POSSIBILITY OF SUPERCONDUCTIVITY TYPE PHENOMENA IN A ONE-DIMENSIONAL SYSTEM

Yu. A. BYCHKOV, L. P. GOR'KOV, and I. E. DZYALOSHINSKIĬ

Institute of Theoretical Physics, Academy of Sciences, U.S.S.R.

Submitted to JETP editor October 6, 1965

J. Exptl. Theoret. Phys. (U.S.S.R.) **50**, 738—758 (March, 1966)

It is shown that there is an instability of the Fermi state in a one-dimensional system with regard to an arbitrarily weak attraction between the particles. In this connection, in contrast to the three-dimensional case, not pairs but quartets of particles near the Fermi surface are characteristic of this instability. The instability causes a rearrangement of the ground state in which a gap appears in the spectrum accompanied by a simultaneous doubling of the period of the structure. The new ground state, however, is capable of conducting current without energy dissipation. The interaction with the lattice is taken into account; its role reduces to the appearance of a certain effective interaction between the electrons. If the effective interaction turns out to be repulsive (remaining weak), then the system remains in the metallic state at all temperatures. The problem of fluctuations is discussed.

INTRODUCTION

THE appearance of the article by Little[1] once again caused a great deal of interest in one-dimensional conducting systems, for example the molecules of certain linear polymers and possibly dislocations. Apparently the mechanism of attraction between the conduction electrons suggested in[1] is quite feasible, and qualitative estimates made by Little with the use of the usual formulas of the Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity[2] indicate that effects of the superconducting pairing type might exist in such systems at rather high temperatures. However, the one-dimensional nature of the problem makes it highly unique, so that a trivial generalization of the BCS theory to this case is impossible. Experimental data concerning the electron spectra in the necessary region of frequencies and temperatures are, for the present, not available. Therefore, below an attempt is made to clarify the theoretical side of the question.

As is well known, one of the major difficulties is associated with the so-called doubling of the period in a one-dimensional system. Peierls[3] has stated a theorem according to which a one-dimensional metallic system is unstable with respect to the doubling of the period of the structure with simultaneous formation of a dielectric gap in the energy spectrum of the electrons. In this case one would hope to be able to observe in an experiment only some trace of the attraction caused by the Little mechanism, since such a gap undoubtedly must be of an atomic order of magnitude. Below, however, we shall show that Peierls' theorem is incorrect. Actually the doubling of the period and Cooper pairing are non-separable parts of one and the same process of the rearrangement of a one-dimensional Fermi system under the influence of attraction between the electrons. The new state which arises is able to conduct current.

The following important consideration is related to the problem of fluctuations. In the one-dimensional case, if the spectrum of vibrations does not contain a gap, calculation of different fluctuating quantities (for example, displacements of the ions) gives a divergent result at large wavelengths. With regard to the ordinary sound fluctuations, they are small at low temperatures since they contain the small parameter of the adiabatic approximation. One of the principal objections introduced by Ferrell[4] and later by Rice[11] consists in the fact that in the one-dimensional model the fluctuations associated with electron sound are large. However, the methods used by Ferrell and Rice and based on the Ginzburg-Landau equation are not applicable to the case of a one-dimensional superconductor, and by the same token it is impossible to regard their conclusion about the absence of a superconducting transition as proved.

In reality, the presence of such fluctuations very strongly complicates the picture of the phenomenon although, in our opinion, it does not suppress superconductivity (see the Supplement). The investigation carried out by us of a model with a weak interaction (including the interaction with phonons) shows that in the case of an effective attraction the system of electrons turns out to be unstable, which indicates the possibility of superconductivity type effects in one-dimensional systems.

1. INSTABILITY OF THE GROUND STATE OF A ONE-DIMENSIONAL FERMI SYSTEM

As stated above, we shall investigate the stability of the Fermi ground state of a system of electrons with regard to the interaction between them. We first digress from the periodic field of a chain and consider the model of a one-dimensional Fermi gas. Later on we shall include the necessary complications.

We choose an interaction of the electrons among themselves in the form

$$\Gamma^{0}_{\alpha\beta\gamma\delta}(p_1p_2p_3p_4) = g(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}), \qquad (1)$$

where the constant g is assumed to be small and corresponds to the Coulomb interaction and to the interaction of Little.[1] This interaction is effective only over interatomic distances, which leads to a cut-off of the corresponding integrals at frequencies on the order of the atomic frequency ϵ_0. Below we shall investigate the singularities of the vertex part associated with a small interaction.

As is well known,[6] two approaches are possible. By investigating the analytic properties of the vertex part for $T = 0$, one can show that if it has a pole in the upper half plane, then this indicates instability of the ground state. Thus, for example, the existence of the Cooper effect in the ordinary case[6] may be demonstrated. Unfortunately, the accuracy with which the calculation below is made is inadequate for these purposes. Therefore we choose another method which consists in an investigation of quantities in a thermodynamical technique.[6] Namely, it will be shown that in the presence of the attraction (1) the thermodynamical vertex part tends to infinity at a certain temperature, which is the temperature of the phase transition. The total picture of the new state at absolute zero will be investigated in Sec. 3.

We consider two diagrams of first order perturbation theory (Fig. 1). The first of them—a diagram of the Cooper type—is proportional to the matrix element

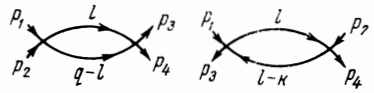

FIG. 1

$$\frac{T}{2\pi}\sum g\int G(l)G(q-l)\,dl(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma})$$

or after summation over frequencies (for $q = 0$)

$$\sim \frac{g}{2\pi}\int \frac{dl}{\xi}\,\mathrm{th}\frac{\xi}{2T}, \qquad (2)^*$$

where $\xi = v(|l| - l_0)$. As usual, in this case a logarithmic integration over l appears, as a consequence of which this diagram is of order $\approx (-g/2\pi v)\ln(\epsilon_0/T)$. At low temperatures the large value of the logarithm may compensate for the smallness of the parameter $g/2\pi v \ll 1$. For later purposes, it is helpful to remind ourselves that in the one-dimensional case the "Fermi surface" is represented by the two points $\pm p_0$. The logarithmic integration in (2) is carried out over the neighborhoods of both points.

Now let us go on to the second diagram of Fig. 1. In the three-dimensional case, for small momentum transfers $p_3 - p_1 = k$, singularities of the "zero sound type" are associated with this diagram. The Coulomb forces, i.e., the condition of electroneutrality, exclude the possibility of electron sound. Therefore, below we consider the properties of this diagram for large momentum transfers $k \approx 2p_0$. It turns out that in the one-dimensional case such a diagram also contains a logarithmic integration and therefore is of the same order of magnitude in the temperature range under consideration as the diagram of the Cooper type.

In fact, the matrix element for this diagram

$$\delta_{\alpha\delta}\delta_{\beta\gamma}gT\sum_{\omega}\int G(l)G(l-k)\frac{dl}{2\pi}$$

after summation over frequencies takes the form

$$\delta_{\alpha\delta}\delta_{\beta\gamma}\frac{g}{2\pi}\int\frac{dl}{\xi(l)-\xi(l-k)-i\omega_0}\left\{\mathrm{th}\frac{\xi(l)}{2T} - \mathrm{th}\frac{\xi(l-k)}{2T}\right\}.$$

If $l \sim p_0$, then $\xi(l-k) = -\xi(l)$, from which it follows that for $\omega_0 = 0$ this term is of order $(g/2\pi v)\ln(\epsilon_0/T)$. If k is fixed (for example, $2p_0$), then the logarithmic factor associated with the integration appears only in the neighborhood of the point $+p_0$ on the Fermi surface, and this is why the coefficient in front of the logarithm in this diagram is one-half the corresponding coefficient for the diagram of the Cooper type. In addition, the signs in front of the logarithms are different, i.e.,

*th ≡ tanh.

we say that in the presence of attraction the contribution from the Cooper diagram increases the vertex part (1), whereas the contribution from the second diagram decreases it.

Here it is appropriate to say that in a one-dimensional lattice, owing to the linear relation between the quasimomentum and the number of states, $2p_0$ is just one-half the period of the reciprocal lattice. The singularity associated with momentum transfer $k = 2p_0$ for the zero-sound diagram therefore means a particular role of the density vibrations with period π/p_0, that is, with double the period of the spatial chain.

Thus, we encounter the phenomenon noted by Peierls—doubling of the period. However, we see that the diagrams corresponding to Cooper pairing have the same order of magnitude. And what is more, it turns out that it is impossible to separately sum the diagrams of both types. Actually, let us consider for example the diagram shown in Fig. 2. This is a Cooper diagram with an internal insert of the zero-sound type. For fixed $p_1 \approx p_0$ the insert gives an additional logarithm in the region when l is integrated in the neighborhood of $-p_0$. The order of magnitude of this diagram is, consequently, $[(g/v)\ln(\epsilon_0/T)]^2$. The singularity of interest to us lies in the region $(g/2\pi v)\ln(\epsilon_0/T) \sim 1$ or

$$T_c \sim \varepsilon_0 \exp(-2\pi v/|g|). \tag{3}$$

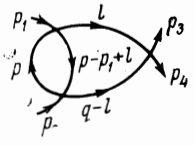

FIG. 2

Therefore, we must sum the entire set of diagrams giving a relative contribution of the form $[(g/v)\ln(\epsilon_0/T)]^n$. A distinctive topological property of this class of diagrams is the fact that one can always divide them into two parts, having cut only the two internal lines corresponding to the logarithmic integration.

Determination of the sum of diagrams of such type, the so-called "parquet," reduces to a solution of the system of nonlinear integral equations established by Dyatlov, Sudakov, and Ter-Martirosyan.[7] The specific form of the system depends on the spin structure of all quantities; therefore, below we shall only schematically reproduce the derivation. Let us denote by $\Phi(p_1p_2; p_3p_4)$[1)] the totality of all diagrams of the type under consideration which are reducible in the direction from the momenta p_1p_2 to the momenta p_3p_4, i.e., they may be separated into two parts by the method indicated above—into parts containing, respectively, the incoming ends p_1p_2 and the outgoing p_3p_4. Figure 2 may serve as a simple example of such a diagram. Similarly two more quantities $\Phi(p_1p_3; p_2p_4)$ and $\Phi(p_1p_4; p_2p_3)$, which are reducible in two other possible directions, may be defined. It is obvious that the total sum of all diagrams $\Gamma(p_1p_2p_3p_4)$ is given by

$$\Gamma(p_1p_2p_3p_4) = \Gamma^0 + \sum_{i=1}^{3}\Phi_i. \tag{4}$$

It is easy to obtain the equation for an isolated "block" by using the reducibility in the appropriate direction. For example, the equation for the "Cooper block" is shown in Fig. 3, or analytically it has the form

$$\Phi(p_1p_2; p_3p_4) = \frac{T}{2\pi}\sum_{\omega}\int \{\Gamma^0 + \Phi(p_1k; p_1+p_2-k, p_2) + \Phi(p_2k; p_1, p_1+p_2-k)\}\, G(k) \times G(p_1+p_2-k)\,\Gamma(k, p_1+p_2-k; p_3, p_4)\, dk. \tag{5}$$

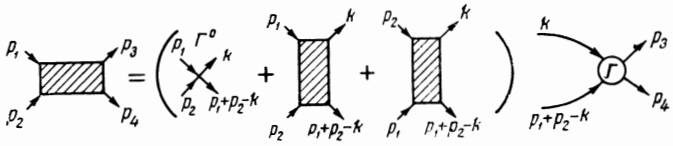

FIG. 3

The system of Eqs. (4) and (5) is exact for the set of diagrams under consideration. It is clear that the "parquet" does not exhaust all possible diagrams for the vertex part; however, one can convince oneself that in the approximation $(g/\pi v)\ln(\epsilon_0/T) \sim 1$ the remaining diagrams (such as, for example, the "envelope" shown in Fig. 4) are smaller by at least the ratio $g/\pi v \ll 1$, since they contain an inadequate number of logarithmic integrations. Strictly speaking, if we return to expression (3) for the critical temperature, then it is evident that in the approximation under consideration the latter is determined only accurate to the factor in front of the exponential. Therefore, in the equation for the pole in the vertex part, which we obtain below

$$1 + \frac{g}{\pi v}\ln\frac{\varepsilon_0}{T} + O(g) = 0,$$

the terms of order $g/\pi v \ll 1$ should be known, i.e., some of the "nonparquet" diagrams should have been included in the equation. Taking account of this circumstance would greatly complicate the

[1)] We temporarily omit the spin structure of all quantities.

FIG. 4

entire calculation. Therefore, below we shall confine our attention only to the approximation $(g/\pi v)\ln(\epsilon_0/T) \sim 1$, which, nevertheless, enables us to investigate the entire physical situation sufficiently completely. In accordance with this, we shall also solve the system of Eqs. (4) and (5) with logarithmic accuracy. The logarithmic nature of the integration of the two G-functions in (5) enables us to change to logarithmic variables, since all quantities are slowly varying functions of the momenta and "frequencies."

In accordance with what has been said above, we shall investigate the vertex part $\Gamma_{\alpha\beta\gamma\delta}(p_1p_2; p_3p_4)$ with small total momentum and momentum transfer of order $2p_0$. For definiteness, we choose

$$|p_1 + p_2| \ll p_0,$$
$$|p_3 - p_1 - 2p_0| \ll p_0.$$

Under these conditions one of the "blocks" corresponding to the direction $(p_1p_4; p_2p_3)$ does not contain a logarithmic integration and is therefore small. The two remaining "blocks" of the Cooper and "zero sound" types depend on

$$\xi = \ln\frac{\varepsilon_0}{v|p_1+p_2|},$$
$$\eta = \ln\frac{\varepsilon_0}{v|p_3-p_1-2p_0|}. \qquad (6)$$

Since the logarithmic integrations in (5) are cut off at the lower limit, then strictly speaking in the definition of ξ and η by "momenta" it is necessary to understand the largest of the following three quantities: the momenta proper, the frequency, or the temperature. The quantities $|p_1 + p_2|$ and $|p_3 - p_1 - 2p_0|$ may be of different order. In this case the vertex part depends on both arguments. As to the "blocks," then it is obvious that if, for example, $\xi > \eta$, then the "Cooper block" depends on both arguments since in it there are integrations extending to the "smallest" momentum $|p_1 + p_2|$, whereas in a "block" of the "zero sound" type only the dependence on the "largest" momentum $|p_3 - p_1 - 2p_0| \gg |p_1 + p_2|$ is retained.

Taking account of all that has been said, and also remembering that in the diagrams of the "zero sound" type the logarithmic integration takes place close to one of the two points of the Fermi surface, one can now write down a system of integral equations (5) for our case. Let us introduce the notation

$$\Gamma_{\alpha\beta\gamma\delta}(p_1p_2p_3p_4) = \gamma_1(\xi,\eta)(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}) + \gamma_2(\xi,\eta)\delta_{\alpha\delta}\delta_{\beta\gamma},$$
$$\Sigma_{\alpha\beta\gamma\delta}(p_1p_2p_3p_4) = \sigma_1(\xi,\eta)(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}) + \sigma_2(\xi,\eta)\delta_{\alpha\delta}\delta_{\beta\gamma},$$
$$S_{\alpha\beta\gamma\delta}(p_1p_2p_3p_4) = S_1(\xi,\eta)(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}) + S_2(\xi,\eta)\delta_{\alpha\delta}\delta_{\beta\gamma}. \qquad (7)$$

In (7) the quantity $S_{\alpha\beta\gamma\delta} - \Gamma^0_{\alpha\beta\gamma\delta}$ is the sum of all diagrams of the Cooper type, i.e., diagrams cut in two by parallel lines with total momentum $p_1 + p_2$. Similarly $\Sigma_{\alpha\beta\gamma\delta} - \Gamma^0_{\alpha\beta\gamma\delta}$ is a "block" of the "zero sound" type, i.e., the sum of the diagrams cut in two by antiparallel lines with difference $p_1 - p_3$. The reason why we included the bare vertex $\Gamma^0_{\alpha\beta\gamma\delta}$ (1) in the definition of the functions (7) is clear from the structure of the right side of the schematic equation (5) for the "blocks." The expression inside the parentheses contains two "blocks"; however, as one can easily verify, in the necessary region of integration over k, only one of them is essential. It is convenient to denote the corresponding combinations by a single letter. In this connection

$$\Gamma_{\alpha\beta\gamma\delta}(p_1p_2p_3p_4) = \Sigma_{\alpha\beta\gamma\delta}(p_1p_2p_3p_4) + S_{\alpha\beta\gamma\delta}(p_1p_2p_3p_4) - \Gamma^0_{\alpha\beta\gamma\delta}. \qquad (8)$$

Substituting expressions (7) into (5) and separating the spinor indices, we obtain

1) in the case $\eta > \xi$

$$S_1(\xi,\eta) \equiv S_1(\xi), \quad S_2(\xi,\eta) \equiv S_2(\xi), \quad \lambda = 1/2\pi v,$$
$$\sigma_1(\xi,\eta) = g - \lambda\int_0^\xi [S_1(t)\gamma_2(t,\eta) + S_2(t)\gamma_1(t,\eta)]\,dt - \lambda\int_\xi^\eta [S_1(\xi)\gamma_2(t,\eta) + S_2(\xi)\gamma_1(t,\eta)]\,dt,$$
$$\sigma_2(\xi,\eta) = -\lambda\int_0^\xi [S_1(t)\gamma_1(t,\eta) + S_2(t)\gamma_2(t,\eta)]\,dt - \lambda\int_\xi^\eta [S_1(\xi)\gamma_1(t,\eta) + S_2(\xi)\gamma_2(t,\eta)]\,dt,$$
$$\gamma_1(\xi,\eta) = \sigma_1(\xi,\eta) + S_1(\xi) - g, \quad \gamma_2(\xi,\eta) = \sigma_2(\xi,\eta) + S_2(\xi). \qquad (9)$$

2) in the case $\eta < \xi$

$$\sigma_1(\xi,\eta) \equiv \sigma_1(\eta), \quad \sigma_2(\xi,\eta) \equiv \sigma_2(\eta),$$
$$S_1(\xi,\eta) = g + \lambda\int_0^\eta [\sigma_1(t)\gamma_2(\xi,t) + \sigma_2(t)\gamma_1(\xi,t) - 2\sigma_1(t)\gamma_1(\xi,t)]\,dt + \lambda\int_\eta^\xi [\sigma_1(\eta)\gamma_2(\xi,t) + \sigma_2(\eta)\gamma_1(\xi,t) - 2\sigma_1(\eta)\gamma_1(\xi,t)]\,dt,$$

$$S_2(\xi,\eta)=\lambda\int_0^{\eta}\sigma_2(t)\gamma_2(\xi,t)\,dt+\lambda\int_{\eta}^{\xi}\sigma_2(\eta)\gamma_2(\xi,t)\,dt,$$

$$\gamma_1(\xi,\eta)=S_1(\xi,\eta)+\sigma_1(\eta)-g,$$

$$\gamma_2(\xi,\eta)=S_2(\xi,\eta)-\sigma_2(\eta)-g. \qquad (10)$$

Equations (9) and (10) are obtained from Eqs. (5), respectively, for the "zero sound" and Cooper "blocks."

Both systems may be solved exactly by the method suggested by Ansel'm.[8] This will be done in the Appendix. Here we present an expression for the vertex part in that case when all the "momenta" are small or of the order of the temperature. In this connection $\xi=\eta=\ln(\epsilon_0/T)$. The expression for the vertex part has the form[2)]

$$\Gamma_{\alpha\beta\gamma\delta}=g\frac{\delta_{\alpha\gamma}\delta_{\beta\delta}-\delta_{\alpha\delta}\delta_{\beta\gamma}}{1+(g/\pi v)\ln(\varepsilon_0/T)}$$

$$-\frac{g^2}{2\pi v}\ln\frac{\varepsilon_0}{T}\frac{\delta_{\alpha\delta}\delta_{\beta\gamma}}{1+(g/\pi v)\ln(\varepsilon_0/T)} \qquad (11)$$

In the case of repulsion ($g>0$) expression (11) does not have any singularity. The pole in (11),

$$T_c\sim\varepsilon_0\exp(-\pi v/|g|) \qquad (11')$$

corresponds to an attraction ($g<0$). As mentioned above, for an accurate determination of T_c, terms of higher order (of the type g and $g\times[g\ln(\epsilon_0/T_c)]^n$) should be taken into consideration. In connection with this, the question may arise as to whether both zeros in the denominators of (11) coincide, when these terms are taken into account. It seems to us that this is obvious, since already in the logarithmic approximation the Cooper effect and the doubling of the period are related to a unique system of equations; however, for the final answer a calculation in a nonlogarithmic approximation is required.

Expression (11) exhibits the presence of a certain critical temperature, below which a system of the usual Fermi type cannot exist in the presence of attraction.

2. INTERACTION WITH THE LATTICE

Before going on to study the properties of the new state, which arises in the system at low temperatures, let us consider those complications which the presence of a periodic structure introduce into the problem. At once we note that, as is evident from the preceding considerations, only the neighborhood of the Fermi "limit" is essential for the singularities investigated. Therefore, we may ignore the fact that electrons in a lattice possess not momentum but quasimomentum. The characteristic properties of Coulomb forces, i.e., the condition of electroneutrality, become apparent in questions associated with the interaction of the conduction electrons with an electromagnetic field. Now we must take into consideration that besides the Coulomb forces between the conduction electrons and the Little interaction[1] there is still the interaction with the ions of the lattice which, in the final account, leads, as we shall see, to an actual doubling of the period. In the BCS theory the attraction between electrons, arising due to the interaction with lattice vibrations, is the basic mechanism responsible for superconductivity. In a one-dimensional lattice, under the assumption that the Little mechanism is dominant, the interaction with the lattice is less essential; however, we include it for completeness of the picture.

Both longitudinal and transverse vibrations (flexure waves) are possible for a linear chain. The conduction electrons interact, generally speaking, both with those and with the others. Without making the form of this interaction specific, let us consider the matrix element of the transition corresponding to the scattering of two conduction electrons by one another due to the exchange of a virtual phonon. The dotted line on Fig. 5 denotes the D-function of the phonon. The corresponding matrix element has the form

$$|a(p_3-p_1)|^2/[(\varepsilon_3-\varepsilon_1)^2-\omega_0^2(p_3-p_1)],$$

where $\omega_0(k)$ is the dispersion law of the phonons, and $a(k)$ is the vertex of the interaction of electrons with the phonon field. For small momentum transfer $k=|p_3-p_1|\ll p_0$ the energy transfer $\epsilon_3-\epsilon_1\sim vk$ whereas, for example, for longitudinal phonons $\omega_0(k)=ck$. Therefore $|\epsilon_2-\epsilon_1|\gg\omega_0(k)$ and, consequently, for small transfers this interaction corresponds to repulsion. On the other hand, if $|p_3-p_1|\sim 2p_0$, then the change of the electron energy is small; in the denominator one can neglect the first term and the effective interaction is an attraction. We note that it follows from this consideration that the magnitude of the

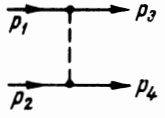

FIG. 5

[2)] It would be possible to obtain this answer more simply by using the method suggested by Sudakov,[7] without consideration of the equations of the "parquet" for the general case ($\xi\neq\eta$).

attraction associated with a large momentum transfer is $v^2/c^2 \sim M/m$ times larger than the repulsion corresponding to the forward scattering, i.e., in general one can neglect[3] scattering processes due to the exchange of phonons with small transfer.

If, as above, we choose the momenta so that

$$|p_1 + p_2| \ll p_0, \quad |p_3 - p_1 - 2p_0| \ll p_0,$$

then the problem of the evaluation of the total vertex part $\Gamma_{\alpha\beta\gamma\delta}(p_1p_2; p_3p_4)$ differs from the preceding calculation by the appearance of a new "bare" interaction of the form $-g_p^2\delta_{\alpha\gamma}\delta_{\beta\delta}$. It is more convenient, however, to write the general form of the "bare" interaction in the following way:

$$\Gamma^0_{\alpha\beta\gamma\delta} = (g - g_p)^2(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}) - g_p^2\delta_{\alpha\delta}\delta_{\beta\gamma}. \quad (12)$$

In accordance with (7) we denote

$$\gamma_{10} = g - g_p^2, \quad \gamma_{20} = -g_p^2. \quad (13)$$

In these formulas the constant g describes, as before, the Coulomb interaction and the Little interaction, g_p describes the constant electron-phonon interaction. Being a long-range action, the latter is cut off at the Debye frequency $\omega_D \sim \epsilon_0\sqrt{m/M}$. Let us denote $\xi_0 = \ln(\epsilon_0/\omega_D) \gg 1$. In the definitions (13) of γ_{10} and γ_{20} it is thus implied that the constant g_p is equal to zero for $\eta > \xi_0$. The derivation of the equations for the vertex part does not differ at all from the previous derivation.

Let us again write down the system of Eqs. (9) and (10):

1) the case $\eta > \xi$

$$\sigma_1(\xi,\eta) = \gamma_{10} - \lambda\int_0^\xi [S_1(t)\gamma_2(t,\eta) + S_2(t)\gamma_1(t,\eta)]\,dt - \lambda\int_\xi^\eta [S_1(\xi)\gamma_2(t,\eta) + S_2(\xi)\gamma_1(t,\eta)]\,dt,$$

$$\sigma_2(\xi,\eta) = \gamma_{20} - \lambda\int_0^\xi [S_1(t)\gamma_1(t,\eta) + S_2(t)\gamma_2(t,\eta)]\,dt - \lambda\int_\xi^\eta [S_1(\xi)\gamma_1(t,\eta) + S_2(\xi)\gamma_2(t,\eta)]\,dt,$$

$$\gamma_1(\xi,\eta) = \sigma_1(\xi,\eta) + S_1(\xi) - \gamma_{10},$$

$$\gamma_2(\xi,\eta) = \sigma_2(\xi,\eta) + S_2(\xi) - \gamma_{20}. \quad (9')$$

2) the case $\eta < \xi$

$$S_1(\xi,\eta) = \gamma_{10} + \lambda\int_0^\eta [\sigma_1(t)\gamma_2(\xi,t) + \sigma_2(t)\gamma_1(\xi,t) - 2\sigma_1(t)\gamma_1(\xi,t)]\,dt + \lambda\int_\eta^\xi [\sigma_1(\eta)\gamma_2(\xi,t) + \sigma_2(\eta)\gamma_1(\xi,t) - 2\sigma_1(\eta)\gamma_1(\xi,t)]\,dt,$$

$$S_2(\xi,\eta) = \gamma_{20} + \lambda\int_0^\eta \sigma_2(t)\gamma_2(\xi,t)\,dt + \lambda\int_\xi^\eta \sigma_2(\eta)\gamma_2(\xi,t)\,dt,$$

$$\gamma_1(\xi,\eta) = S_1(\xi,\eta) + \sigma_1(\eta) - \gamma_{10},$$

$$\gamma_2(\xi,\eta) = S_2(\xi,\eta) + \sigma_2(\eta) + \gamma_{20}. \quad (10')$$

The solution of this system is given in the Appendix. Here we present only the final result for the vertex part, from which, in analogy to (11), the existence of a singularity with respect to the temperature in the presence of an effective attraction follows.

For $\xi < \xi_0$ expression (11) for $\Gamma(\xi,\xi)$ is retained; for $\xi > \xi_0$ we have

$$\Gamma(\xi,\xi) = (\bar g - g_p^2)(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}) \Big/ \left[1 + \frac{\bar g - g_p^2}{\pi v}(\xi - \xi_0)\right] + \left\{\frac{1}{2}(\bar g - g_p^2) \Big/ \left[1 + \frac{\bar g - g_p^2}{\pi v}(\xi - \xi_0)\right] - \frac{1}{2}(\bar g + g_p^2) - \frac{\bar g g \xi_0}{2\pi v}\right\}\delta_{\alpha\delta}\delta_{\beta\gamma}, \quad (14)$$

$$\bar g = g \Big/ \left[1 + \frac{g}{\pi v}\ln\frac{\varepsilon_0}{\omega_D}\right]. \quad (14')$$

From formulas (11) and (14) it follows that provided that

$$(g/\pi v)\ln(\varepsilon_0/\omega_D) < -1$$

the old expression (11) is preserved for T_c; in this case $T_c > \omega_D$. In the opposite case

$$T_c \sim \omega_D \exp\left(\frac{\pi v}{\bar g - g_p^2}\right), \quad (15)$$

so that the combination

$$\bar g - g_p^2 < 0 \quad (16)$$

plays the role of an effective attraction.

Thus, the phonon interaction enters into the transition temperature T_c only if $T_c < \omega_D$.

In the presence of an interaction with phonons, at the transition point there are also singularities of the quantities characterizing the lattice. For this, we consider in greater detail the D-function of the lattice vibrations, which we define as the Fourier component[6] $D(\omega_n, k)$ of the thermodynamical average $\langle T(\hat u(\mathbf{r}_1, T_1)\hat u(\mathbf{r}_2, T_2))\rangle$ of the displacement operators $\hat u(r, \tau)$. Without any interaction

[3] For simplicity, we assume that there are no optical branches in the chain.

$$D_0(\omega_n, k) = 1/4Mp_0[\omega_n^2 + \omega_0^2(k)] \qquad (17)$$

($\pi/2p_0$ is the period of the lattice).

After including both the electron interactions among themselves and the interactions of the latter with the lattice, one can relate the phonon Green's function with the total vertex part by means of the exact equation graphically shown in Fig. 6:

$$D(k) = D_0(k) - 2D_0^2(k) g_p^2 \omega_{2p_0}^2 4Mp_0 \frac{T}{2\pi} \times \sum_{\omega} \int G(p_+) G(p_-)\, dp - 4g_p^2 Mp_0 \omega_{2p_0}^2 D_0^2(k) \times \sum_{\omega_1 \omega_2} \int\int G(p_{1+}) G(p_{1-}) \Gamma_{\alpha\beta\gamma\delta}(p_{1+} p_{2-}; p_{1-} p_{2+}) \times G(p_{2+}) G(p_{2-})\, dp_1\, dp_2, \qquad (18)$$

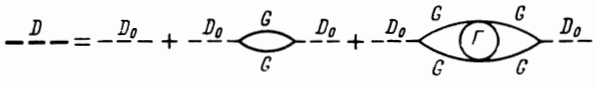

FIG. 6

where $p_\pm = p \pm (k/2)$, and the combination $4Mp_0 g_p^2 \omega 2p_0$ appeared as a result of the fact that above in Eq. (12) we denoted the ratio $a^2(2p_0)/\omega_0^2(2p_0)$ by g_p^2. Expression (18) contains a singularity for $k = \pm 2p_0$. Let, for example, $k = 2p_0$. The second and third terms in (18) contain logarithmic integrations of a pair of G-functions. Let us denote

$$D(2p_0, T) = -d(\eta)/4Mp_0\omega^2_{2p_0},$$

where $\eta = \ln(\epsilon_0/T)$. Then, by changing to logarithmic variables, we obtain

$$d(\eta) = 1 - \frac{g_F^2}{\pi v}\eta + 2g_p^2 \int_0^\eta \int_0^\eta \frac{d\eta_1}{2\pi v}\frac{d\eta_2}{2\pi v}\gamma^+(\min \eta_1, \eta_2, \eta) = 1 - \frac{g_p^2}{\pi v}\eta + 2\frac{g_p^2}{\pi v}\int_0^\eta (\eta - \eta_1)\frac{d\eta_1}{2\pi v}\gamma^+(\eta_1 \eta)$$

(see the Appendix for the definition of γ^+). Without carrying out the final evaluation of the last integral of formulas (A.8b) and (A.8c), we obtain the result that near (15) the D-function tends to infinity:

$$D(2p_0) \infty (T - T_c)^{-3/4}.$$

The nature of the results obtained above is not modified even if we set $g = 0$. Then it becomes clear from the previous consideration that the instability mentioned by Peierls with regard to the formation of a dielectric gap associated with the displacement of ions through one of their own positions of equilibrium is only one side of the phenomenon, since the interaction of the electrons with the lattice vibrations leads to the appearance of an effective attraction, i.e., to the possibility of pairing.

3. GROUND STATE OF THE SYSTEM AT ABSOLUTE ZERO TEMPERATURE

Above we indicated that in the one-dimensional case the effective attraction between electrons leads to an instability of the system or, more accurately speaking, to the appearance of a certain phase transition at a sufficiently low temperature.[4] However, in contrast to the usual theory of superconductivity, now not pairs but quartets of particles exist: two electrons and two holes on opposite sides of the Fermi "surface."

Now let us try to construct the total set of equations describing the properties of the ground state of the system. It is essential, of course, that we assume that there is only one transition, i.e., the denominators in expressions (11) and (14) tend to zero simultaneously—a fact proved by us only with logarithmic accuracy.

Following the usual approach to the theory of superconductivity (see, for example, [6]), we must assume that in our system for $T = 0$ the average values of the field operators

$$\langle a_p a_{\bar{p}}^+ \rangle, \quad \langle a_p^+ a_{-p}^+ \rangle, \quad \langle a_{p\pm 2p_0}\, a_p^+ \rangle$$

are different from zero.

The average of the first type is associated with ordinary G-functions, while the two of the second type reflect the appearance of bound pairs of the Cooper type and a bound state of an electron and a hole. As is well known,[6] the analogy with Bose condensation is the basis for such a procedure. At the condensation temperature a pole appears in the Green's function $G(p)$ of a Bose particle for $p = 0$, corresponding to the fact that at lower temperatures a portion of the particles are found in the ground level. From a formal point of view, this corresponds to the introduction into the Green's function of a term with a δ-function structure of the form $n_0\delta(p)$, where n_0 is the density of condensed particles. The vertex part $\Gamma_{\alpha\beta\gamma\delta}(p_1 p_2 p_3 p_4)$ which we investigated above is proportional to the two-particle Green's function

$$\langle T(a_{p_1\alpha}(t_1) a_{p_2\beta}(t_2) a^+_{p_3\gamma}(t_3) a^+_{p_4\delta}(t_4))\rangle, \qquad (19)$$

[4] In order to avoid misunderstandings, we indicate that the well known proof about the impossibility of the existence of phase transitions in the one dimensional case, cited in[9], concerns only phase transitions of the first kind.

which for two particles or for a particle-hole pair has boson-like properties. Therefore, the origin of two poles of different spin structure in (11) or (14) requires lower temperatures, and in particular at absolute zero, the introduction in (19) of δ-functions of two types:

$$A(p_1p_3)\delta(p_1+p_2)(\delta_{\alpha\gamma}\delta_{\beta\delta}-\delta_{\alpha\delta}\delta_{\beta\gamma}) + \sum_{i\neq l\neq k} B(p_i,p_l)\delta_{\alpha\gamma}\delta_{\beta\delta}\delta(p_i-p_k\pm 2p_0).$$

In writing down these terms there is still some arbitrariness, which is eliminated by the requirement that all momenta are found close to the Fermi surface. With regard to the form of A and B in the model with a weak interaction, it is natural to again assume [6] that expression (19) may be described in terms of pair averages, including the new Green's functions as well, which confirms the final results.

We introduce the following notation:

$$\hat{G}_{\omega\alpha\beta}(p) = -i\int_{-\infty}^{+\infty} dt\,\langle T(a_{p\alpha}(t)a_{p\beta}^{+}(0))\rangle e^{i\omega t}$$

$$\hat{G}^{(\pm)}_{\omega\alpha\beta}(p) = \int_{-\infty}^{+\infty} dt\,\langle T(a_{p\pm 2p_0,\alpha}(t)a^{+}_{p\beta}(0))\rangle e^{i\omega t},$$

$$\hat{F}^{+}_{\omega\alpha\beta}(p) = \int_{-\infty}^{+\infty} dt\,\langle T(a^{+}_{-p\alpha}(t)a_{p\beta}(0))\rangle e^{i\omega t},$$

$$\hat{F}^{-}_{\omega\alpha\beta}(p) = \int_{-\infty}^{+\infty} dt\,\langle T(a_{-p\alpha}(t)a_{p\beta}(0))\rangle e^{i\omega t}. \qquad (20)$$

Having written down the equations for the Heisenberg operators a(t) and $a^+(t)$:

$$\left\{i\frac{\partial}{\partial t}-\frac{p^2}{2m}+\mu\right\}a_{p\alpha}-g\sum_{p_1+p=p_3+p_2}a^{+}_{p_1\sigma}a_{p_2\sigma}a_{p_3\alpha}=0,$$
$$\left\{i\frac{\partial}{dt}+\frac{p^2}{2m}+\mu\right\}a^{+}_{p\beta}+g\sum_{p+p_3=p_1+p_2}a^{+}_{p_1\beta}a^{+}_{p_2\sigma}a_{p_3\sigma}=0$$

and applying them to the averages defined above, and then describing the operator products in terms of pair averages according to the generalized Wick theorem, similar to the way this is described in [6], one can obtain a system of equations for the Green's functions of interest to us. Here we omit the entire derivation of the equations in coordinate space and present only the results for the Fourier components (20):

$$(\omega-\xi)G_\omega(p)-i\Delta F_\omega^{+}(p)-i\varkappa G_\omega^{-}(p)=1,$$
$$(\omega+\xi)G_\omega^{-}(p)+i\varkappa^{*}G_\omega(p)=0,$$
$$(\omega+\xi)F_\omega^{+}(p)+i\Delta^{*}G_\omega(p)=0. \qquad (21)$$

The solutions of this system are given by

$$G_\omega(p)=\frac{\omega+\xi}{\omega^2-\xi^2-|\Delta|^2-|\varkappa|^2},$$

$$G_\omega^{-}(p)=-\frac{i\varkappa^{*}}{\omega^2-\xi^2-|\Delta|^2-|\varkappa|^2}$$

$$F_\omega(p)=-\frac{i\Delta^{*}}{\omega^2-\xi^2-|\Delta|^2-|\varkappa|^2}. \qquad (22)$$

Thus, the energy spectrum of the system has the form

$$\varepsilon=(\xi^2+|\Delta|^2+|\varkappa|^2)^{1/2}. \qquad (23)$$

The gap in the spectrum is the sum of the "dielectric" ($|\varkappa|^2$) and of the "superconducting" ($|\Delta|^2$) parts.

Before determining the quantities Δ and κ, it is convenient to graphically represent the system of equations (21) (see Fig. 7). In this connection, we shall associate arrows with both ends of the lines: An arrow from a point corresponds on the average for the Green's function to an annihilation operator, whereas an arrow directed toward a point corresponds to a creation operator. Therefore, on Fig. 7 one arrow enters into the G-function, and the other arrow comes out; in F both arrows are directed toward each other, where in F^+ they are directed away from each other. In addition, we shall mark each end with the sign (±) in accordance with whether the momenta p in (21) lie near the points $+p_0$ or $-p_0$. The square in Fig. 7 denotes the quantity Δ, the circle denotes $-\kappa$, the quantities appearing in Eqs. (21).

In a three-dimensional system Δ is connected with F by the relation $|g|F(xx)$; in the one-dimensional case this is no longer so. Actually the term $|g|F(xx)$ is represented graphically in Fig. 8. It is easy to understand that corrections of the type shown in Fig. 8b give exactly the same contribution, since in our approximation

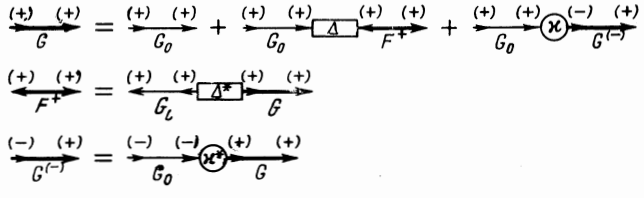

FIG. 7

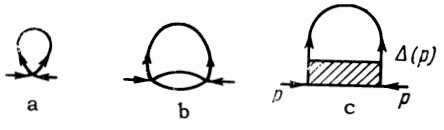

FIG. 8

$(|g|/\pi v)\ln(\epsilon_0/T_c) \sim 1$. In this connection, of course, in all internal lines it is sufficient to substitute only the Green's functions giving logarithmic integration, i.e., not to assume in the calculation the existence of the functions F and $G^{\pm}$. Owing to this, as shown in Fig. 8c, the "zero sound block" or, more precisely, the combination $\Sigma(7)$ enters into the definition of Δ. For similar reasons, a "block" of the Cooper type enters into the definition of κ, as shown in Fig. 9.

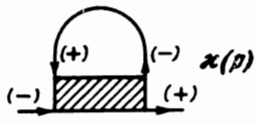

FIG. 9

After these preliminary remarks, let us write down the equations for Δ and κ, which thus arise during the derivation of (21):

$$\hat{\Delta}^{+}_{\alpha\beta}(p) = \frac{1}{2(2\pi)^2}\iint \Sigma_{\gamma\delta\alpha\beta}(p'p';\ pp)\,\hat{F}^{+}_{\gamma\delta}(p')\,dp'd\omega',$$

$$\kappa_{\alpha\beta}(p) = \frac{1}{(2\pi)^2}\iint S_{\alpha\gamma\beta\delta}(pp';\ pp')\,G^{-}_{\gamma\delta}(p')\,dp'd\omega'.$$

The matrix structure of G, $\hat{F}$, and $\hat{G}^{-}$ can be chosen in the form

$$\hat{G}_{\alpha\beta} = \delta_{\alpha\beta}G, \qquad \hat{G}_{\alpha\beta}{}^{-} = \delta_{\alpha\beta}G^{-}, \qquad \hat{F}_{\alpha\beta}{}^{+} = I_{\alpha\beta}F^{+},$$

where

$$(I^2)_{\alpha\beta} = \delta_{\alpha\beta}, \qquad I_{\alpha\beta} = -I_{\beta\alpha}.$$

Substituting expressions (7) into the relations presented above and taking into consideration that the logarithmic integration in these definitions is cut off for $v(p' - p_0) \sim \delta = (\Delta^2 + \kappa^2)^{1/2}$, we obtain the following equations for $\Delta(p)$ and $\kappa(p)$:

$$\Delta^{+}(\eta) = \sigma_3(\eta)\lambda\int_{\eta}^{\xi_{\text{eff}}}\Delta^{+}(t)\,dt + \lambda\int_{0}^{\eta}\sigma_3(t)\Delta^{+}(t)\,dt,$$

$$\kappa(\eta) = -S^{+}(\eta)\lambda\int_{\eta}^{\xi_{\text{eff}}}\kappa(t)\,dt - \lambda\int_{0}^{\eta}S^{+}(t)\kappa(t)\,dt. \qquad (24)$$

Here

$$\eta = \ln\frac{\varepsilon_0}{\{v(p-p_0),\,\omega\}}, \qquad \xi_{\text{eff}} = \ln\frac{\varepsilon_0}{\delta},$$

and see the Appendix for the definition of the quantities σ_3 and S^{+}.

Thus, $\Delta^{+}(\eta)$ and $\kappa(\eta)$ are slowly varying functions of the distance to the Fermi surface. Upon glancing at Eqs. (24) it becomes clear that they coincide with the equations of the "parquet" for the vertex parts γ^{+} (A.1) and γ_3(A.2), in which the inhomogeneous terms σ_3 and S^{+} are omitted. Since according to Eq. (A.3) γ^{+} and γ_3 have the form of a product of functions $A(\xi)B(\eta)$, then it is obvious that a solution of (24) exists if

$$1 + \frac{g}{\pi v}\xi_{\text{eff}} = 0, \qquad \delta_0 \sim T_c,$$

since here one can actually omit the inhomogeneous terms in Eqs. (A.1) and (A.2). The behavior of the solution of Eqs. (24) in this region, i.e., for $\xi_{\text{eff}} > \eta$ follows from expressions (A.8) and (A.9):

$$\kappa(\eta) \infty \delta_0\left[1 + \frac{g}{\pi v}\eta\right]^{-1/4}, \qquad \Delta(\eta) \infty \delta_0\left[1 + \frac{g}{\pi v}\eta\right]^{-1/4}.$$

Associated with the convergence $\eta \to \xi_{\text{eff}}$, the functions $\kappa(\eta)$ and $\Delta(\eta)$ from (25) increase. However, in order to find the values of both functions and the exact value of the gap δ_0 in the spectrum at the Fermi surface, it is necessary to go outside the limits of the logarithmic approximation used above. From the results (25) one can nevertheless derive the conclusion that in the region $v(p - p_0) \sim \delta_0$ both functions $\kappa(p)$ and $\Delta(p)$ become functions rapidly varying over distances of order δ_0.

The last circumstance strongly hampers an investigation of the superconducting properties of the system at absolute zero, if the obtained results are applied to the conduction electrons in a linear molecule, since the evaluation for example of the electron current arising in such a system during the application to it of an electromagnetic field is related to integration of the Green's functions in a neighborhood of the Fermi surface, precisely of order δ_0. Nevertheless, in order to obtain a qualitative representation concerning the singularities of our system, we assume that a weak constant electric field with frequency $\omega_0 \to 0$ is applied along the molecule, and we calculate the current in this gap, assuming that Δ and κ do not depend on p. In this case, in equations (21) we must introduce the vector potential $A = -iE/\omega_0$, having everywhere made the replacement $p \to p - eA$. In this connection, we seek all quantities in the form $G = G_0 + G_1$, where G_1 is the correction to the corresponding Green's function which is linear in the field.

The equations determining these corrections will have the form

$$\begin{aligned}
&(\omega - \xi)G_1 - i\Delta F_1{}^{+} - i\kappa G_1{}^{-} = -veAG_0,\\
&(\omega + \xi)G_1{}^{-} + \Delta F_{2p_0} + i\kappa^{*}G_1 = veAG_0{}^{-},\\
&(\omega + \xi)F_1{}^{+} + i\Delta^{*}G_1 - \kappa F_{2p_0} = -veAF_0{}^{+},\\
&(\omega - \xi)F_{2p_0} + \Delta^{+}G_1{}^{-} - \kappa^{*}F_1{}^{+} = 0. \qquad (26)
\end{aligned}$$

In this system the new function F_{2p_0} has appeared, which we defined in the following way:

$$F_{2p_0,\omega\alpha\beta}(p)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\langle T(a_{-2p_0-p,\alpha}(t),\quad a_{p\beta}(0))\rangle e^{i\omega t}\,dt.$$

Quantities of such form, i.e., modulations of the "superconducting" functions F and F^+, should, strictly speaking, have been written down in the system of equations (21), if its derivation were carried out with complete rigor. In the absence of an external field, however, as a result of the solution it turns out that these functions are equal to zero by virtue of the relation

$$\varkappa^* F^+ = \Delta^* G^-,$$

which earlier enabled us to conceal this circumstance.

In the system (26), F_{2p_0} also drops out of the combination $\Delta F_1^+ + \kappa G_1^-$ entering into the equation for the correction to the Green's function, so that we can write

$$G_1 = -\frac{veA\,[(\omega+\xi)^2+\Delta^2-\varkappa^2]}{[\omega^2-\xi^2-\Delta^2-\varkappa^2]^2}. \tag{27}$$

With the aid of the well known quantum mechanical formula for the current

$$j = -\frac{e}{m}(\nabla_x - \nabla_{x'})G_1(xx') - \frac{ne^2}{m}A,\quad x\to x'$$

(n is the number of electrons in the chain per unit length), after integration of (27) over frequencies and momenta[5] we obtain the expression

$$j_\omega = -\frac{ne^2}{m}A\frac{\Delta^2}{\delta^2},$$

which it is more convenient to write as

$$\frac{dj}{dt} = \frac{ne^2}{m}\frac{\Delta^2}{\delta^2}E. \tag{28}$$

Thus, the number of "superconducting" electrons at absolute zero temperature is proportional, as one would naturally expect, to the square of the ratio of the "superconducting" gap to the total gap in the energy spectrum. Formula (28) is, of course, qualitative in nature.

In the conclusion of this section, we note that the linear density of the electrons has Fourier components corresponding to $\pm 2p_0$, since the averages $\langle a_p^+ a_{p\pm 2p_0}\rangle$ do not vanish. The creation of an electron density varying with the double period of the lattice gives rise to a corresponding modulation in the distribution of the ions. We shall not dwell on this question in more detail.

[5] Calculation of the current is carried out here with the specific properties of the Coulomb interaction taken into account, in analogy to the way this is done in the three-dimensional case (see, for example, [6], Sec. 37).

4. FLUCTUATIONS OF THE DISPLACEMENTS OF THE IONS

Now let us estimate the effect of fluctuations of the displacements of the ions, for which we calculate the mean square fluctuation of the distance between two ions whose equilibrium distance apart is L:

$$\overline{\Delta^2(L)} = \langle (u(0)-u(L))^2\rangle = 2[\langle u(0)\rangle^2 - \langle u(0)u(L)\rangle]$$
$$= 2(D(0,0)-D(L,0)).$$

Substituting here the expression for the phonon Green's function, we obtain

$$\overline{\Delta^2(L)} = \frac{1}{2\pi}\frac{T}{Mp_0}\sum_\omega\int_0^\infty dk\,\frac{1-\cos kL}{\omega_n^2+\omega_0^2(k)}$$

or after summation over the frequencies $\omega_n = 2n\pi T$ we have*

$$\overline{\Delta^2(L)} = \frac{1}{2\pi Mp_0}\int_0^{2p_0}\frac{dk}{\omega_0(k)}\operatorname{cth}\frac{\omega_0(k)}{2T}(1-\cos kL).$$

By atomic displacements, in these formulas we understand either longitudinal or transverse displacements. Depending on this, we must introduce here either the spectrum of longitudinal vibrations, which is linear for small values of k ($\omega_0 = ck$), or flexure waves, whose spectrum is proportional to k^2 ($\omega_0 \sim ak^2$, where $a \sim \omega_D/p_0^2 \sim c/p_0$).

At the absolute zero of temperature $\coth(\omega_0/2T) = 1$, and we obtain

$$\overline{\Delta^2(L)} \sim \frac{1}{Mp_0}\int_{1/L}^\infty \frac{dk}{\omega_0(k)} = \begin{cases} \sim p_0^{-2}\sqrt{m/M}\ln p_0 L, & \omega_0 = ck \\ \sim p_0^{-1}L\sqrt{m/M}, & \omega_0 = ak^2. \end{cases}$$

Thanks to the adiabatic parameter $\sqrt{m/M} \ll 1$, the fluctuations are always sufficiently small. At high temperatures or large lengths L, the criterion for which is the condition

$$L \gg \sqrt{\frac{m}{M}}\frac{v}{T},\quad \omega_0 = ck;$$

$$L^2 \gg \sqrt{\frac{m}{M}}\frac{v}{p_0 T},\quad \omega_0 = ak^2,$$

for the longitudinal displacements we have

$$\overline{\Delta^2(L)} \sim p_0^{-2}LT/v.$$

Now let us determine the distances L_0 at which the system of ions and with it the self-consistent field lose periodicity. The periodicity (or one-dimensionality for transverse displacements) is violated if the relative displacement Δ of the ions becomes of the order of the distances between ions: $\overline{\Delta} \sim 1/p_0$. The formulas obtained above give the

*cth ≡ coth.

following expressions for L_0 at low temperatures:

$$L_0 \sim p_0^{-1} \exp \sqrt{M/m}, \quad \omega_0 = ck;$$
$$L_0 \sim p_0^{-1} \sqrt{M/m}, \quad \omega_0 = ak^2.$$

At high temperatures

$$L_0 \sim \frac{1}{p_0} \frac{\varepsilon_0}{T} \quad (\omega_0 = ck).$$

In the case of transverse vibrations a divergent result is obtained for $\overline{\Delta^2}$ at high temperatures. This circumstance means that the displacements of atoms located at sufficiently large distances from each other become large because of the bending of the molecule. Considering the molecule as an elastic thread,[9] one can show in this case that the radius of curvature still remains large. Thus, the relative fluctuation displacements become on the order of atomic only, either at temperatures on the order of the temperature of decay of the molecule ω_D, or at large distances. Since the wavelength of the electrons is of atomic order, a long molecule always possesses a quasi-one-dimensional character in the sense that the latter is violated only over sufficiently large regions. (For a molecule imbedded in a condensed medium, all of this is correct, of course, since the molecule has sufficient rigidity.)

APPENDIX

Combining all the equations of system (9′) and introducing the notation

$$\gamma^{\pm}(\xi, \eta) = \gamma_1(\xi, \eta) \pm \gamma_2(\xi, \eta), \qquad S^{\pm}(\xi) = S_1(\xi) \pm S_2(\xi),$$

we rewrite them in the more convenient form:

$$\gamma^{+}(\xi, \eta) = S^{+}(\xi) - \lambda \int_0^{\xi} S^{+}(t)\gamma^{+}(t, \eta)\,dt - \lambda \int_{\xi}^{} S^{+}(\xi)\gamma^{+}(t, \eta)\,dt,$$

$$\gamma^{-}(\xi, \eta) = S^{-}(\xi) + \lambda \int_0^{\xi} S^{-}(t)\gamma^{-}(t, \eta)\,dt + \lambda \int_{\xi}^{} S^{-}(\xi)\gamma^{-}(t, \eta)\,dt \tag{A.1}$$

($\eta > \xi$). In analogy to (10′) we introduce

$$\gamma_3(\xi, \eta) = \gamma_2(\xi, \eta) - 2\gamma_1(\xi, \eta), \quad \sigma_3(\eta) = \sigma_2(\eta) - 2\sigma_1(\eta).$$

As a result we obtain ($\eta < \xi$)

$$\gamma_2(\xi, \eta) = \sigma_2(\eta) + \lambda \int_0^{\eta} \sigma_2(t)\gamma_2(\xi, t)\,dt + \lambda \int_{\eta}^{\xi} \sigma_2(\eta)\gamma_2(\xi, t)\,dt,$$

$$\gamma_3(\xi, \eta) = \sigma_3(\eta) + \lambda \int_0^{\eta} \sigma_3(t)\gamma_3(\xi, t)\,dt + \lambda \int_{\eta}^{\xi} \sigma_3(\eta)\gamma_3(\xi, t)\,dt. \tag{A.2}$$

Being reduced to such form, Eqs. (A.1) and (A.2) are easily solved by the method of separation of variables proposed by Ansel'm.[8] We shall, for example, seek $\gamma^{+}(\xi, \eta)$ in the form

$$\gamma^{+}(\xi, \eta) = A^{+}(\xi)B^{+}(\eta). \tag{A.3}$$

Having divided both sides of the equation by $B^{+}(\eta)$, then differentiating with respect to η, we obtain

$$-B^{+\prime}(\eta)/B^{+2}(\eta) - \lambda A^{+}(\eta) = 0.$$

From which it follows that

$$B^{+}(\eta) = \exp\left(-\lambda \int_a^{\eta} \gamma^{+}(t)\,dt\right), \qquad \gamma^{+}(t) \equiv \gamma^{+}(t, t). \tag{A.4}$$

Differentiating the equation for $\gamma^{+}(\xi, \eta)$ with respect to ξ, and then assuming $\xi = \eta$, we find

$$A^{+\prime}(\xi)B^{+}(\xi) = S^{+\prime}(\xi).$$

Combining this with the previous relation, we obtain an equation for $\gamma^{+}(\xi)$:

$$\gamma^{+\prime}(\xi) = S^{+\prime}(\xi) - \lambda\gamma^{+2}(\xi). \tag{A.5a}$$

Applying the same method, we write down several more relations of the type (A.5a):

$$\gamma'^{-}(\xi) = S'^{-}(\xi) + \lambda\gamma^{-2}(\xi), \tag{A.5b}$$
$$\gamma_2'(\xi) = \sigma_2'(\xi) + \lambda\gamma_2^2(\xi), \tag{A.5c}$$
$$\gamma_3'(\xi) = \sigma_3'(\xi) + \lambda\gamma_3^2(\xi). \tag{A.5d}$$

Expressing from here the derivatives for γ_1 and γ_2 separately, we find the equations determining the total vertex part for $\xi = \eta$:

$$\gamma_1' = -2\lambda\gamma_1^2, \qquad \gamma_2' = -\lambda\gamma_1^2(\xi).$$

Or with the boundary conditions (13) taken into account, we obtain

$$\gamma_1(\xi) = g - 2\lambda \int_0^{\xi} \gamma_1^2\,d\eta, \qquad \gamma_2(\xi) = -\int_0^{\xi} \gamma_1^2\,d\eta,$$
$$\xi < \xi_0 = \ln(\varepsilon_0/\omega_D); \tag{A.6a}$$

$$\gamma_1(\xi) = \bar{\gamma_1}(\xi_0) - g_p^2 - 2\lambda \int_{\xi_0}^{\xi} \gamma_1^2\,d\eta,$$

$$\gamma_2(\xi) = \bar{\gamma_2}(\xi_0) - g_p^2 - \lambda \int_{\xi_0}^{\xi} \gamma_1^2\,d\eta, \qquad \xi > \xi_0. \tag{A.6b}$$

(The vertex part changes discontinuously at $\xi_0 = \ln(\epsilon_0/\omega_D)$.) Hence

$$\gamma_1(\xi) = \frac{g}{1 + (g/\pi v)\xi},$$
$$\gamma_2(\xi) = -\frac{g^2}{2\pi v} \frac{\xi}{1 + (g/\pi v)\xi} \quad (\xi < \xi_0). \tag{A.7a}$$

For $\xi > \xi_0$ it is convenient to introduce

$$\bar{g} = \bar{\gamma}_1(\xi_0) = g\Big/\Big(1+\frac{g}{\pi v}\xi_0\Big).$$

Then

$$\bar{\gamma}_2(\xi_0) = -g\bar{g}\xi_0/2\pi v.$$

Therefore, for $\xi > \xi_0$ we obtain from (A.7a)

$$\gamma_1(\xi) = (\bar{g} - g_p^2)\Big/\Big[1+\frac{\bar{g}-g_p^2}{\pi v}(\xi-\xi_0)\Big],$$

$$\gamma_2(\xi) = -\frac{\bar{g}g\xi_0}{2\pi v} - \frac{1}{2}(\bar{g}+g_p^2) + \frac{1}{2}(\bar{g}-g_p^2)\Big/\Big[1+\frac{\bar{g}-g_p^2}{\pi v}(\xi-\xi_0)\Big]. \qquad (A.7b)$$

In order to find $\gamma^+(\xi,\eta)$ for $\xi \neq \eta$, we return to formulas (A.3) and (A.4):

$$\gamma^+(\xi,\eta) = \gamma^+(\xi)B^+(\eta)/B^+(\xi). \qquad (A.3a)$$

It is obvious that the values of $B^+(\eta)$ are different for $\eta > \xi_0$ and $\eta < \xi_0$.

We denote by

$$\gamma^+_{\ll}(\xi,\eta),\quad \gamma^+_{<>}(\xi,\eta),\quad \gamma^+_{\gg}(\xi,\eta)$$

respectively, $\gamma^+(\xi,\eta)$ in the regions

$$\xi < \xi_0,\ \eta < \xi_0;\ \xi < \xi_0,\ \eta > \xi_0;\ \xi > \xi_0,\ \eta > \xi_0.$$

The continuity of $\gamma^+(\xi,\eta)$ with respect to η for $\eta = \xi_0$ follows from Eq. (A.1). With the aid of this condition, from Eqs. (A.3a) and (A.7a, b) we obtain

$$\gamma^+_{\ll}(\xi,\eta) = \gamma^+_<(\xi)\exp\Big[\frac{g}{4\pi v}(\eta-\xi)\Big] \times\Big[\Big(1+\frac{g}{\pi v}\xi\Big)\Big/\Big(1+\frac{g}{\pi v}\eta\Big)\Big]^{1/4}, \qquad (A.8a)$$

$$\gamma^+_{\gg}(\xi,\eta) = \gamma^+_>(\xi)\exp\Big\{\frac{1}{4\pi v}(\eta-\xi)\Big[\bar{g}+g_p^2+\frac{\bar{g}g\xi_0}{\pi v}\Big]\Big\} \times\Big[\Big(1+\frac{\bar{g}-g_p^2}{\pi v}(\xi-\xi_0)\Big)\Big/\Big(1+\frac{\bar{g}-g_p^2}{\pi v}(\eta-\xi_0)\Big)\Big]^{1/4}, \qquad (A.8b)$$

$$\gamma^+_{<>}(\xi,\eta) = \gamma^+_<(\xi)\exp\Big\{\frac{g}{4\pi v}(\xi_0-\xi) + \frac{1}{4\pi v}(\eta-\xi_0)\Big[\bar{g}+g_p^2+\frac{\bar{g}g\xi_0}{\pi v}\Big]\Big\}\cdot \Big[1+\frac{g}{\pi v}\xi\Big]^{1/4}\Big/\Big[1+\frac{\bar{g}-g_p^2}{\pi v}(\eta-\xi_0)\Big]^{1/4}\Big[1+\frac{g}{\pi v}\xi_0\Big]^{3/4}, \qquad (A.8c)$$

where $\gamma^+_{<>}$ is given by

$$\gamma^+_<(\xi) = \frac{3}{2}g\Big/\Big(1+\frac{g}{\pi v}\xi\Big) - \frac{g}{2},$$

$$\gamma^+_>(\xi) = -\frac{\bar{g}g\xi_0}{2\pi v} - \frac{1}{2}(\bar{g}+g_p^2) + \frac{3}{2}(\bar{g}-g_p^2)\Big/\Big[1+\frac{\bar{g}-g_p^2}{\pi v}(\xi-\xi_0)\Big]. \qquad (A.8d)$$

An expression for $S^+(\xi)$ may be obtained by simple integration of relation (A.5a). Similarly one can deal with all remaining equations of the system (A.1) and (A.2). Without going into the details, we present only the result for $\gamma_3(\xi,\eta)$ in the simplest region:

$$\gamma_3(\eta,\xi) = \gamma_3(\eta)\Big[\Big(1+\frac{g}{\pi v}\eta\Big)\Big/\Big(1+\frac{g}{\pi v}\xi\Big)\Big]^{3/4} \times\exp\Big[-\frac{g}{4\pi v}(\xi-\eta)\Big], \qquad (A.9a)$$

$$\gamma_3(\eta) = -{}^3/_2\gamma_1(\eta) - {}^1/_2 g. \qquad (A.9b)$$

SUPPLEMENT
(February 9, 1966)

FLUCTUATIONS OF THE ELECTRON DENSITY

In contrast to the fluctuations of the displacements of the ions the fluctuations of the electron density are not small. (We retract the opposite assertion, which was made by us earlier.[10]) However, the assertion of Ferrel[4] and Rice[11] about the vanishing of the superconducting gap in this case is, in our opinion, incorrect. Their consideration is based either on the fact that for $T = 0$ the gap $\Delta(x,t)$ is the pair wave function and as such it satisfies some wave equation, or else for T close to T_c it satisfies the Ginzburg-Landau equation. It is possible, however, to show that the Ginzburg-Landau equation (and with it the equation for the pair wave function) does not exist in the one-dimensional case. The condition for its existence would be $T - T_c \gg T_c$ (whereas in the three-dimensional case, this condition has the form $T_c \gg T_c^5/\epsilon_F^4$).

In actual fact, the situation is considerably more complicated. The force of action of the fluctuations in a one-dimensional system is associated with the singular "infrared catastrophe." An electron with energy $\xi = v(p - p_0)$ may emit an arbitrary number of real quanta of density oscillations with energy $\omega = vk$, since upon fulfillment of the momentum conservation law, the law of energy conservation in the one-dimensional case is identically satisfied. The calculational situation here is extremely complicated, and we postpone its discussion to a subsequent article; however, here we shall very briefly describe it.

The "infrared catastrophe" correction of order n corresponding to the self-energy has the form

$$\Sigma^{(n)} \sim g^n\xi^n/v^n(\omega-\xi)^{n-1}.$$

It becomes substantial ($\sim \Sigma^{(0)} = \omega - \xi$) only for $\omega - \xi \sim g\xi/v$. (Therefore they are completely negligible in the calculational technique with imag-

inary frequencies for $T \neq 0$, as in Sec. 2, when $\Sigma^{(n)} \lesssim g^n \xi / v^n$.) On the other hand, in the logarithmic integrals in the equations for the gaps (24) a considerably larger region ω of order ξ is essential, where one can use the zero-order approximation for g. In addition, it is physically obvious that associated with the presence of a gap in the spectrum for $\xi \leq \Delta$ no emission of phonons is possible. Therefore, the "infrared catastrophe" and, by the same token also the density fluctuations cannot affect the form of the superconducting gaps Δ and κ for $\omega \sim \xi$ (but $\omega \neq \xi$!) for $\xi, \omega > \Delta, \kappa$ and for $\omega, \xi \lesssim \Delta, \kappa$. The gap undergoes an essential change only in the region $\omega - \xi \sim g\xi$ for $\omega, \xi \gg \Delta, \kappa$. Therefore, in our opinion the fluctuations of the density cannot destroy superconductivity.

[1] W. A. Little, Phys. Rev. **A134**, 1416 (1964).

[2] J. Bardeen, L. Cooper, and J. Schrieffer, Phys. Rev. **108**, 1175 (1957).

[3] R. E. Peierls, Quantum Theory of Solids, Oxford, 1955 (Russ. Transl., IIL, M. 1956).

[4] R. A. Ferrell, Phys. Rev. Letters **13**, 330 (1964).

[5] L. D. Landau and E. M. Lifshitz, Kvantovaya mekhanika (Quantum Mechanics), Fizmatgiz, 1963.

[6] A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, Metody kvantovoĭ teorii polya v statisticheskoĭ fizike (Methods of Quantum Field Theory in Statistical Physics), Fizmatgiz, 1962.

[7] I. T. Dyatlov, V. V. Sudakov, and K. A. Ter-Martirosyan, JETP **32**, 767 (1957), Soviet Phys. JETP **5**, 631 (1957).

[8] A. A. Ansel'm, JETP **38**, 1887 (1960), Soviet Phys. JETP **11**, 1356 (1960).

[9] L. D. Landau and E. M. Lifshitz, Statisticheskaya fizika (Statistical Physics), Idz. "Nauka", 1964, Sec. 151.

[10] Yu. A. Bychkov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, JETP Letters **2**, 146 (1965), transl. p. 92.

[11] T. M. Rice, Phys. Rev. **A140**, 1889 (1965).

Translated by H. H. Nickle
93