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QUASILINEAR THEORY OF CURRENT INSTABILITY IN A PLASMA

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The problem of the passage of a current in a two-temperature plasma ($T_e \gg T_i$) is solved. The electric field strength E is assumed to be given and sufficiently large so that Coulomb collisions do not interfere with the free acceleration of most of the electrons. Ion-sound oscillations are excited in the plasma as a result of the current instability. The interaction between the electrons and ion-sound oscillations is considered in the quasilinear approximation. In this approximation the electrons are scattered by the oscillations almost elastically with a frequency $\sim v^{-3}$. Thus, during a period of time $t < t_0 \approx \sqrt{T_e M}/eE$ the current remains stationary and is equal to $j \approx -e\sqrt{T_e/M}$. Then, owing to heating of the electrons, runaway electrons appear and the current begins to grow.

SEVERAL recent experimental papers report observation of anomalously large resistance of plasma to electric current.[1-5] The anomalous resistance occurs under conditions when the electric field in the plasma exceeds a critical value $E_{cr} \approx T_e \ln \Lambda/8\pi e n r_D^4$ at which all electrons can be freely accelerated. The authors of these papers are unanimous in the opinion that the anomalous resistance is a consequence of the instability of the current. There is experimental evidence in favor of this hypothesis. In some investigations[1, 2] intense microwave plasma noise was registered during the time of current flow.

To explain the experiment it is necessary to develop a nonlinear theory of current instability. This is the subject of a paper by Field and Fried.[6] It contains a numerical calculation of the initial stage of development of the current instability in a plasma situated in an electric field, and shows that as soon as the current velocity exceeds the velocity of the ion sound and ion-sound oscillations begin to build up, the anomalous resistance sets in. However, in our opinion, sight has been lost in this paper of several important features of the phenomenon. In the present article we present in the quasilinear approximation an analytic solution of the problem of instability of a current in a plasma in a specified electric field.

The initial system of equations is the same as in [6]. We solve a very simple problem: in a uniform plasma with Maxwellian particle velocity distribution, and with temperatures T_i and T_e ($T_e \gg T_i$), there is a constant electric field E. Coulomb collisions are completely neglected. The external magnetic field is equal to zero, and the magnetic field of the current is negligibly small. Under these assumptions, the instability of the current is described by the system of equations

$$\frac{\partial f}{\partial t} + \frac{e}{m} E \frac{\partial f}{\partial v_z} = \frac{\partial}{\partial v_\alpha} D_{\alpha\beta} \frac{\partial f}{\partial v_\beta},$$

$$D_{\alpha\beta} = \pi \frac{e^2}{m^2} \int k_\alpha k_\beta |\varphi_k|^2 \delta(\omega - \mathbf{kv})\, d\mathbf{k}; \tag{1}$$

$$\gamma \equiv \frac{1}{2}\frac{\partial}{\partial t} \ln |\varphi_k|^2 = \frac{\pi}{2}\frac{\omega^3}{k^2}\frac{M}{mn} \int \mathbf{k} \frac{\partial f}{\partial \mathbf{v}} \delta(\omega - \mathbf{kv})\, d\mathbf{v}$$

$$- \frac{\pi^{1/2}\omega^4}{k^3 c_i^3} \exp\left[-\left(\frac{\omega}{kc_i}\right)^2\right]; \tag{2}$$

$$\omega = \frac{kc_s}{(1 + k^2 r_D^2)^{1/2}}, \quad r_D^2 = \frac{T_e}{m\omega_{pe}^2}, \quad c_s = \left(\frac{T_e}{M}\right)^{1/2}. \tag{3}$$

Here f(v, t) is the electron distribution function, φ_k the amplitude of the Fourier component of the disturbance of the potential, ω and **k** the frequency wave vector, E the external electric field, and $\omega_{p\alpha}^2 = 4\pi n e^2/m_\alpha$. In deriving (2) and (3) we have assumed that the electron velocity is much lower than their thermal velocity $c_e = (2T_e/m)^{1/2}$, and the ion temperature T_i is much lower than the average electron energy. In (2) we have left out terms that take into account the change in the oscillation energy as a result of nonlinear scattering of the waves by plasma particles and wave-wave scattering. The influence of these processes will be estimated after the solution is obtained.

The oscillations determined by the dispersion relation (3) are ion-sound oscillations. From (2) and (3) it follows that the damping of the oscillations by the ions gives rise to a critical velocity

$v_0 \approx 3(2T_i/M)^{1/2}$ below which the ion-sound is stable.

A wave with given ω and $\mathbf{k}$ is excited by resonant particles whose velocity in the direction of the wave vector is equal to the phase velocity of the wave, $\omega = \mathbf{k}\cdot\mathbf{v}$. In (2) this is taken into account by the δ-function. Excitation or damping of the wave can be understood as being the summary effect of a large number of Cerenkov emission and absorptions acts of ion-sound plasmons by the resonant electrons.

The change in the electron distribution function as a result of interaction with the ion-sound is described by (1). Since the maximum phase velocity of the ion-sound is c_s, the resonant electrons moving with a velocity of the order of thermal velocity, emit (absorb) waves whose wave vector is orthogonal, accurate to a quantity c_s/v, to the velocity of the particle $\mathbf{v}$. The velocity increment $\Delta\mathbf{v}$ is obviously parallel to the wave vector. Therefore the interaction between electrons and the ion-sound oscillations is a process in which the electron energy is conserved in the zeroth approximation in the parameter ω/kv.

Let us rewrite (1) in spherical coordinates (v, θ, φ). The z axis is chosen in the direction of the average electron velocity. We use the spherical coordinates (k, θ', φ') and when integrating with respect to $\mathbf{k}$ we measure the angle θ' from the same axis. We have

$$\frac{\partial f}{\partial t}+\frac{e}{m}E\left(\cos\theta\frac{\partial f}{\partial v}-\frac{\sin\theta}{v}\frac{\partial f}{\partial\theta}\right) = \frac{1}{v^3\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\left(A\frac{\partial f}{\partial\theta}-B\frac{\partial f}{\partial v}\right) + \frac{1}{v^2}\frac{\partial}{\partial v}\frac{1}{v}\left(D\frac{\partial f}{\partial v}-B\frac{\partial f}{\partial\theta}\right). \tag{4}$$

Here

$$A=\pi\frac{e^2}{m^2}\int k_\theta^2\,|\varphi_k|^2\delta(\omega-\mathbf{kv})\,v\,d\mathbf{k}, \tag{5}$$

$$B=-\pi\frac{e^2}{m^2}\int \omega k_\theta\,|\varphi_k|^2\delta(\omega-\mathbf{kv})\,v\,d\mathbf{k}, \tag{6}$$

$$D=\frac{\pi e^2}{m^2}\int \omega^2|\varphi_k|^2\delta(\omega-\mathbf{kv})\,v\,d\mathbf{k}; \tag{7}$$

k_θ is the projection of the wave vector in the direction of the unit vector $\mathbf{n}_\theta$. Equation (4) can be readily derived by writing the right side of (1) in the form of the divergence of the flux $j_\alpha = D_{\alpha\beta}\partial f/\partial v_\beta$ in spherical coordinates, and using the equality $k_v = \mathbf{k}\cdot\mathbf{v}/v = \omega/v$ and the obvious symmetry of f with respect to the angle φ.

Equation (2) in terms of the same variables is rewritten in the form

$$\gamma=\frac{\pi}{2}\frac{\omega^3}{k^2}\frac{M}{mn}\left\{\omega\int\frac{1}{v}\frac{\partial f}{\partial v}\delta(\omega-\mathbf{kv})\,dv + \int\frac{k_\theta}{v}\frac{\partial f}{\partial v}\delta(\omega-\mathbf{kv})\,dv\right\}. \tag{8}$$

If we compare the values of the coefficients A, B, and D we find that

$$B\sim A\omega/k,\quad D\sim(\omega/k)^2A.$$

Therefore the term with A is the principal one for an overwhelming majority of the particles, with the exception of a small vicinity $v \sim \omega/k \sim c_s$ about the origin, containing approximately $(m/M)^{3/2}$ particles from among the total number. It describes energy-conserving electron diffusion over the angle θ. If we leave out of (4) small terms containing B and D and ω in the argument of the δ-function, then the resultant equation is similar to that describing the behavior of a Lorentz plasma in an electric field.[7] The role of the ions is played in our problem by ion-sound oscillations. To be sure, the effective frequency of the electron scattering

$$\nu_{ef}=\frac{\omega_p^2}{v^3}\int\frac{k^2\,|\varphi_k|^2}{4nm}\delta(\mathbf{kv})\,v\,d\mathbf{k} \tag{9}$$

depends on the angle θ.

After turning off the electric field, all the electrons begin to acquire the same velocity as the current, and the instability sets in only when this velocity exceeds the critical value $v_0 \sim c_s$. At first, the energy of the ion-sound oscillations will increase exponentially with time. It is perfectly clear that the nature of the growth should change after the lapse of a time of order of $1/\gamma \sim (M/m)^{1/2}\omega_{pi}^{-1}$, since the work performed by the electric field on the particles increases not faster than $eEt^2/2$, and in final analysis the energy of the oscillations comes from the work of the field. This means that the electron distribution function and the level of the vibrations should become such that the increment γ_k becomes practically equal to zero for all values and orientations of the vector $\mathbf{k}$.

The initial state of development of the instability was calculated in [6] with a computer. It is the most difficult stage to solve analytically. Let us consider the properties of the steady state just referred to. We assume that the steady level of the amplitudes of the ion-sound oscillations is so high that the force of friction between electrons and the vibrations does not allow the bulk of electrons to accelerate freely in the electric field, that is, $A \gg (e/m)Ev^2$.[1)] Then (4) can be solved by suc-

[1)]We shall show below that if there were no runaway electrons initially, then they appear after a time $t \sim (eE/m)^{-1}c_e^2/v_0$.

cessive approximations in terms of the quantity A^{-1}.

As the zeroth approximation we choose an arbitrary function $f_0(v)$ which does not depend on the angle θ. The correction $f_1(v, \theta)$ to the function $f_0(v)$ is of the form

$$f_1(v,\theta)=\int^{\theta}\left[\frac{e}{2m}\frac{E}{A}v^3\frac{\partial f_0}{\partial v}\frac{\sin^2\theta+G(v)}{\sin\theta}+\frac{B}{A}\frac{\partial f_0}{\partial v}\right]d\theta. \quad (10)$$

The constant $G(v)$ should be set equal to zero, so that the flux of the particles vanishes on the axis $\theta = 0, \pi$. If we multiply (4) by $\sin\theta$, integrate with respect to θ from 0 to π, and use the expression for f_1, we obtain for f_0 a differential equation that is accurate to terms of the order A^{-1}:

$$\frac{\partial f_0}{\partial\tau}=\frac{a}{u^2}\frac{\partial}{\partial u}u^5\frac{\partial f_0}{\partial u}+\frac{b}{u^2}\frac{\partial}{\partial u}\frac{1}{u}\frac{\partial f_0}{\partial u},$$

$$a=-\frac{1}{4}\frac{e}{m}E\frac{c_e^3}{v_0}\int_0^{\pi}\sin\theta\cos\theta\,d\theta\int^{\theta}\frac{\sin\theta'\,d\theta'}{A(\theta')},$$

$$b=\frac{1}{2}\left(\frac{e}{m}Ev_0c_e^3\right)^{-1}\int_0^{\pi}\left(D-\frac{B^2}{A}\right)\sin\theta\,d\theta. \quad (11)$$

We have introduced here the following convenient dimensionless variables. The velocity is measured in units of c_e and $u = v/c_e$, and the time in units of $t_0 = (mc_e/eE)c_e/v_e$. The sequence in which the systems (8) and (10) should be solved is as follows: it is necessary to substitute the solution (10) in (8) and integrate with respect to the velocity. This yields a nonlinear integral-differential equation with respect to $|\varphi_k|^2$ (see (21)).

ON THE LAW GOVERNING THE GROWTH OF ION-SOUND OSCILLATIONS IN THE CASE OF CURRENT INSTABILITY

It is natural to attempt to obtain first solutions corresponding to stationary states, when ion-sound oscillations of constant amplitude are excited in the plasma and hinder the free acceleration of most or all electrons. If the distribution function of the electrons is isotropic, as in our case, then it is necessary that the amplitudes of the oscillations differ from zero in the half-space $k > 0$ for at least one value of $|k|$. In the half-space $k < 0$ the oscillations certainly attenuate and there should be none there.

Let us turn to Eq. (8). The first term in this equation contains a small quantity $\omega/kv \sim c_s/v$, so that it is sufficient to substitute in it for f the quantity $f_0(v, t)$ and the integral with respect to the velocity can be readily calculated. The second term in the expression for the increment, on the contrary, is determined by the asymmetrical part $f_1(v, \theta)$ of the distribution function. In this term we can replace $\delta(\omega - \mathbf{k}\cdot\mathbf{v})$ by $\delta(\mathbf{k}\cdot\mathbf{v})$. This means that for each fixed $\mathbf{k}$ the contribution to the integral is made by electrons whose velocities lie, accurate to a small angle $\sim \omega/kv$, in a plane perpendicular to the wave vector. This condition can be used and integration carried out over the angle θ. The quantity k_θ, at the accuracy chosen

$$k_\theta=-\cos\theta'/\sin\theta.$$

If we take into account these simplifying circumstances, then the expression for the increment can be presented in the form

$$\gamma=-\frac{\pi}{2}\frac{\omega^3}{k^2}\frac{M}{mn}\left(2\pi\frac{\omega}{k}f\right)\frac{\omega}{k}$$
$$+2\cos\theta'\int_0\frac{d\theta}{(\sin^2\theta-\cos^2\theta')^{1/2}}\frac{\partial}{\partial\theta}\int_0^{v_{max}}f_1dv\Big\}. \quad (12)$$

The upper limit of integration v_{max} is determined by the condition of applicability of the formula (10) for f_1, namely, $A \gg eEv^2/m$. The runaway electrons, $v^2 \gg mA/eE$, are not taken into account in this formula. Their contribution to the increment will be estimated below. The damping of the oscillations by the ions is allowed for by the condition that the value of $|k|$ be bounded from above. From (12) we can see, in particular, that the ion-sound instability has the following property. If the asymmetrical part of the distribution function $f_1(v, \theta)$ decreases with increasing v more rapidly than v^{-1}, then the particles with the larger velocities will make no contribution to the increment.

A similar simplification of the calculation can be made in the expressions for the coefficients A, B, and D, which can be rewritten in the form

$$\left.\begin{matrix}A\\-B\\D\end{matrix}\right\}=2\pi\frac{e^2}{m^2}\int k\,dk\int_0^1\frac{dx}{(1-x^2)^{1/2}}|\varphi_k(\cos\theta')|^2\left\{\begin{matrix}k^2x^2\\\omega kx;\\\omega^2\end{matrix}\right.$$

$$x=\frac{\cos\theta'}{\sin\theta}. \quad (13)$$

We have to find a solution of (12) for which

$$\gamma_{k,\theta'}=0,\quad \pi/2\geqslant\theta'\geqslant 0;$$
$$\gamma_{k,\theta'}<0,\quad \theta'>\pi/2. \quad (14)$$

The first term in (12) does not depend on the angle θ'. It can be readily checked directly that if

$$\frac{\partial}{\partial\theta}\int_0^{v_{max}}f_1(v,\theta)\,dv=\frac{C}{\sin\theta}, \quad (15)$$

then the second term in (12) is a step-like function of the angle θ', which reverses sign at $\theta' = \pi/2$.

Indeed,

$$2\cos\theta' \int\limits_{\pi/2-\theta'}^{\pi/2} \frac{d\theta}{\sin\theta(\sin^2\theta - \cos^2\theta')^{1/2}}$$

$$= 4\,\mathrm{sign}\cos\theta' \tan^{-1}\left[\cos\theta\left(\frac{\sin^2\theta}{\cos^2\theta'} - 1\right)^{-1/2}\right]\Bigg|_{\pi/2-\theta'}^{\pi/2}$$

$$= -2\pi\,\mathrm{sign}\cos\theta'.$$

Since the constant C does not depend on k, the condition (14) can be derived for only one value of k, and in order for all the k to be stable, we must set this value of k equal to $k_{max} \equiv k_0 \approx \omega_{pi}/v_0$ and $C = v_0 f(v_0)$. This means that if steady-state solutions of this type exist, then the values of the wave vectors of the ion-sound oscillations should lie in a narrow interval about the values $k_0 = \omega_{pi}/v_0$ and the frequency should be equal to $\sim\omega_{pi}$. Therefore, the distribution function of the waves $N_{k,\theta'} = (\partial\epsilon/\partial\omega)k^2|\varphi_k|^2/8\pi$ can be approximated by the formula

$$N_{k,\theta'} = N_{\theta'}\delta(k - k_0). \quad (16)$$

Formula (15) is an integral equation for the quantity $N_{\theta'}$. We substitute in it the explicit expression (10) for f_1 and for the constant $C = v_0 f(v_0)$, and integrate with respect to v. As a result we obtain an equation relating the quantities A and B:

$$v_0 A = B\sin\theta + \frac{3}{8\pi}\frac{e}{m}E\frac{n_c}{f(v_0)}\sin^2\theta, \quad n_c = 4\pi\int\limits_0^{v_{max}} f_0 v^2 dv. \quad (17)$$

If we substitute $N_{k,\theta'}$ in the form of (16) in A and B and integrate with respect to k, we obtain

$$\int\limits_0^1 \frac{x^2 - xy}{(1-x^2)^{1/2}} N(yx)\,dx = \frac{3\pi}{4}y^2, \qquad N_{k\theta'} = N(z)N_0,$$

$$x = \frac{\cos\theta'}{\sin\theta}, \quad y = \sin\theta, \quad N_0 = \frac{1}{16\pi^4\omega_{pi}^2}\frac{m}{e}E\frac{n_c}{f(v_0)}. \quad (18)$$

We have sought the solution of this equation in the form of a series in $z = xy$:

$$N(z) = z^2(1 + a_1 z + \ldots).$$

The coefficients of this series satisfy the recurrence relation

$$a_n = a_{n-1}\frac{n+4}{n+3} = \frac{n+4}{4}.$$

Such a series can be readily contracted and we obtain

$$N_{\theta'} = z^2\frac{4-3z}{(1-z)^2}N_0, \quad z = \cos\theta'. \quad (19)$$

The result obtained has no physical meaning, since the oscillation energy $\int\limits_0^1 \omega_{pi} N_{\theta'} dz$ vanishes at infinity.

In order to eliminate the divergence of N(z) at z = 1, we must forego the requirement that the amplitudes of the oscillations do not change in time. Since $N_{\theta'}$ diverges at small angles, it is natural to assume that the fastest to grow are oscillations in just this direction and that outside a small vicinity of $\theta' = 0$ the oscillations are stationary. This assumption is satisfied by the function N(z, t) chosen in the form

$$N(z,t) = z^2(4-3z)\left[1 - (1-\varepsilon(z,t))z\right]^{-2}, \quad (20)$$

where $\epsilon(z,t)$ is assumed to be small compared with unity at any instant of time and for arbitrary z. Therefore N(z, t) is close to N(z) if $1 - z \gg \epsilon(z,t)$.

Let us consider a small vicinity near the point z = 1. The equation for N(z, t) can be obtained by substituting the expression (10) for $f_1(v,\theta)$ in formula (12) for $\gamma_{k,\theta'}$ and using expressions (13), (16) and (20) for the quantities A, B, $N_{k\theta'}$, and N(z, t). We have

$$\frac{1}{2}\frac{d}{d\tau'}\ln N(z,\tau) = \frac{1}{\sqrt{2}\pi}\int\limits_z^1 \frac{y\,dy}{(1-y^2)^{1/2}(y^2-z^2)^{1/2} I(y,t)}$$

$$\times\left[\int\limits_0^y \frac{z'(z-z')N(z',t)\,dz'}{(y^2-z'^2)^{1/2}} + \frac{3\pi}{4}N_0 z y^2\right], \quad (21)$$

$$I(y,t) = \int\limits_0^y \frac{z'^2 dz'}{(y^2-z'^2)^{1/2}} N(z',t), \quad y = \sin\theta,$$

$$\tau' = t\left(\frac{\pi^2}{2\sqrt{2}}\frac{\omega^4}{k^3}\frac{M}{nm}f(v_0)\right) \approx t\omega_{pi}\frac{1}{16}\sqrt{\frac{m}{M}}\left(\frac{v_0}{c_s}\right)^3.$$

We see from this equation why there are no stationary solutions. Indeed, when z = 1 the right side of the equation is always positive. Since we have assumed that $\epsilon(z,t) \ll 1$, the values of the integrals of $N(z',t)$ in (21) are determined by the behavior of the integrand at the outer limit. We shall use this circumstance and transform (21) in the region of $1 - z \ll 1$ in the form

$$\frac{d}{d\tau'}\ln N(z,\tau') = -\frac{1}{\varepsilon+\xi}\frac{\partial\varepsilon}{\partial\tau'} = \frac{1}{\pi}\int\limits_0^\xi \frac{d\eta}{[\eta(\xi-\eta)]^{1/2}}$$

$$\times\int\limits_\eta^1 \frac{(\xi'-\xi)\,d\xi}{(\xi'-\eta)^{1/2}(\xi'+\varepsilon)^2}\left[\int\limits_\eta^1 \frac{d\xi'}{\sqrt{\xi'-\eta}\,(\xi'+\varepsilon)^2}\right]^{-1}, \quad (22)$$

where we have introduced new variables $\xi = 1 - z$, $\xi' = 1 - z'$, and $\eta = 1 - y$. The last term in (21) is small when $\xi \le \epsilon$ compared with the first term, like $\epsilon^{1/2}$, and we have left it out from (22).

It is easy to show that (22) has solutions of the form $\epsilon(\xi, t) = \epsilon_0(\xi t)/t$. Indeed, the equation for ϵ_0 does not depend explicitly on the time:

$$\frac{1}{\varepsilon_0+\psi}\left(\psi\frac{d\varepsilon_0}{d\psi}-\varepsilon_0\right)=-\frac{1}{\pi}\int_0^{\psi}\frac{d\chi}{[\chi(\psi-\chi)]^{1/2}}$$

$$\times\int_{\chi}^{\infty}\frac{(\psi'-\psi)\,d\psi'}{(\psi'-\chi)^{1/2}(\psi'+\varepsilon_0)^2}\left[\int_{\chi}^{\infty}\frac{d\psi'}{(\psi'-\chi)^{1/2}(\psi'+\varepsilon_0)^2}\right]^{-1},$$

$$\psi=\xi t. \quad (23)$$

Let us express the energy of the ion-sound oscillations in a unit volume by ϵ_0:

$$W=2\pi\int_0^1\omega_{pi}k_0^2N_{\theta'}\,dz=2\pi\,\omega_{pi}N_0k_0^2\int_0^1\frac{d\xi}{(\xi+\varepsilon(\xi,t))^2}$$

$$=2\pi\omega_{pi}N_0k_0^2\int_0^{\infty}\frac{d\psi}{(\psi+\varepsilon_0)^2}\,t. \quad (24)$$

We have obtained the important result that the energy of the oscillations increases in proportion to the time. We must still verify that the coefficient of t is finite. Let us assume that the integral diverges at a point ψ_0. Then the ratio of the similar diverging integrals in (23) would be equal to $\psi_0-\psi$ and the equation for ϵ_0 would take the form

$$\psi\frac{d(\varepsilon_0+\psi)}{d\psi}=(\varepsilon_0+\psi)(\psi-\psi_0)+(\varepsilon_0+\psi).$$

Its solution $\epsilon_0+\psi=Ce^{\psi}\psi^{1-\psi_0}$ is not equal to zero only at the point $\psi=\psi_0$, thus contradicting the assumption made. The exact value of the coefficient can be obtained by solving (23).

RUNAWAY ELECTRONS

If we know the distribution function $N_{k\theta'}$ for the oscillations, then we can obtain explicit expressions for the quantities A, B, and D and for the particle distribution function $f_0+f_1(v,\theta)$, and then calculate the electric current density j in the steady state. All these quantities are determined in final analysis by the function $f_0(v,t)$. The time variation of the function $f_0(v,t)$ is described by (11). Let us write it out once more

$$\frac{\partial f}{\partial\tau}=\frac{a}{u^2}\frac{\partial}{\partial u}u^5\frac{\partial f}{\partial u}+\frac{b}{u^2}\frac{\partial}{\partial u}\frac{1}{u}\frac{\partial f}{\partial u}, \quad (25)$$

where

$$a=-\frac{1}{4}\frac{e}{m}E\frac{c_e^3}{v_0}\int_0^{\pi}\sin\theta\cos\theta\,d\theta\int^{\theta}\frac{\sin\theta'\,d\theta'}{A(\theta')}\approx\frac{2f(v_0)c_e^3}{n_c},$$

$$b=\frac{1}{2}\left(\frac{e}{m}Ev_0c_e^3\right)^{-1}\int_0^{\pi}\left(D-\frac{B^2}{A}\right)\sin\theta\,d\theta\approx 5\cdot 10^{-3}\frac{n_c}{fc_e^3}. \quad (26)$$

In calculating the coefficients a and b we have used formula (20). The first term on the right side of (25) describes the heating of the electrons by an electric field. It is obvious that the particles are heated principally when their velocity is antiparallel to the field. The angles $\theta\approx\pi/2$ make no contribution to the coefficient a. It therefore does not depend explicitly on the time. The second term takes into account the heating due to the small inelasticity of the scattering. The order of magnitude of a is unity.

For large velocities $u\gg 1$ the principal term in (25) is the first one. An equation of the form

$$\frac{\partial f}{\partial\tau}=\frac{1}{u^2}\frac{\partial}{\partial u}u^5\frac{\partial f}{\partial u} \quad (27)$$

describes, in particular, the heating of electrons in a Lorentz plasma. Its solution is described in detail in a paper by Kruskal and Bernstein.[7] We shall use the results of this paper. Rapid heating, which leads to the appearance of runaway electrons, begins with $\tau\sim 1$. The electron distribution function has in the region $u\tau>1$ the form

$$f=\frac{n}{24\pi^{3/2}c_e^3}\,\frac{1}{(u\tau)^4}\,\tau^{2/3}\exp(-\tau^{-2/3})\equiv\frac{g(\tau)}{u^4}. \quad (28)$$

When $\tau\gtrsim 1$ the solution (28) becomes incorrect, since particles with $u\lesssim 1$, for which (27) is not valid, begin to influence the heating described by this solution. It is physically clear that the additional heating described by the second term in (26) can only increase the particle flux in a high-velocity region. We shall therefore use formula (28) for estimates even when $\gtrsim 1$.

When $\tau<1$ ($t<mc_e^2/eEv_0$) the number of heated particles ($v>c_e$) is exponentially small and we can neglect the variation of the initial distribution function. The electron current for $\tau<1$ does not depend on the time and is equal to

$$j=-e\int f_1v\cos\theta\,d\mathbf{v}\approx -env_0 \quad (29)$$

when $\tau\gtrsim 1$ ($t>mc_e^2/dEv_0$) the current of the heated particles ($(c_e/v_0)^{1/2}c_e>v>c_e$) increases rapidly and after several τ units it becomes equal to

$$j=-e2\pi\int_0^{\pi}\int_0^{v_{max}}v\cos\theta\frac{e}{2m}Ev^3\frac{\partial f}{\partial v}\int^{\theta}\frac{\sin\theta_1\,d\theta_1}{A(\theta_1)}\sin\theta\,d\theta\,v^2dv$$

$$\approx -enc_e. \quad (30)$$

Further increase in the current is connected with the appearance of runaway electrons.

So long as the number of runaway electrons is small, we can disregard their contribution to the increment, and when solving (4) for the runaway electrons we can assume the oscillations to be specified and describe them by means of the distribution function (20). At velocities $v \gg c_e$ the largest of the diffusion terms in (4) is the first term. Its order of magnitude is $eEc_e^3/v_0 m$ and is comparable with the last term, which takes into account the change in the electron distribution function under the influence of the electric field when $v^2 \approx c_e^3/v_0 \equiv v_{max}^2$. At high particle velocities the scattering by the oscillations loses its effectiveness, and the particles begin to acquire velocity freely in the electric field. If we neglect the scattering completely, then the distribution function f of the runaway electrons is an arbitrary function of $s = (eEt/m - v\cos\theta)$ and $v_\perp = v\sin\theta$. Its form can be established by equating the diffusion flux of the particles in u-space at $u < v_{max}/c_e$,

$$j_u = -u^5 \partial f / \partial u = 5g(\tau), \tag{31}$$

and the flux of the runaway electrons, which in our variables is equal to $\tilde{f} c_e/v_0$. Therefore

$$\tilde{f} = \frac{v_0}{c_e} 5g\left(\tau - \frac{v_0 u}{c_e}\cos\theta\right) F(u_\perp). \tag{32}$$

Since the velocity of the runaway electrons increases predominantly only along the field $u_{||} = u\cos\theta$, and the component $u_\perp$ can change only as a result of scattering by the oscillations and is small, we can state the following with respect to $F(u_\perp)$: the function $F(u_\perp)$ should decrease rapidly with decreasing $u_\perp$ when $u_\perp > v_{max}/c_e$. If in (4) we take into account the diffusion with respect to the angle θ as being a small correction, then the function F will depend little on $u_{||}$.

Let us estimate the contribution of the runaway particles to the increment (8). To calculate the integral we must know the dependence of $\tilde{f}$ on $u_\perp$. We choose it in the form $\tilde{f} \sim \ln u_\perp$, so that the function becomes continuous with $f_1(v, \theta)$ when $u \approx v_{max}/c_e$. In light of the foregoing, this choice is natural. Thus,

$$\delta\gamma \approx -\frac{\gamma}{n} \int_{v_{max}/c_e}^{c_e\tau/v_0} du \int_{\theta''}^{v_{max}/c_e u} \frac{\partial \tilde{f}}{\partial \theta}\left(\frac{\sin^2\theta}{\sin^2\theta''} - 1\right)^{-1/2} d\theta$$

$$\approx \frac{\gamma}{n}\frac{v_0}{c_e} \int_{v_{max}/c_e}^{v_{max}/c_e\theta''} \left[\frac{\pi}{2} - \tan^{-1}\left(\frac{v_{max}^2}{c_e^2 u^2 \theta''^2} - 1\right)^{-1/2}\right]$$

$$\times 5g\left(\tau - \frac{v_0 u}{c_e}\right) du \approx \gamma \frac{\tilde{n}}{n}\frac{v_0}{c_e}, \tag{33}$$

where $\theta'' = (\pi/2) - \theta'$. We see from this estimate that the contribution $\delta\gamma$ of the runaway electrons to the increment γ is small. The current of the runaway electrons is equal to

$$\tilde{j} = -\frac{1}{2} e\tilde{n} \frac{e}{m} Et. \tag{34}$$

This formula, strictly speaking, is applicable when $\tau \lesssim 1$, so long as $\tilde{n} \ll n$. When $\tau \approx 1$ ($t \approx mc_e^2/v_0 eE$) the runaway-electron current is approximately equal to $-enc_e^2/v_0$, and the thermal velocity scattered is $\Delta v_{||} \approx c_e^2/v_0$.

OVER-ALL PICTURE OF THE PHENOMENON AND DISCUSSION OF THE LIMITS OF APPLICABILITY OF THE OBTAINED SOLUTIONS

On the basis of the results obtained we can describe the over-all picture of the phenomenon. After a time $t = mv_0/eE_0$ following the application of the electric field E, the electrons acquire a velocity v_0 which exceeds by several times the thermal velocity of the ions, and ion-sound oscillations begin to be excited in the plasma. During a time of the order of $(M/m)^{1/2}\omega_{pi}^{-1}$, so long as the instability is developing, the electron current will grow. Then the oscillation amplitude will reach a value such that the frequency of the scattering of electrons by the oscillations becomes larger than $eE/c_e m$, $c_e = (2T_e/m)^{1/2}$, and the electron acceleration will cease. Some kinetic energy in excess of $mv_0^2/2$ will go over into oscillations and thermal motion, the electrons will decelerate to an average velocity v_0, and the states which we have investigated in detail will set in. The initial phase of development of instability, as already noted, was investigated with the aid of a computer.[6]

During a time $t < t_0$ ($t_0 = mc_e^2/eEv_0$) after the start of the instability the electron current does not change and is equal to $-env_0$. The work of the electric field $enEv_0t$ during this period goes to increase the energy of the oscillations and to heat the electrons. The energy density of the oscillations increases in proportion to the time and reaches by the instant $t = t_0$ a value $\sim nT_e$. The growth is confined predominantly to oscillations that travel in a direction strictly opposite to the field E. Simultaneously the electrons are heated, the current of the heated electrons increases, and the runaway electrons appear. All these processes have a characteristic time t_0, and after the lapse of several times t_0 the average energy of the electrons which experience frequent collisions increases by a factor $(c_e/v_0)^{1/2}$, the rms scatter of the velocities $v_{||}$ of the runaway electrons reaches a value $\sim Mc_e^2/m$, and their current reaches $-enc_e^2/v_0$. Of course, the current of the runaway electron increases with time even further, but the solution obtained by us ceases to be valid when $t > t_0$. Thus the ion-sound instability limits the growth of the current only in a time

$$t \approx \frac{mc_e^2}{eEv_0} \approx \frac{mc_e}{eE}\left(\frac{M}{m}\right)^{1/2}$$

Let us investigate the limits of applicability of the obtained solutions. It is necessary first that the time t_0 be larger than $\gamma^{-1} \approx \omega_{pi}^{-1}(M/m)^{1/2}$. Only then is it meaningful to consider the steady state. This condition is satisfied if $E^2/8\pi nT_e < m/M$. Further, we have not taken into account in (2) the change in the amplitudes of the ion-sound oscillations, due to the nonlinear interaction of the waves. The principal nonlinear process in our conditions is the scattering of the ion-sound waves by ions. For oscillations described by (20), it has a frequency[8]

$$\nu \approx \omega_{pi}\frac{W}{nT_e}\frac{c_i^2}{v_0^2}\left[\sin^2\theta' + \frac{c_i^2}{v_0^2}\right], \quad c_i = \left(\frac{2T_i}{M}\right)^{1/2}.$$

The maximum value of c_i^2/v_0^2 at which ion-sound instability can occur is numerically equal to ~ 0.1. Therefore, strictly speaking, we cannot neglect nonlinear scattering of the waves by the ions if W/nT_e is of the order of unity, that is, the solution (20) is valid for times shorter than $t_0 = mc_e^2/eEv_0$.

Qualitatively, in the stationary state, after the lapse of a time $t \approx t_0$, when the energy of the oscillations propagating strictly opposite to the field E grows to a value $\sim nT_e$, the nonlinear scattering of the waves by the ions should cause a new state to be established, in which the linear increment becomes comparable with the scattering frequency $\nu_{\theta'}$. We hope to investigate this state in a forthcoming paper. But it is clear even now that all the results pertaining to the heating of the electrons remain qualitatively in force.

A few words about ion heating. The minimum velocity v_0 is determined for the condition that the damping of the oscillations with phase velocity $\omega/k = v_0$ by the ions is somewhat smaller than the buildup of these oscillations by the electrons. Therefore, even during the process of creation of the oscillations, a noticeable part of the work of the field goes to heating the ions. The ions should be heated also when ion-sound oscillations are scattered by ions. It can be shown that for each scattering act the ions obtain a fraction of wave energy approximately equal to $c_i^2/v_0^2 \approx 0.1$, so that the rate at which the ions are heated by this process is

$$\frac{dT_i}{dt} \approx 10^{-2}\,\omega_{pi}\frac{W}{nT_e}\frac{W}{n} \approx 10^{-2}\,\omega_{pi}T_e.$$

INSTABILITY OF CURRENT IN A BOUNDED PLASMA AND IN A STRONG MAGNETIC FIELD

An interesting result is obtained if for some reason the oscillations can be excited only in a cone with apex angle θ_0 about the direction of the electric field. Such a situation takes place when there is a strong magnetic field parallel to E, and the plasma frequency of electrons is comparable with or smaller than the electron cyclotron frequency eH/mc. Another example is when the current flows along a narrow tube filled with a plasma. In such experiments the oscillation energy can be taken out to the wall with group velocity

$$\frac{\partial\omega}{\partial\mathbf{k}} = c_s(kr_D)^{-2}\frac{\mathbf{k}}{k}.$$

The oscillations traveling radially are therefore more stable.

For oscillations inside the cone $\theta' < \theta_0$ we can use the solution (20). The particles whose velocity vector falls in the interval of values $\theta = \pi/2 \pm \theta_0$ are scattered by the waves and diffuse over the angle θ, but as soon as they fall in the cone $\theta < \pi/2 - \theta_0$ they begin to accelerate freely. Since the characteristic value of the diffusion flux

$$j_\theta \approx \frac{A}{c_e^2}f \approx \frac{e}{m}E\frac{c_e}{v_0}f$$

is much larger than the flux $\tilde{j}_\theta = eE\tilde{f}\sin\theta/m$ of the freely accelerating particles, the velocity of the runaway electrons is determined by the value of the flux $\tilde{j}_\theta$, where $\tilde{f}$ must be replaced by f. So long as the number of runaway electrons is small, their current will increase like

$$\tilde{j} = -2\pi e\int_0^{c_e\cos\theta_0} v_\perp dv_\perp \int_0^{eEt/m} v_\parallel \tilde{f}\, dv_\parallel \approx -\frac{en}{c_e}\left(\frac{e}{m}Et\right)^2\cos^2\theta_0,$$

until all the electrons run away after a time $t \approx (mc_e/eE)\cos^{-2}\theta_0$.

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