ZERO SOUND OSCILLATIONS IN A SYSTEM OF INTERACTING BOSE AND FERMI PARTICLES

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It is shown that the interaction between Fermi and Bose particles can lead to the appearance of a new acoustic branch of excitations in the Bose system, the velocity of propagation and anisotropy of which are determined by the shape of the Fermi surface.

It is well known that the account of interactions between Bose and Fermi systems in a metal leads to such an essential rearrangement of the spectrum of Fermi particles (electrons) that a qualitatively new phenomenon appears—superconductivity. The reaction of a Fermi system on the excitation spectrum of a Bose system (phonons) has received much less attention; it usually reduces to the renormalization of the sound velocity and to its attenuation. Kohn[1] and Afanas'ev and Kagan[2] have noted that the electron-phonon interaction leads to interesting singularities in the phonon spectrum. We shall show here, by the example of zero sound oscillations,[1)] that the effect of the Fermi system on the Bose system can be more significant: in the spectrum of the latter, complete new branches of excitations can appear which do not exist if the interaction with the Fermi system is absent.

Let us consider the complete Green's function of the phonons:

$$D^{-1} = D_0^{-1} - \Pi, \quad D_0 = c^2\mathbf{k}^2(\omega^2 - c^2\mathbf{k}^2)^{-1},$$

$$\Pi = -2ig\int G(p+k)G(p)\Gamma(p+k, p, k)\frac{d^4p}{(2\pi)^4},$$

$$k = (\mathbf{k}, \omega). \tag{1}$$

The function $\Pi(\omega, \mathbf{k})$ completely describes the effect of the Fermi system on the Bose system. If we look for the pole of the D function close to the pole of the zeroth approximation $\omega = c|\mathbf{k}|$, then, as Migdal has shown,[7] the electron-phonon vertex and the electron Green's function G under the integral sign in Π can be replaced by their values g and $G^{(0)}$ in the zeroth approximation. Then the renormalization of the sound velocity and its attenuation are determined by the real and imaginary parts of the function.

$$\Pi_2 = -2ig^2\int G^{(0)}(p+k)G^{(0)}(p)\frac{d^4p}{(2\pi)^4}. \tag{2}$$

For $\omega \ll v_F|\mathbf{k}|$, it has been shown[7] that $\mathrm{Re}\,\Pi \approx -2\xi$ and $|\mathrm{Im}\,\Pi| \ll |\mathrm{Re}\,\Pi|$ ($\xi = mp_F g^2/2\pi^2$).

As will be shown below,[2)] the function (2) is large in the vicinity of the frequency $\omega = v_F|\mathbf{k}|$, where v_F is the velocity of the electrons on the Fermi surface. This suggests that the D function can have a new pole in the region of high frequencies.

To clarify this question, it is necessary to obtain a more exact expression for the electron-phonon vertex Γ which, as is well known,[3] differs at high frequencies from its zeroth approximation g. We shall entirely ignore the Coulomb interaction between the electrons; that is, we shall assume the Fermi system to be neutral.[3)] Then the integral equation for the vertex part Γ can be represented graphically as in Fig. 1.

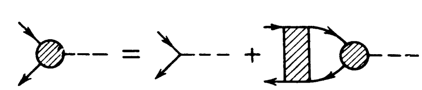

FIG. 1

The shaded rectangle $\Gamma^{(1)}$ denotes the set of "irreducible" graphs which do not contain the horizontal singular lines $G(p+k)G(p)$ and $D_0(k)$. For transitions near the Fermi surface, the function $\Gamma^{(1)}$, as estimates show, can be replaced by

[1)]We recall that the zero sound oscillations are Fermi-system oscillations that can propagate at low temperatures, when the hydrodynamic sound waves are strongly damped. Zero sound was first investigated by Landau[3,4] and later by Silin[5] and Gor'kov and Dzyaloshinskii.[6]

[2)]The exact expression for $\Pi_2(\omega, \mathbf{k})$ for spherical and cylindrical Fermi surfaces is derived in Appendices A and B. Attention is called to the fact that the function Π_2 changes sign at $\omega \sim v_F|\mathbf{k}|$ and becomes large in absolute value.

[3)]For the role of Coulomb interaction, see below.

the constant $\Gamma^{(1)} = 2g^{(1)}g^2(g^{(1)} \sim 1/2)$, and the integral equation (Fig. 1) has the solution

$$\Gamma(p+k, p, k) = g \,/\, (1 - g^{(1)}\widetilde{\Pi}), \tag{3}$$

$$\widetilde{\Pi} = -2ig^2 \int G(p)G(p+k)\, d^4p/(2\pi)^4. \tag{4}$$

Substituting (3) in (1), we find

$$\Pi = \widetilde{\Pi}/(1 - g^{(1)}\widetilde{\Pi}) \tag{5}$$

and

$$D = \frac{D_0(1 - g^{(1)}\widetilde{\Pi})}{1 - (g^{(1)} + D_0)\,\widetilde{\Pi}}\,. \tag{6}$$

In Eqs. (5), (6) and below, the function $\widetilde{\Pi}$ can be replaced by (2) with accuracy to $\sqrt{m/M}$. We see that the self-energy part of the phonons Π is described by the integral (2) only when this integral is small. In the region of frequencies where this integral is large, the D function has a pole described by the equation

$$\widetilde{\Pi} = (g^{(1)} + D_0)^{-1}. \tag{7}$$

This pole is identical with the zero-sound of the four-fermion vertex part Γ_F, the integral equation for which is shown in Fig. 2. In the approximation $\Gamma^{(1)}$ = const, this equation has the solution

$$\Gamma_F = \frac{2g^2(g^{(1)} + D_0)}{1 - (g^{(1)} + D_0)\widetilde{\Pi}}. \tag{8}$$

The existence of the high-frequency pole of the D function means that the zero sound (collective oscillations of the Fermi system) is accompanied by excitations of the Bose system (lattice vibrations). It can be shown that the observation of the latter is experimentally simpler than the direct observation of zero sound.

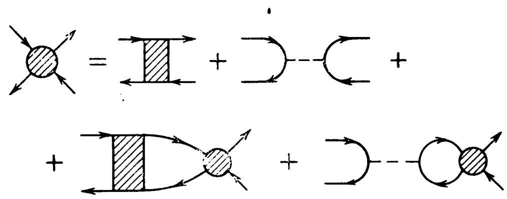

FIG. 2

If $\omega = \omega(k)$ is a solution of Eq. (7), then the intensity of the new branch of Bose excitations is characterized by the value of the numerator in the representation of the D function (6) close to the pole in the form

$$D = \frac{A}{\omega^2 - \omega^2(k)}, \quad A = -\frac{D_0^2}{(g^{(1)} + D_0)^2}\frac{1}{d\widetilde{\Pi}/d\omega^2}. \tag{9}$$

Inasmuch as the function $D_0 \sim (c/v_F)^2$, for $\omega \sim v_F|k|$, the intensity of the new branch is generally speaking, very small and consequently its observation is difficult. However, if the shape of the Fermi surface possesses an essential anisotropy, then for certain directions of the wave vector the spectrum $\omega(k)$ approaches the spectrum of acoustical vibrations, and the value of A for these directions increases sharply. Actually for small values of the 4-vector $k = (\mathbf{k}, \omega)$ the function (2) can be represented in the following form:

$$\Pi_2 = 2\zeta\left[-1 + \frac{1}{S}\int \frac{\omega^2}{\omega^2 - (\mathbf{vk})^2}\, dS\right],$$

$$\zeta = g^2 S, \quad S = \int \frac{d^3p}{(2\pi)^3}\delta(E_p - \mu),$$

$$v_\alpha = \frac{\partial E}{\partial p_\alpha}, \quad d^3p = (2\pi)^3\, dE\, dS. \tag{10}$$

For the solution of Eq. (7), the principal role is played by the region of values of Π_2 close to the "extremal point" $\omega = \omega_0$, at which Π_2 goes to infinity. If the Fermi surface is close to spherical, then, for small values of $\xi = (\omega^2 - \omega_0^2)/\omega_0^2$, we get

$$\Pi_2 = \zeta \ln(\gamma_a/\xi), \qquad \omega_0 = v_F|k|. \tag{11a}$$

If the Fermi surface has regions of cylindrical shape, then

$$\Pi_2 = \zeta\sqrt{\gamma_b/\xi}, \qquad \omega_0 = v_F|k_\perp| \tag{11b}$$

($k_\perp$ is the projection of the vector k on the plane perpendicular to the cylinder).

Finally, if the Fermi surface contains regions of low curvature, then

$$\Pi_2 = \zeta\gamma_c/\xi, \qquad \omega_0 = v_F|k_z| \tag{11c}$$

(the z axis is perpendicular to the plane part of the Fermi surface).

In Eq. (11a), the value of γ_a is of the order of unity. In Eqs. (11b), (11c), the values of $\sqrt{\gamma_b}$ and $\sqrt{\gamma_c}$ are of the order of the ratio of the size of the region of the Fermi surface of cylindrical (plane) shape to the size of the entire Fermi surface.

Substituting (11) in (7) and neglecting D_0 in comparison with $g^{(1)}$, we find the spectrum of zero-sound excitations corresponding to the cases a, b, and c:

$$\omega^2(k) = \omega_0^2(1 + \xi),$$

$$\xi_a = \gamma_a \exp(-1/\zeta g^{(1)}), \qquad \omega_0 = v_F|k|, \tag{12a}$$

$$\xi_b = \gamma_b(\zeta g^{(1)})^2, \qquad \omega_0 = v_F k_\perp, \tag{12b}$$

$$\xi_c = \gamma_c \zeta g^{(1)}, \qquad \omega_0 = v_F k_z. \tag{12c}$$

The expressions (12) are derived under the assumption that $\zeta g^{(1)} \ll 1$ and $\omega_0 \gg c|k|$. Equations (12) indicate the sharp anisotropy of the zero-sound oscillations in the case of an anisotropic Fermi surface.

For a spherical Fermi surface with a quadratic dispersion law, we also found (see Appendix A) the damping of zero-sound oscillations, which begins at $k = 2p_F\xi_a$ and becomes comparable with the real part of the frequency for $k \sim p_F \zeta g^{(1)}$.

Our conclusions are directly applicable only to neutral systems of the type of a solution of He^3 in He^4. In metals, for wavelengths

$$k^2 < k_c^2 = p_F^2 (4\pi^2 \zeta g^{(1)})^{-1}$$

the Coulomb interaction transforms zero-sound oscillations into plasma oscillations. Nevertheless, in cases with an anisotropic Fermi surface, a region of directions of zero-sound oscillations can exist for which the modulus of the wave vector k is large enough for the Coulomb interaction to be weak and the projection of the wave vector $k_\perp$ (or k_z in the plane case) is such that the frequency $\omega(k)$ remains small in comparison with the Fermi energy. The latter is necessary, inasmuch as all the conclusions were drawn under the assumption that the 4-vector k does not move the Fermi particle far from the Fermi surface.

In conclusion we note that the preliminary estimates show that the new branch of phonon oscillations leads to an essential change in the temperature dependence of the heat capacity of metals with anisotropic Fermi surfaces.

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APPENDIX A

Let us find the solution of the dispersion equation (7) for a spherical Fermi surface. We start from an electron spectrum of the form

$$E_p = \varepsilon_F + \frac{1}{2m}(p^2 - p_F^2). \tag{A.1}$$

This allows us to determine not only the form of the spectrum of zero-sound oscillations but also to find their damping in the end region of the spectrum.

For real values of ω, the integral (2) is equal to

$$\operatorname{Re}\Pi_2 = -\frac{\zeta}{4\varkappa}\left\{4\varkappa + (1-\alpha_-^2)\ln\left|\frac{1-\alpha_-}{1+\alpha_-}\right| + (1-\alpha_+^2)\ln\left|\frac{1+\alpha_+}{1-\alpha_-}\right|\right\},$$

$$\operatorname{Im}\Pi_2 = -\zeta\frac{\pi\operatorname{sign}\omega}{4\varkappa}\{(1-\alpha_-^2)\theta(1-\alpha_-^2) - (1-\alpha_+^2)\theta(1-\alpha_+^2)\}. \tag{A.2}$$

Here

$$\varkappa = k/2p_F, \qquad \alpha_\pm = \alpha \pm \varkappa, \qquad \alpha = \omega / v_F k.$$

The shape of the functions $\operatorname{Re}\Pi_2$, $\operatorname{Im}\Pi_2$ is shown in Fig. 3.

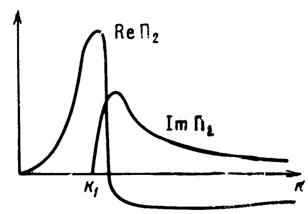

FIG. 3

Analytic continuation of (A.2) in the complex plane gives

$$\Pi_2 = \frac{\zeta}{4\varkappa}\left\{(1-\alpha_-^2)\ln\frac{\alpha_-+1}{\alpha_--1} - (1-\alpha_+^2)\ln\frac{\alpha_++1}{\alpha_+-1} - 4\varkappa\right\},$$

$$\Pi_2(\alpha) = \Pi_2(-\alpha). \tag{A.3}$$

The expression (A.3) satisfies the condition $\operatorname{Im}\Pi_2 = 0$ for $\alpha_-^2 > 1$ and $\alpha_+^2 > 1$, and is analytic in the complex plane of the variable α with the two cuts

$$(-1-\varkappa+i\delta, \quad 1-\varkappa+i\delta),$$

$$(-1+\varkappa-i\delta, \quad 1+\varkappa-i\delta),$$

where $\delta \to +0$. In view of the parity of (16) relative to α, one can limit oneself to consideration of the case $\operatorname{Re}\alpha > 0$. For $\kappa \to 0$, (A.3) transforms into the well known expression[8]

$$\Pi_2 = \zeta\left\{\frac{\alpha}{2}\ln\frac{\alpha+1}{\alpha-1} - 1\right\}. \tag{A.4}$$

In this limit, the dispersion equation of zero-sound oscillations (see Eq. (7))

$$\Pi_2 = 1/g^{(1)} \tag{A.5}$$

has a real solution, found by Landau[3,4]

$$\alpha = 1 + \frac{2}{e}\varkappa_1, \qquad \varkappa_1 = \exp\left\{-\left(\frac{1}{\zeta g^{(1)}} + 1\right)\right\}. \tag{A.6}$$

Equation (A.4) was obtained under the assumption that $\alpha - 1 \gg \kappa$ and, as is seen from (A.6), is valid for $\kappa \ll \kappa_1$. For $\kappa \gtrsim \kappa_1$, it is necessary to seek a solution of (A.5) from the exact expression (A.3) for Π_2. It is evident that in this case (as before, $\kappa \ll 1$) the solution is close to unity. Therefore, it is natural to look for a solution (A.5) in the form

$$\alpha = 1 + \varkappa z. \tag{A.7}$$

With respect to the unknown quantity z, Eq. (A.5) is rewritten in the form

$$\omega \equiv \frac{1}{2}[(z-1)\ln(z-1)-(z+1)\ln(z+1)] + \ln 2 = \ln\frac{\varkappa}{\varkappa_1}. \tag{A.8}$$

This function is analytic in the z plane with the cuts $(-\infty, -1+i\delta)$ and $(-\infty, 1-i\delta)$. For $z \gg 1$, Eq. (A.8) has the solution $z = 2\kappa_1/e\kappa$, which is identical with (A.6).

The solution of Eq. (A.8) remains real with increase in κ up to the value $\kappa = \kappa_1$, which corresponds to the solution $z = 1$. For $\mathrm{Re}\, z < 1$, Eq. (A.8) has only complex roots. It is convenient to introduce the new real variables x and y:

$$z = 1 - 2(x + iy). \tag{A.9}$$

In the variables x, y, Eq. (A.8) takes the following form:

$$\mathrm{Re}\,\omega = -\frac{x}{2}\ln(x^2+y^2) - \frac{1}{2}(1-x)\ln[(1-x)^2+y^2] + y\left(\pi + \mathrm{arctg}\frac{y}{x}\right) + y\,\mathrm{arctg}\frac{y}{1-x} = \ln\frac{\varkappa}{\varkappa_1}, \tag{A.10*}$$

$$\mathrm{Im}\,\omega = \frac{1}{2}y\ln\frac{y^2+(1-x)^2}{y^2+x^2} - x\left(\pi + \mathrm{arctg}\frac{y}{x}\right) + (1-x)\,\mathrm{arctg}\frac{y}{1-x} = 0. \tag{A.11}$$

Equation (A.11) defines (in implicit fashion) a monotonically increasing function $y = y(x)$, and $y(0) = 0$, $y(1/4) = \infty$. Close to the end of the spectrum, where $y \gg 1$, the system (A.10), (A.11) has the solution

$$y = \frac{1}{2\pi}\ln\frac{\varkappa}{\varkappa_1}, \quad x = \frac{1}{4}\left(1 - 1\Big/\ln\frac{\varkappa}{\varkappa_1}\right). \tag{A.12}$$

The spectrum and the damping of zero sound at values of the wave vector $k \gg 2p_F\kappa_1$ are determined by the formulas

$$\omega = v_F k[1 + \varkappa(1-2x)], \tag{A.13}$$

$$\gamma = 2v_F k\varkappa y. \tag{A.14}$$

It follows from (A.12)–(A.14) that the ratio of the damping decrement to the frequency becomes of the order of unity as the value of κ approaches the value of κ_2, which is determined by the equation

$$\varkappa_2 \ln\frac{\varkappa_2}{\varkappa_1} \sim 1.$$

Then $\kappa_2 \sim \zeta g^{(1)}$ and the spectrum of zero sounds ends for

$$k \sim k_2 = 2p_F\zeta g^{(1)},$$

$$\omega \sim \omega_2 = 4\mu\zeta g^{(1)}, \quad \mu = p_F^2/2m. \tag{A.15}$$

An example is shown in Fig. 4 for the dependence of the phase velocity ω/k and the relative damping coefficient γ/ω on the wave vector divided by $2p_F$.

*arctg ≡ tan⁻¹.

APPENDIX B

The integral (2) in the case of a cylindrical Fermi surface ($E_p = p_\perp^2/2m$) is equal to ($\omega > 0$, $\mathrm{Re}\,\Pi_2(\omega, k_\perp) = \mathrm{Re}\,\Pi_2(-\omega, k_\perp)$, $\mathrm{Im}\,\Pi_2(\omega, k_\perp) = -\mathrm{Im}\,\Pi_2(-\omega, k_\perp)$):

$$\mathrm{Re}\,\Pi_2(\omega, k_\perp) = -2\zeta - \frac{\zeta}{k_\perp^2}\begin{cases} \sqrt{S_{12}} - \sqrt{S_{34}}, & k_\perp < q_1 \\ -\sqrt{S_{34}}, & q_1 < k_\perp < q_3 \\ 0, & q_3 < k_\perp < q_4 \\ -\sqrt{S_{34}}, & q_4 < k_\perp < q_2 \\ \sqrt{S_{12}} - \sqrt{S_{34}}, & q_2 < k_\perp \end{cases}, \tag{B.1}$$

$$\mathrm{Im}\,\Pi_2(\omega, k_\perp) = -\frac{\zeta}{k_\perp^2}\begin{cases} 0, & k_\perp < q_1 \\ \sqrt{-S_{12}}, & q_1 < k_\perp < q_3 \\ \sqrt{-S_{12}} - \sqrt{-S_{34}}, & q_3 < k_\perp < q_4 \\ \sqrt{-S_{12}}, & q_4 < k_\perp < q_2 \\ 0, & q_2 < k_\perp \end{cases}, \tag{B.2}$$

where

$$S_{12} = (k_\perp^2 - q_1^2)(k_\perp^2 - q_2^2),$$
$$S_{34} = (k_\perp^2 - q_3^2)(k_\perp^2 - q_4^2); \tag{B.3}$$

$$q_1 = -p_F + \sqrt{p_F^2 + 2m\omega},$$
$$q_2 = p_F + \sqrt{p_F^2 + 2m\omega},$$
$$q_3 = p_F - \sqrt{p_F^2 - 2m\omega},$$
$$q_4 = p_F + \sqrt{p_F^2 - 2m\omega}. \tag{B.4}$$

A typical form of the function Π_2 is shown in Fig. 5.

In the special case $\omega \ll v_F k_\perp$, the expressions for Π_2 were obtained by Kagan and Afanas'ev.[2] In Fig. 5, $k_\perp > q_3$ corresponds to this region.

Equation (7), which determines the dispersion law and the phonon damping, has the form

$$\Pi_2 = \frac{\omega^2 - c^2k^2}{\omega^2 g^{(1)} + c^2k^2(1-g^{(1)})}. \tag{B.5}$$

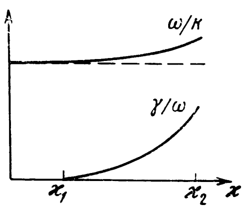

FIG. 4

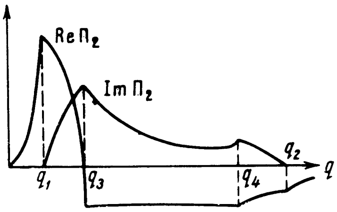

FIG. 5

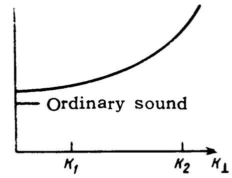

FIG. 6

We are interested in the solution of this equation for values $\omega \sim v_F k_\perp$, corresponding to the zeroth acoustic branch of the phonon excitations. Such a solution can only be for $\omega > ck$ and $k_\perp < q_1$. This is easily understood by using Fig. 5. For values $k_\perp > q_3$, the solution (B.5) gives the usual acoustic branch of phonon excitations, while in the branch $q_1 < k_\perp < q_3$, excitations cannot exist because of the large damping ($|\mathrm{Im}\,\Pi_2| \sim |\mathrm{Re}\,\Pi_2|$).

We therefore consider $k_\perp$ in the interval $0 < k_\perp < q_1$. So far as the component $k_{||}$ parallel to the axis of the cylinder is concerned, we shall assume its value to be fixed and not small ($k_{||} \lesssim p_F$), since the role of Coulomb interaction decreases for large values of $|k|$.

In the region of small $k_\perp$ ($k_\perp \ll q_1$), where $\Pi_2 = \zeta (p_F/m\omega)^2 k_\perp^2$ the only solution of Eq. (B.5) is

$$\omega^2 = c^2k^2 + \zeta v_F^2 k_\perp^2. \qquad (B.6)$$

For values

$$k_\perp \sim k_1 = \frac{c}{v_F}\frac{k_{||}}{\sqrt{g^{(1)}\zeta}}$$

the second term in (B.6) becomes important, and the explicit form of the spectrum for such values of $k_\perp$ is determined by the exact Eq. (B.5). The termination point of the spectrum $k_\perp \sim k_2$ is determined from the condition

$$\max \Pi_2 = 1 / g^{(1)},$$

which gives the value $k_2 = 8 p_F (\zeta g^{(1)})^2$.

Close to the point k_2, the dependence $\omega(k_\perp)$ is determined by the expression (12b) given earlier.

A typical form of the phonon spectrum is plotted in Fig. 6.

[1] W. Kohn, Phys. Rev. Lett. **3**, 393 (1959).

[2] A. Afanas'ev and Yu. Kagan, JETP **43**, 1456 (1962), Soviet Phys. JETP **16**, 1030 (1963).

[3] L. D. Landau, JETP **30**, 1058 (1956), Soviet Phys. JETP **3**, 920 (1956).

[4] L. D. Landau, JETP **32**, 59 (1957), Soviet Phys. JETP **5**, 101 (1957).

[5] V. P. Silin, JETP **35**, 1243 (1958), Soviet Phys. JETP **8**, 870 (1959).

[6] L. P. Gor'kov and I. E. Dzyaloshinskiĭ, JETP **44**, 1650 (1963), Soviet Phys. JETP **17**, 1111 (1963).

[7] A. B. Migdal, JETP **34**, 1438 (1958), Soviet Phys. JETP **7**, 996 (1958).

[8] A. A. Abrikosov, L. P. Gor'kov and I. E. Dzyaloshinskiĭ, Metody kvantovoĭ teorii polya v statisticheskoĭ fizike, (Quantum Field Theory Methods in Statistical Physics) (Fizmatgiz, 1962).

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