiently small:

$$\gamma \ll |\omega - k_\parallel v_\alpha - n\omega_\alpha|,$$

where $v_\alpha = (T_\alpha/m_\alpha)^{1/2}$ is the thermal velocity of the particles of species α.

The interval of the Coulomb collisions $St^\alpha\{f^\beta\}$ is of the form[4]

$$St^\alpha\{f^\beta\} = -\frac{2\pi e_\alpha^2 \Lambda}{m_\alpha}\sum_{\beta=e,\,i} e_\beta^2 \left[\frac{8\pi f^\alpha f^\beta}{m_\beta} + \frac{1}{m_\alpha}\frac{\partial^2 f^\alpha}{\partial v_j \partial v_k}\right.$$
$$\left.\times \frac{\partial^2 \Psi^\beta}{\partial v_j \partial v_k} + \left(\frac{1}{m_\alpha} - \frac{1}{m_\beta}\right)\frac{\partial f^\alpha}{\partial v_j}\frac{\partial^3 \Psi^\beta}{\partial v_j \partial v_k^2}\right], \tag{2.3}$$

where Λ is the Coulomb logarithm and

$$\Psi^\beta = \int f^\beta(\mathbf{v}')\,|\mathbf{v} - \mathbf{v}'|\,d\mathbf{v}'.$$

For narrow wave packets ($\Delta k \ll k$), when the number of resonant particles is small, $v_1 < v_\parallel < v_2$ and $\Delta v = v_2 - v_1 \ll v_\alpha$, we can neglect the collisions between the resonant particles. In this case the collision integral becomes simpler. For resonant electrons with velocity $v_\parallel$ much larger than the thermal velocity v_i of the ions, the collision integral is written in the form

$$St^e\{f^\beta\} = \frac{\partial}{\partial v_\parallel}\left\{D_c \frac{\partial (f^e - f_M^e)}{\partial v_\parallel}\right\},$$
$$f_M^e = \frac{\exp(-v^2/2v_e^2)}{(2\pi)^{3/2} v_e^3}, \tag{2.4}$$

where f_M^e is the Maxwellian distribution function

$$D_c = \frac{v_e^3}{\tau_c}\left(\frac{v_\perp^2}{v^3} + \frac{\partial^2 \Psi_M^e}{\partial v_\parallel^2}\right), \quad \tau_c = \frac{m_e^2 v_e^3}{2\pi e^4 \Lambda n_0}; \tag{2.5}$$

$$\Psi_M^e = \int f_M^e(\mathbf{v}')\,|\mathbf{v} - \mathbf{v}'|\,d\mathbf{v}' = \sqrt{\frac{2}{\pi}}\, v_e \exp\left(-\frac{v^2}{2v_e^2}\right)$$
$$+ \left[\frac{(v_e^2 + v^2)}{v}\right]\Phi\left(\frac{v}{\sqrt{2}\,v_e}\right),$$
$$\Phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt. \tag{2.6}$$

The first term of D_c in (2.5) takes into account the collision of the resonant electrons with the ions and the second, with the electrons. If $v_\parallel \gg v_e$, then the diffusion coefficient (2.5) takes the form

$$D_c = (2v_e^3/\tau_c v_\parallel^3)(v_\perp^2 + v_e^2). \tag{2.7}$$

The collisions of resonant ions with the electrons can be neglected. We then obtain

$$St^i\{f^i\} = \frac{\partial}{\partial v_\parallel}\left\{D_c \frac{\partial (f^i - f_M^i)}{\partial v_\parallel}\right\},$$
$$D_c = \left(\frac{v_i^3}{\tau_c}\right)\frac{\partial^2 \Psi_M^i}{\partial v_\parallel^2}, \tag{2.8}$$

where $\tau_c = m_i^2 v_i^3/2\pi e^4 \Lambda n_0$, and Ψ_M^i is determined by expression (2.6), in which v_e must be replaced by v_i. When $v_\parallel \gg v_i$, expression (2.8) simplifies to

$$D_c = (v_i^3/\tau_c v_\parallel^3)(v_\perp^2 + 2v_i^2). \tag{2.9}$$

In the derivation of the collision integral (2.4)–(2.8) it was assumed that the electron and ion distribution functions in the resonant region (but not their derivatives with respect to $v_\parallel$) differ little from Maxwellian distributions, and we neglected the quantities $\partial(f^\alpha - f_M^\alpha)/\partial v_\perp \sim (f^\alpha - f_M^\alpha)/v_\alpha$ compared with $\partial(f^\alpha - f_M^\alpha)/\partial v_\parallel \sim (f^\alpha - f_M^\alpha)/\Delta v$.

The time dependence of the oscillation amplitude is determined from the equation

$$\partial|\mathbf{E}_\mathbf{k}|^2/\partial t = 2\gamma_\mathbf{k}|\mathbf{E}_\mathbf{k}|^2, \tag{2.10}$$

where the damping decrement γ_k is determined by the linear-theory dispersion equation, in which the distribution function is taken from (2.9).

As follows from (2.1) and (2.4)–(2.8), the relaxation of the distribution function in the resonant region, due to collisions, occurs when $v_\parallel \sim v_\alpha$ during a time of the order of

$$\tau_{rel} \sim \tau_c(\Delta v/v_\alpha)^2 \ll \tau_c.$$

In the absence of collisions, as can be readily shown, the "diffusion on the waves" leads to establishment of a certain stationary state. Let us multiply Eq. (2.1) by f^α and integrate over the velocity-space volume occupied by the resonant particles. We then obtain

*$[\mathbf{k}_\perp \mathbf{E}_\mathbf{k}] \equiv \mathbf{k}_\perp \times \mathbf{E}_\mathbf{k}$.

$$-\frac{\partial}{\partial t}\int (f^\alpha)^2 d\mathbf{v} = -\frac{\pi e_\alpha^2}{|\omega_\alpha| m_\alpha^2}\int d\mathbf{v}\sum_{\mathbf{k},n}(Rf^\alpha)^2\cdot\left|E_1\frac{nJ_n}{a} - iE_2J_n' + E_3\frac{v_\|}{v_\perp}J_n\right|^2\delta(b+n)\leqslant 0. \quad (2.11)$$

It follows therefore that inasmuch as $\int (f^\alpha)^2 d\mathbf{v} > 0$, we get as $t \to \infty$

$$\frac{\partial}{\partial t}\int (f^\alpha)^2 d\mathbf{v} = 0.$$

Then we obtain from (2.11) either

$$|E_1 nJ_n / a - iE_2J_n' + E_3 v_\| J_n / v_\perp|^2 = 0,$$

or

$$Rf^\alpha = 0$$

($\omega = k_\| v_\| + n\omega_\alpha$). Since the quantity γ_k is proportional to the anti-hermitian parts of the tensor ϵ_{ij}, which have the form of integrals of Rf^α with respect to $v_\perp$ at $\omega = R_\| v_\| + n\omega_\alpha$, [5] we get in the final state $\gamma_k = 0$ or $|\mathbf{E}_k|^2 = 0$. Thus, in the absence of collisions the quasi-linear relaxation leads either to the damping of the oscillations, or to the formation of a state with a "plateau," in which $Rf^\alpha = 0$. This result was obtained by Andronov and Trakhtengerts for narrow one-dimensional wave packets.[6] The quasi-linear relaxation of longitudinal oscillations of a plasma in a magnetic field was considered in [7].

As shown in [6], the state in which $Rf^\alpha = 0$ is generally speaking unstable. The examination carried out on the present work pertains to the initial state of quasi-linear relaxation, when instabilities of this type have not yet time to develop.

3. CERENKOV DAMPING

Let us consider first Cerenkov absorption of narrow wave packets propagating at a certain angle ϑ to the magnetic field in a low-pressure plasma ($4\pi n_0 T_\alpha \ll B_0^2$), when the wavelength is much larger than the Larmor radius of the particles with thermal velocities ($kv_\alpha \ll \omega_\alpha$). The condition for Cerenkov resonance, $\omega = k_\| v_\|$, can be satisfied only for slow waves ($\omega/k_\| \ll c$). In a magnetoactive plasma such waves are longitudinal electrostatic oscillations, and also the Alfven and fast magnetic-sound waves.[5] Since the phase velocity of these waves is much larger than the thermal velocity of the ions, it is sufficient to take into account the absorption of the energy of the waves by the plasma electrons only. (Exceptions are only the Alfven wave in the region of low frequencies, $\omega \ll \omega_i$, and the fast magnetic-sound wave at $\vartheta \ll 1$ and $\omega \ll \omega_i$, which attenuate weakly even when $\omega/k_\| \sim v_i$. However, we shall not consider these cases.)

Retaining in (2.2) only one term with n = 0, we represent (2.1) in the form

$$\frac{\partial f^e}{\partial t} = \frac{\partial}{\partial v_\|}\left\{D\frac{\partial f^e}{\partial v_\|} + D_c\frac{\partial(f^e - f_M^e)}{\partial v_\|}\right\}, \quad (3.1)$$

where D_c is given by (2.5) and

$$D = \frac{\pi e^2}{m_e^2}\sum_k \delta(\omega - k_\| v_\|)\,|E_3 + iv_\perp aE_2/2v_\||^2. \quad (3.2)$$

Equation (3.1) is the usual one-dimensional equation describing the process of "diffusion on waves."

In the quasi-equilibrium state, when we can neglect $\partial f^e/\partial t$ in (3.1), we get

$$\frac{\partial f^e}{\partial v_\|} = \frac{\partial f_M^e}{\partial v_\|}\Big/\left(1 + \frac{D}{D_c}\right). \quad (3.3)$$

Since D and D_c depend on $v_\perp$, the quantity $\partial f^e/\partial v_\|$ will be different for particles with different $v_\perp$. From (3.3) it follows that f^e does not split up into a product $f(v_\perp)f(v_\|)$. Such splitting is possible only when $D/D_c \gg 0$ and when $D/D_c \gg 1$, but in the latter case $f(v_\perp)$ is not a Maxwellian function.

Let us determine first the damping decrement of longitudinal oscillations of the plasma in a magnetic field. Since the anti-hermitian term in the dispersion equation of the longitudinal oscillations (see [5]), together with the damping decrement, is proportional to

$$\left(\int_0^\infty dv_\perp\, v_\perp\frac{\partial f^e}{\partial v_\|}\right)_{\omega = k_\| v_\|},$$

we find, by substituting expression (3.3) for $\partial f^e/\partial v_\|$ into this integral, that the damping decrement of the longitudinal oscillations is equal to

$$\gamma = \gamma_M\Phi, \quad (3.4)$$

where γ_M is the damping decrement in a plasma with Maxwellian particle velocity distribution, and

$$\Phi = \int_0^\infty dx\frac{1}{(1+\eta)}e^{-x}, \quad x = \frac{v_\perp^2}{2v_e^2},$$

$$\eta = \frac{D(v_\perp, v_\|)}{D_c(v_\perp, v_\|)}. \quad (3.5)$$

If the resonant electrons are in the tail of a Maxwellian distribution ($v_\| \gg v_e$), then $D/D_c = y/(x + 1/2)$, where

$$y = \frac{D}{D_0}, \quad D_0 = \frac{4v_e^5}{\tau_c v_\|^3}, \quad D = \pi\left(\frac{e}{m}\right)^2\sum_{\mathbf{k}}|E_3|^2\delta(\omega - k_\| v_\|).$$

In this case

$$\Phi = 1 + y\exp({}^1/_2 + y)\,\mathrm{Ei}(-{}^1/_2 - y),$$

$$\mathrm{Ei}(-x) = -\int_x^\infty dt\,\frac{e^{-t}}{t}. \qquad (3.6)$$

For oscillations with large amplitude ($y \gg 1$) we find that $\Phi \approx 3y/2$. This result follows also from expression[1-3] for $\Phi = (1 + 2y/3)^{-1}$, which is obtained by averaging the collision integral with respect to $v_\perp$ with a Maxwellian distribution function.

The frequencies of the Alfven and fast magnetic-sound waves in a dense plasma ($v_A = B_0/(4\pi n_0 m_i)^{1/2} \ll c$) are determined by the expression[5]

$$\omega^2(k,\vartheta) = {}^1/_2 k^2 v_A{}^2\{1 + \cos^2\vartheta + r\cos^2\vartheta \pm [(1 + \cos^2\vartheta + r\cos^2\vartheta)^2 - 4\cos^2\theta]^{1/2}\},$$

$$v_A{}^2 = H_0{}^2/4\pi n_0 m_i, \quad r = k^2c^2/\Omega_i{}^2, \quad \Omega_i{}^2 = 4\pi e^2 n_0/m_i. \qquad (3.7)$$

Using formula (4.3) from [5] for the dielectric tensor of the plasma, we obtain the damping decrement of the oscillations with frequencies (3.7):

$$\gamma/\omega = \sqrt{\pi}\,\Omega_i{}^2 k_\perp{}^2 v_s{}^2 Q / 2\omega^2\omega_i{}^2\varepsilon_1 P, \qquad (3.8)$$

where

$$v_s = \sqrt{T_e/m_i}, \quad \varepsilon_1 = \Omega_i{}^2/(\omega_i{}^2 - \omega^2),$$

$$P = (1 + \cos^2\vartheta)k^2c^2\omega_i{}^2/\omega^2(\omega_i{}^2 - \omega^2) - \varepsilon_1(2 - \omega^2/\omega_i{}^2),$$

$$Q = 2\left(\frac{k_\parallel{}^2c^2}{\omega^2} - \varepsilon_1\right)\Phi_1 + (\psi^2 + \pi\Phi_2{}^2)^{-1}\left[\frac{k_\parallel{}^2c^2}{\omega^2 - \omega_i{}^2}\times\left(2\Phi_3 - \Phi_2 - \frac{k^2c^2\omega_i{}^2}{\omega^2\Omega_i{}^2}\Phi_2\right) + (2\psi^2\Phi_3 - \psi^2\Phi_2 + \pi\Phi_2\Phi_3{}^2)\left(\varepsilon_1 - \frac{k_\parallel{}^2c^2}{\omega^2}\right)\right],$$

$$\Phi_1 = -\frac{\pi^{3/2}}{4v_e{}^2}\int_0^\infty dv_\perp\, v_\perp{}^5\frac{\partial f^e}{\partial v_\parallel},$$

$$\Phi_2 = -\frac{2\pi^{3/2}}{v_e{}^3}\int_0^\infty dv_\perp\, v_\perp\frac{\partial f^e}{\partial v_\parallel}, \quad \Phi_3 = -\pi^{3/2}\int_0^\infty dv_\perp\, v_\perp{}^3\frac{\partial f^e}{\partial v_\parallel},$$

$$\psi = 1 - 2z_e\exp(-z_e{}^2)\int_0^{z_e} e^{t^2}dt, \quad z_e = \frac{\omega}{\sqrt{2}\,k_\parallel v_e}. \qquad (3.9)$$

Here $\partial f^e/\partial v_\parallel$ is determined by expression (3.3). For a Maxwellian distribution formula (3.8) goes over into the well known expression[5, 8] for the damping decrement.

If the phase velocity of the Alfven wave is considerably larger than the thermal velocity of the electrons, then expression (3.9) for Q assumes the simple form

$$Q = \frac{2\varepsilon_1\omega^2 z_e{}^3}{v_e{}^2\omega_i{}^2}\left(\Phi + \frac{k^2c^2\omega_i{}^2}{\omega^2\Omega_i{}^2}\Phi - 2\tilde{\Phi}\right)\exp(-z_e{}^2), \quad (3.10)$$

where $\Phi = \Phi(y)$ is determined by formula (3.6) and

$$\tilde{\Phi}(y) = 1 - y - y(y + {}^1/_2)e^{y+1/2}\mathrm{Ei}(-y - {}^1/_2),$$

$y = D/D_0$, and D is given by formula (3.2), in which we discard the term $\sim v_\perp a E_2/v_\parallel$.

For weak fields, $D/D_0 \ll 1$, expressions (3.4) and (3.7) go over into the expressions of the linear theory. In the case of strong fields, $D/D_0 \gg 1$, the damping decrements (3.4) and (3.8) are reduced by a factor D/D_0.

4. CYCLOTRON DAMPING

We now consider the damping of electromagnetic waves under conditions of cyclotron resonance, when collisions can be neglected. In the case of narrow wave packets ($\Delta k \ll k$), retaining in (2.1) only the resonant term and introducing new variables

$$\xi_{1,2} = v_\parallel{}^2/(\omega - n\omega_\alpha) \pm v_\perp{}^2/n\omega_\alpha, \qquad (4.1)$$

we represent Eq. (2.1) in the form

$$\frac{\partial f^\alpha}{\partial t} = \frac{\partial}{\partial\xi_1}\left\{D\frac{\partial f^\alpha}{\partial\xi_1}\right\}; \qquad (4.2)$$

$$D = 8\left(\frac{e_\alpha}{m_\alpha}\right)^2\frac{|E_3v_\parallel J_n(a) - iE_2v_\perp J_n'(a) + E_1 nJ_n(a)\omega_\alpha/k_\perp|^2}{\omega^2|v_{gr} - v_\parallel\cos\vartheta|},$$

$$v_{gr} = \partial\omega/\partial k, \quad v_\parallel = (\omega - n\omega_\alpha)/k_\parallel. \qquad (4.3)$$

The diffusion coefficient (4.3) can vanish at certain points $\xi_1 = q_m(\xi_2)$ ($m = 0, 1, 2, \ldots$), $q_{m+1} > q_m$. In the region $q_m < \xi_1 < q_{m+1}$ the number of particles is conserved:

$$\frac{\partial}{\partial t}\int_{q_m}^{q_{m+1}} f^\alpha d\xi_1 = D\frac{\partial f^\alpha}{\partial\xi_1}\Bigg|_{\xi_1=q_m}^{\xi_1=q_{m+1}} = 0,$$

i.e.,

$$\int_{q_m}^{q_{m+1}} f^\alpha d\xi_1 = \int_{q_m}^{q_{m+1}} f_M{}^\alpha d\xi_1.$$

If the oscillation energy differs from zero in the final state, then it follows from (4.2) that $\partial f^\alpha/\partial\xi_1 = 0$ and thus

$$f^\alpha = \frac{4\exp[-(\omega - 2n\omega_\alpha)\xi_2/4v_\alpha{}^2]}{(2\pi)^{3/2}(q_{m+1} - q_m)v_\alpha\omega}\times\left\{\exp\left(-\frac{\omega q_m}{4v_\alpha{}^2}\right) - \exp\left(-\frac{\omega q_{m+1}}{4v_\alpha{}^2}\right)\right\}. \qquad (4.4)$$

Let us find the zeroes of the diffusion coefficient for electromagnetic waves with a phase velocity much larger than the thermal velocity of the electrons ($kv_\alpha/\omega_\alpha \ll 1$) propagating in a low-pressure plasma at $\omega \approx n\omega_\alpha$. Expressing the components E_1 and E_3 in terms of E_2 by means of Maxwell's equations, we find that

$$D=8\left(\frac{e_\alpha}{m_\alpha}\right)^2\frac{v_\perp^2|E_2|^2\,|J_n'+nJ_n\lambda/a|^2}{\omega^2|v_{\rm gr}-v_\parallel\cos\vartheta|},\qquad(4.5)$$

where

$$\lambda=(\varepsilon_1-k^2c^2/\omega^2)/\varepsilon_2,$$

$$\varepsilon_1=1-\sum_{\beta=e,\,i}\frac{\Omega_\beta^2}{\omega^2-\omega_\beta^2},\qquad\varepsilon_2=-\sum_{\beta=e,\,i}\frac{\Omega_\beta^2\omega_\beta}{\omega(\omega^2-\omega_\beta^2)}.$$

For the narrow wave packet under consideration, for which the velocity of the resonant particle lies in the range $v_1 < v_{||} < v_2$, the variable ξ_1 at fixed ξ_2 varies between the limits defined from the inequalities

$$v_1^2\leqslant{}^1\!/_2(\omega-n\omega_\alpha)(\xi_1+\xi_2)\leqslant v_2^2.\qquad(4.6)$$

The diffusion coefficient (4.5) vanishes at the point $v_\perp = 0$, i.e., $\xi_1 = \xi_2$, and also at the point $v_\perp = v \gtrsim \omega_\alpha/k_\perp \gg v_\alpha$, determined from the equation $J_n' + nJ_n\lambda/a = 0$. However, if $|\vartheta - \pi/2| \gg kv_\alpha/\omega_\alpha$, then the points $v_\perp = v_\nu$ do not flow in the interval (4.6) for which $D \neq 0$ (we confine ourselves here to an examination of this most interesting case).

The point $\xi_1 = \xi_2$ falls in the interval (4.6) if

$$(v_\parallel-v_1)v_\parallel/v_\perp^2>(\omega-n\omega_\alpha)/2n\omega_\alpha$$

$$\text{for}\quad\omega-n\omega_\alpha>0,\qquad(4.7a)$$

or else

$$(v_2-v_\parallel)v_\parallel/v_\perp^2>(n\omega_\alpha-\omega)/2n\omega_\alpha$$

$$\text{for}\quad\omega-n\omega_\alpha<0.\qquad(4.7b)$$

If the inequalities (4.7) are not satisfied, then in the final state[1)]

$$f^\alpha=\frac{1}{(2\pi)^{3/2}v_\alpha^3}\exp\left[-\frac{v_\perp^2}{2v_\alpha^2}-\frac{v_1^2}{2v_\alpha^2}+\frac{v_1(v_\parallel-v_1)n\omega_\alpha}{v_\alpha^2(\omega-n\omega_\alpha)}\right]\times\frac{1-e^{-x}}{x}\qquad(4.8)$$

when $\omega - n\omega_\alpha > 0$, and

$$f^\alpha=\frac{1}{(2\pi)^{3/2}v_\alpha^3}\exp\left[-\frac{v_\perp^2}{2v_\alpha^2}-\frac{v_2^2}{2v_\alpha^2}+\frac{v_2(v_2-v_\parallel)n\omega_\alpha}{v_\alpha^2(n\omega_\alpha-\omega)}\right]\times\frac{1-e^{-x}}{x}\qquad(4.9)$$

when $\omega - n\omega_\alpha < 0$. Here $x = \omega(v_2 - v_1)/k_{||}v_\alpha^2$.

Thus, in this case the diffusion leads to a redistribution of the particles only with respect to $v_{||}$ in a narrow velocity interval.

[1)]In the case of an extraordinary wave propagating along a magnetic field ($\vartheta = 0$) in a dense plasma ($\Omega_e^2 \gg \omega(|\omega_e| - \omega)$ for $\omega \approx |\omega_e|$) expression (4.9) can be obtained from the relations given in[2].

If on the other hand the point $\xi_1 = \xi_2$ falls in the interval (4.6), then in the equilibrium state

$$f^\alpha=\frac{2\exp[-v_\parallel^2/2v_\alpha^2+(\omega-n\omega_\alpha)v_\perp^2/2n\omega_\alpha v_\alpha^2]}{(2\pi)^{3/2}v_\alpha\omega[(v_2^2-v_\parallel^2)/(\omega-n\omega_\alpha)+v_\perp^2/n\omega_\alpha]}\times\left\{1-\exp\left[-\frac{\omega}{2v_\alpha^2}\left(\frac{v_2^2-v_\parallel^2}{\omega-n\omega_\alpha}+\frac{v_\perp^2}{n\omega_\alpha}\right)\right]\right\}\qquad(4.10)$$

for $\omega - n\omega_\alpha > 0$ and

$$f^\alpha=\frac{2\exp[-v_\parallel^2/2v_\alpha^2+(\omega-n\omega_\alpha)v_\perp^2/2n\omega_\alpha v_\alpha^2]}{(2\pi)^{3/2}v_\alpha\omega[(v_1^2-v_\parallel^2)/(\omega-n\omega_\alpha)+v_\perp^2/n\omega_\alpha]}\times\left\{1-\exp\left[-\frac{\omega}{2v_\alpha^2}\left(\frac{v_1^2-v_\parallel^2}{\omega-n\omega_\alpha}+\frac{v_\perp^2}{n\omega_\alpha}\right)\right]\right\}\qquad(4.11)$$

for $\omega - n\omega_\alpha < 0$.

Let us consider the equilibrium distributions (4.8)–(4.11) obtained above in the particular cases of "narrow" ($v_2 - v_1 \ll v_\alpha k_{||}v_\alpha/\omega_\alpha$) and "broad" ($v_2 - v_1 \gg v_\alpha k_{||}v_\alpha/\omega_\alpha$) wave packet.

For a "narrow" wave packet the inequalities (4.7) cannot be satisfied for $v_\perp \sim v_\alpha$; the "diffusion on the waves" leads in this case, as can be seen from expressions (4.8) and (4.9), to the formation of a "plateau" with respect to $v_{||}$ in a narrow interval, and does not change the distribution with respect to $v_\perp$ in the region of not very small $v_\perp$.

For "broad" wave packets the inequalities (4.7) are satisfied practically for all $v_{||}$ and $v_\perp \lesssim v_\alpha$ (these inequalities can be violated only for values of $v_{||}$ that are close to v_1 or to v_2); the "diffusion on the waves" leads in this case to a strong change of the distribution function of the particles, both with respect to $v_{||}$ and with respect to $v_\perp$. It is precisely the case of "broad" wave packets which is most interesting from the point of view of heating of a plasma under cyclotron-resonance conditions.

Let us consider the relaxation process for "broad" wave packets in greater detail. During the initial state we can neglect in the right side of Eq. (4.2) the terms $k_{||}\partial/\partial v_{||} \sim k_{||}/\Delta v$ compared with the terms $(n\omega_\alpha/v_\perp)\partial/\partial v_\perp \sim n\omega_\alpha/v_\alpha^2$. Then Eq. (4.2) takes the form

$$\frac{\partial f^\alpha}{\partial t}=\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\left(D_\perp\frac{\partial f^\alpha}{\partial v_\perp}\right),$$

$$D_\perp=\frac{1}{2}\left(\frac{e_\alpha}{m_\alpha}\right)^2\frac{v_\perp|E_1nJ_n/a-iE_2J_n'+E_3v_\parallel J_n/v_\perp|^2}{|v_{\rm gr}-v_\parallel\cos\vartheta|}.\qquad(4.12)$$

Thus, diffusion of particles with respect to $v_\perp$, due to cyclotron acceleration of the particles, occurs for "broad" packets during the initial stage.

For long waves and resonant particles with not too large a velocity $v_\perp$ ($a \ll 1$) Eq. (4.2) can be represented in the form

$$\frac{\partial f^{\alpha}}{\partial \tau}=\frac{\partial}{\partial x}\left(x^{n}\frac{\partial f^{\alpha}}{\partial x}\right), \tag{4.13}$$

where

$$x=\frac{v_{\perp}^{2}}{2v_{\alpha}^{2}}, \quad \tau=\int_{0}^{t}D(t)\,dt,$$

$$D(t)=\left(\frac{e_{\alpha}}{m_{\alpha}}\right)^{2}\left(\frac{k_{\perp}v_{\alpha}}{\omega_{\alpha}}\right)^{2(n-1)} \times\frac{|E_{1}-iE_{2}+k_{\perp}v_{\parallel}E_{3}/n\omega_{\alpha}|^{2}}{2^{n+1}[(n-1)!]^{2}v_{\alpha}^{2}|v_{\mathrm{gr}}-v_{\parallel}\cos\vartheta|}.$$

We present the solutions of Eq. (4.13) for n = 1, n = 2, and n = 4:

$$f^{\alpha}=\frac{\exp[-v_{\parallel}^{2}/2v_{\alpha}^{2}-x/(1+\tau)]}{(2\pi)^{3/2}v_{\alpha}^{3}(1+\tau)}, \quad n=1; \tag{4.14}$$

$$\frac{\partial f^{\alpha}}{\partial x}=-\frac{\sqrt{\pi}\exp(-v_{\parallel}^{2}/2v_{\alpha}^{2})}{(2\pi)^{3/2}v_{\alpha}^{3}x\sqrt{\tau}}\int_{0}^{\infty}dx'\,e^{-x'}\left(\frac{x}{x'}\right)^{-[\tau-\ln(x/x')]^{2}/4\tau}, \quad n=2; \tag{4.15}$$

$$\frac{\partial f^{\alpha}}{\partial x}=-\frac{\sqrt{\pi}\exp(-v_{\parallel}^{2}/2v_{\alpha}^{2})}{(2\pi)^{3/2}v_{\alpha}^{3}x^{3}\sqrt{\tau}}\int_{0}^{\infty}dx'\,x'e^{-x'} \times\left\{\exp\left[-\left(\frac{1}{x'}-\frac{1}{x}\right)^{2}\frac{1}{4\tau}\right]+\exp\left[-\left(\frac{1}{x'}+\frac{1}{x}\right)^{2}\frac{1}{4\tau}\right]\right\}, \quad n=4. \tag{4.16}$$

It follows from (4.14) that in the case of single cyclotron resonance an increase takes place in the "transverse" temperature of the resonant particles. The damping of the oscillations occurs in this case just as in the linear theory ($\epsilon(t) = \epsilon(0)\exp(2\gamma_{M}t)$).

In the case of double resonance the damping of the field is given by $\gamma = -|\gamma_{M}|e^{8\tau}$ and

$$\varepsilon(t)=\varepsilon(0)\frac{1+\zeta}{\exp[2|\gamma_{M}|(1+\zeta)t]+\zeta},$$

$$\zeta=\frac{(e_{\alpha}/m_{\alpha})^{2}(k_{\perp}/\omega_{\alpha})^{2}|E_{2}(0)|^{2}|\lambda+1|^{2}}{8|\gamma_{M}|\,|v_{\mathrm{gr}}-v_{\parallel}\cos\vartheta|}. \tag{4.17}$$

(An expression for the linear damping decrement γ_{M} is given in [5].) It follows from (4.17) that in the nonlinear case the damping is more rapid than in the linear case.

Let us consider now the effect of Coulomb collisions on the damping of electromagnetic waves under the cyclotron-resonance conditions $\omega \approx n\omega_{\alpha}$. For simplicity we confine ourselves to examination of "narrow" wave packets. Since in this case $\partial/\partial v_{\parallel} \approx (4/k_{\parallel})\partial/\partial\xi_{1}$, expression (4.2) must be replaced when account is taken of the collision integral in the form (2.4) or (2.8), by the equation

$$\frac{\partial f^{\alpha}}{\partial t}=\frac{\partial}{\partial\xi_{1}}\left\{D\frac{\partial f^{\alpha}}{\partial\xi_{1}}+16\frac{D_{c}}{k_{\parallel}^{2}}\frac{\partial(f^{\alpha}-f_{M}^{\alpha})}{\partial\xi_{1}}\right\};$$

$$D=\tilde{D}\left(\frac{v_{\perp}}{\sqrt{2}\,v_{\alpha}}\right)^{2n},$$

$$\tilde{D}=\frac{(e_{\alpha}/m_{\alpha})^{2}(k_{\perp}v_{\alpha}/\omega_{\alpha})^{2(n-2)}v_{\alpha}^{4}|1+\lambda|^{2}|E_{2}|^{2}}{2^{n-3}[(n-1)!]^{2}\omega^{2}|v_{\mathrm{gr}}-v_{\parallel}\cos\vartheta|}. \tag{4.18}$$

For the "equilibrium" state we find from this that

$$\frac{\partial f^{\alpha}}{\partial\xi_{1}}=\frac{\partial f_{M}^{\alpha}/\partial\xi_{1}}{1+\eta}, \quad \eta=\frac{Dk_{\parallel}^{2}}{16D_{c}}. \tag{4.19}$$

Since the anti-hermitian terms in the dielectric tensor are integrals with respect to $v_{\perp}$, of the form

$$\int_{0}^{\infty}v_{\perp}^{2n+1}\exp(-v_{\perp}^{2}/2v_{\alpha}^{2})(\partial f^{\alpha}/\partial\xi_{1})\,dv_{\perp},$$

the damping decrement of the "narrow" wave packets in the "equilibrium" state is determined by the expression

$$\gamma=\gamma_{M}F, \tag{4.20}$$

where

$$F=\frac{1}{n!}\int_{0}^{\infty}e^{-x}x^{n}\frac{1}{1+\eta}\,dx, \quad x=\frac{v_{\perp}^{2}}{2v_{\alpha}^{2}}. \tag{4.21}$$

In the case of electron cyclotron resonance $\omega \approx n|\omega_{e}|$ we have when $v_{\parallel} \gg v_{e}$

$$\eta=\eta_{0}x^{n}/(1+2x), \quad \eta_{0}=\tilde{D}k_{\parallel}^{2}\tau_{c}v_{\parallel}^{3}/32v_{e}^{5}. \tag{4.22}$$

When n = 1, using (4.22), we get

$$F=x[3-2x+x(1-2x)e^{x}\mathrm{Ei}(-x)], \tag{4.23}$$

where $x = 1/(2+\eta_{0})$. For weak fields ($\eta_{0} \to 0$) $F \approx 1$ and $\gamma = \gamma_{M}$. For strong fields ($\eta_{0} \gg 1$) the damping decrement decreases strongly: $F = 3/\eta_{0}$.

An explicit expression for the function F can be obtained also for n = 2:

$$F=\frac{3}{2\eta_{0}}\left\{1-\frac{4}{3\eta_{0}}-\frac{2(\eta_{0}-2)}{\eta_{0}\sqrt{1-\eta_{0}}}[(x_{1}+p)e^{-x_{1}}\mathrm{Ei}(x_{1}) -(x_{2}+p)e^{-x_{2}}\mathrm{Ei}(x_{2})]\right\}, \tag{4.24}$$

where

$$p=(\eta_{0}-4)/4(\eta_{0}-2), \quad x_{1,2}=(-1\pm\sqrt{1-\eta_{0}})/\eta_{0}.$$

When $\eta_{0} \ll 1$ it follows from (4.24) that F = 1 and for $\eta_{0} \gg 1$ we get $F = 3/2\eta_{0}$. The function F approaches asymptotically the value $F = 3/\eta_{0}n!$ for n = 1, 2, ... when $\eta_{0} \gg 1$ and $v_{\parallel} \ll v_{e}$.

For ion cyclotron resonance, $\omega \approx n\omega_i$, we get for $v_{\parallel} \gg v_i$

$$\eta = \eta_0 x^n / (1+x), \qquad \eta_0 = \bar{D} k_{\parallel}^2 \tau_c v_{\parallel}^3 / 32 v_i^5. \quad (4.25)$$

Using expression (4.25) for η, we find from (4.21) that

$$F = x[2 - x + e^x \mathrm{Ei}(-x) x (1-x)],$$

$$x = 1/(1+\eta_0), \quad n = 1, \qquad (4.26)$$

$$F = \frac{1}{\eta_0}\Big\{1 - \frac{1}{2\eta_0} + \frac{(2\eta_0 - 1)}{\eta_0 \sqrt{1-4\eta_0}}[(x_1 + p)e^{-x_1}\mathrm{Ei}(x_1)$$

$$-(x_2 + p)e^{-x_2}\mathrm{Ei}(x_2)]\Big\}, \quad n = 2, \qquad (4.27)$$

where

$$x_{1,2} = (-1 \pm \sqrt{1-4\eta_0})/2\eta_0, \quad p = (\eta_0 - 1)/(2\eta_0 - 1).$$

In the case of strong fields ($\eta_0 \gg 1$) and when $\omega \approx n\omega_i$ and $v_{\parallel} \gg v_i$ we have $F = 2/\eta_0 n!$.

Thus, for narrow wave packets the diffusion of the resonant particles by the waves under conditions of cyclotron resonance always leads to a decrease in the damping decrement, and in strong fields the cyclotron damping is determined by the collisions: $\gamma \sim \gamma_M/\eta_0 \sim 1/\tau_c$.

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