THEORY OF THE HYDROMAGNETIC DYNAMO

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The Cowling theorem regarding the impossibility of a stationary axially-symmetrical hydromagnetic dynamo is formulated as a theorem stating the impossibility of a short-circuited axially-symmetrical dynamo, defined as a hydromagnetic dynamo with a zero electric field. Formulated in this way, Cowling's theorem can be extended to include the arbitrary three-dimensional case. It is concluded that generation of a magnetic field in the stationary case must necessarily involve separation of the electric charges in the fluid; in other words an electric field must also appear in space along with the magnetic field.

In magnetohydrodynamics, there is a theorem by Cowling[1,2], according to which a stationary axially symmetrical hydromagnetic dynamo is impossible. In the initial formulation of the problem, Cowling assumed that the azimuthal components of the magnetic field H_φ and of the fluid velocity v_φ are equal to zero. Backus and Chandrasekhar[2] have generalized Cowling's theorem to include the case when H_φ and v_φ differ from zero, while Braginskiĭ[3] extended it to the nonstationary case.

As initially formulated, Cowling's theorem can be phrased differently. Namely, from the condition $H_\varphi = 0$ and $v_\varphi = 0$ it follows that the components of the electric field in the meridional planes are equal to zero, and from the axial symmetry condition it follows also that $E_\varphi = 0$ and consequently $\mathbf{E} \equiv 0$. A hydromagnetic dynamo with zero electric field can be called "short-circuited," since the density $\mathbf{j}$ at each point of the fluid is determined by the value of the emf and the conductivity σ at the same point. Thus, Cowling's theorem can be formulated as follows: a short-circuited, axially symmetrical hydromagnetic dynamo is impossible.

In such a formulation, Cowling's theorem can be generalized to an arbitrary three-dimensional case.

The equations of a short-circuited hydromagnetic dynamo ($\mathbf{E} = 0$) in the kinematic formulation[3] are

$$\operatorname{div} \mathbf{H} = 0, \tag{1}$$

$$\operatorname{rot} \mathbf{H} = [\mathbf{uH}], \tag{2*}$$

where $\mathbf{u} = \mathbf{v}/D_m$, $\mathbf{v}$ is the velocity field of the conducting liquid and D_m the diffusion coefficient of the magnetic field. It is assumed that the liquid occupies a certain finite volume V of space. Outside this volume $\mathbf{u} \equiv 0$. It is required to find a continuous solution of equations (1) and (2) with zero boundary condition for H at infinity.

It follows from (2) that $\mathbf{H} \operatorname{curl} \mathbf{H} \equiv 0$, and therefore $\mathbf{H}$ can be represented in the form

$$\mathbf{H} = \psi \nabla \Phi. \tag{3}$$

Outside the volume V Eq. (2) goes over into $\operatorname{curl} \mathbf{H} = 0$, and consequently, outside V the field $\mathbf{H}$ can be represented in the form $\mathbf{H} = \nabla\Phi'$. Without loss of generality we can assume that $\Phi' \equiv \Phi$. Then the representation (3) will hold true in all space if we put $\psi = 1$ outside V. The zero condition for $\mathbf{H}$ at infinity then yields

$$\lim \Phi = 0 \quad \text{as} \quad r \to \infty. \tag{4}$$

Substituting (3) in (1) and (2) we obtain

$$\psi \Delta \Phi + (\nabla \psi \nabla \Phi) = 0, \tag{5}$$

$$[\nabla \psi \nabla \Phi] = \psi [\mathbf{u} \nabla \Phi]. \tag{6}$$

The application of the divergence operation to (3) necessitates by the same token that Φ be some twice-differentiable function. As regards the function of ψ, its properties are determined in many respects by the properties of the vector function $\mathbf{u}$.

Let us assume that the system (5) and (6) has nontrivial solutions. We can then show that, subject to certain assumptions concerning the vector function $\mathbf{u}$, there should be satisfied in the volume V the conditions

$$\psi \neq 0, \quad |\psi| \neq \infty, \quad |(\nabla\psi)_\tau| \neq \infty, \tag{7}$$

where $(\nabla\psi)_\tau$ is the projection of the gradient on the direction of $\mathbf{H}$. Indeed, assume that the conditions (7) are not satisfied at some point inside V. We

*rot ≡ curl; $[\mathbf{uH}] = \mathbf{u} \times \mathbf{H}$.

introduce at this point a Cartesian coordinate system with z axis directed along **H**. In this coordinate system, Eq. (6) can be written in the form

$$\frac{1}{\psi}\frac{\partial\psi}{\partial x}=u_x, \quad \frac{1}{\psi}\frac{\partial\psi}{\partial y}=u_y. \tag{8}$$

Integrating the system (8), we obtain

$$\psi=\varphi(z)\exp\left\{\int u_x dx+\int u_y dy-\iint\frac{\partial u_x}{\partial y}dx\,dy\right\}, \tag{9}$$

$$(\nabla\psi)_\tau=\frac{\partial\psi}{\partial z}=\left(\frac{d\varphi}{dz}+\varphi\frac{\partial F}{\partial z}\right)e^{F(x,y,z)}, \tag{10}$$

where $\varphi(z)$ is an arbitrary function of the coordinate z and F(x, y, z) is the expression in the curly brackets of (9).

We note that if we reverse the order of integration of (8), then $\partial u_x/\partial y$ will be replaced in (9) by $\partial u_y/\partial x$. This does not change the values of ψ, since $\partial u_x/\partial y = \partial u_y/\partial x$, as follows from (2) to which we apply the divergence operation **H** curl **u** = 0, that is, $(\text{curl}\,\mathbf{u})_z = 0$ in the chosen coordinate system.

If the vector function **u** is such that

$$F(x,y,z)=\int u_x\,dx+\int u_y\,dy-\iint\frac{\partial u_x}{\partial y}dx\,dy\neq\pm\infty,$$

$$\partial F/\partial z\neq\pm\infty, \tag{11}$$

then, as follows from (9) and (10), violation of condition (7) can result only if the arbitrary function $\varphi(z)$ assumes values zero or $\pm\infty$, or else if its derivative becomes equal to $\pm\infty$. These values of $\varphi(z)$ or $d\varphi/dz$ will remain the same at all points of a surface orthogonal to the magnetic field, independently of the value of **u**. (The proof of the existence of surfaces orthogonal to **H**, for fields for which the condition **H** curl **H** = 0 is satisfied, can be found in[4].) If the surface goes outside the region V, we arrive at a contradiction, for outside V we have $\psi = 1$ and $\partial\psi/\partial z = 0$. Consequently, the conditions (7) are satisfied at those points of the volume V through which it is possible to pass surfaces orthogonal to **H** and going outside V.

We shall show that the surfaces orthogonal to **H** go outside the limits of V. Indeed, assume that there is inside V an orthogonal surface that does not go outside the limits of the region V. This surface should be closed and smooth. The intersection of the orthogonal surfaces is excluded by the requirement that the magnetic field be unique. For an arbitrary point lying inside a closed orthogonal surface, the orthogonal surface should also be closed, etc. These surfaces which are imbedded in one another should contract to a certain point, which will be a singular point of the magnetic field. But this is impossible because of the continuity of **H**.

Thus the conditions (7) are satisfied everywhere in space if **u** is such that the conditions (11) are satisfied inside V, the condition u = 0 is satisfied outside V, and **H** is a continuous function.

But when conditions (7) are satisfied together with boundary conditions (4), Eq. (5) has as a twice-differentiable function only the trivial solution[5] $\Phi \equiv 0$. This contradicts the assumption made that the system (5) and (6) has a nontrivial solution. Consequently, a short circuited hydromagnetic dynamo is impossible.

The theorem proved leads to the following conclusion: when a magnetic field is generated in the stationary case, there must occur a separation of electric charges in the liquid, that is, along with the magnetic field there should be produced in space also an electric field. Indeed, in the stationary case, curl **E** = 0; further, div **E** = $4\pi\rho$, and if $\rho(r) = 0$ then **E** = 0 and a hydromagnetic dynamo is impossible.

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