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FLOW OF A SUPERFLUID LIQUID IN POROUS MEDIA

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A modified equilibrium equation of phenomenological superfluidity theory is proposed for describing helium II in porous media. It is shown that in the vicinity of a large volume of helium II the density of the superfluid component in a porous substance increases, the "penetration depth" of the influence being the larger the narrower the pores. The flow of the superfluid component through a porous medium is considered and the critical fluxes at which superfluidity is destroyed are evaluated.

1. IN an earlier paper[1] we proposed to use for the description of the behavior of a superfluid situated in a porous medium a phenomenological wave function[2] $\psi = fe^{i\varphi}$ averaged over a volume containing many pores. In the same paper we considered particular examples pertaining to the case of a liquid at rest, and proposed a modified Ginzburg-Pitaevskiĭ equation for the determination of its average wave function. In this article we derive a more general equilibrium equation, and consider problems connected with the flow of the liquid. In particular, we discuss the possibility of observing in helium II the analog of the Josephson direct current.[3,4]

2. We shall use dimensionless units in which the local density of the superfluid component is equal to the square of the modulus of the unaveraged function ψ, and the velocity of the superfluid component is equal to the gradient of its phase. The density is measured in units of $m\alpha/\beta$ and f in units of $(\alpha/\beta)^{1/2}$, where m is the mass of the helium atom, $\alpha \approx 4.5 \times 10^{-17}(T_\lambda - T)$ erg, and $\beta \approx 4 \times 10^{-40}$ erg-cm^3;[2] the distances are measured in units of $a_0 = \hbar/\sqrt{2m\alpha} \approx 4.3 \times 10^{-8} \times (T_\lambda - T)^{-1/2}$ cm, the symbols ∇ and Δ denote differentiation with respect to the dimensionless coordinates, the free energy per unit volume is measured in units of α^2/B, the liquid flow in units of $\alpha\sqrt{2m\alpha}/\beta \approx 2.8 \times 10^3(T_\lambda - T)^{3/2}$ g-cm^{-2} sec^{-1}, and the velocity in units of $\hbar/ma_0 = \sqrt{2\alpha/m} \approx 3.7 \times 10^3(T_\lambda - T)^{1/2}$ cm/sec.

3. It is advantageous to consider first the flow of a superfluid liquid in one channel, without resorting to the averaging proposed in [1]. We assume that the channel is sufficiently narrow, so that the normal component of the helium II remains immobile in it. Then the equilibrium equation[2] has the form

$$\Delta f + f - f^3 - f(\nabla\varphi)^2 = 0. \tag{1}$$

Let the channel be a plane slit of width δ in the direction of the y axis, extending without limits in the two remaining directions, and let the flow be in the direction of the x axis ($\nabla\varphi = (v,0,0)$, $v = \text{const}$). Then $f = f(y)$ and the solution of (1) satisfying the boundary conditions $f(\pm\delta/2) = 0$ is

$$\frac{\delta}{2} - y = \frac{\sqrt{2}}{[2(1-v^2)-f_0^2]^{1/2}} \times F\left(\arcsin\frac{f}{f_0}, \frac{f_0}{[2(1-v^2)-f_0^2]^{1/2}}\right), \tag{2}$$

where $f(0) \equiv f_0 = f_{max}$ is determined by the equality

$$\frac{\delta}{2} = \frac{\sqrt{2}}{[2(1-v^2)-f_0^2]^{1/2}} K\left(\frac{f_0}{[2(1-v^2)-f_0^2]^{1/2}}\right), \tag{3}$$

F and K are elliptic integrals of the first kind (see [2], where the case $v = 0$ is considered).

According to (3) f_0 vanishes, i.e., superfluidity stops, when

$$1 - v^2 - \pi^2/\delta^2 = 0. \tag{4}$$

This equation determines the critical dimensions for the slit for a given flow velocity (see [2])

$$\delta_c^2 = \pi^2/(1 - v^2) \tag{4a}$$

or the critical (maximal) velocity for a slit of given width

$$v_m^2 = 1 - \pi^2/\delta^2. \tag{4b}$$

For an unbounded medium this result is obtained by Lifshitz and Kaganov.[5]

It must be noted, however, that the critical velocity at which the density of the superfluid component vanishes cannot be realized in an experiment in which the liquid flux is specified, rather than the flow velocity. In fact, the critical quantity is the flux corresponding to a velocity much lower than v_m. Let us explain the reason for this circumstance in the two extreme cases $\delta \gg \pi$ and $\delta \gtrsim \pi$. In the first case—for a sufficiently wide slit—we have in accordance with (3)

$$f_0^2 = 1 - v^2, \tag{5}$$

and $f = f_0$ almost everywhere, except for thin layers of width of the order of several times a_0 next to the wall. Under these conditions—the conditions of an unbounded liquid—the flux of the superfluid component j is equal to

$$j = (1 - v^2)\, v. \tag{6}$$

Only at relatively small v does an increase in velocity correspond to an increase in the flux. When $v = v_c$, where

$$v_c^2 = {}^1/_3, \tag{7}$$

the flux reaches its maximum value

$$j_c^2 = {}^4/_{27}. \tag{8}$$

From thereon the increase in velocity corresponds not to an increase but to a decrease in the flux (in connection with the decrease in density in accordance with (5)). Being the maximal value of the flux compatible with the presence of superfluidity, j_c is a critical flux, above which (that is, at a velocity $v_c = 1/\sqrt{3}$, and not $v_m = 1$) a radical change takes place in the density of the superfluid component, from a value

$$f_c^2 = {}^2/_3 \tag{9}$$

to zero.

It must be noted that to each value of $0 < j < j_c$ there correspond two values of v ($0 < v_1 < v_c$, $1 > v_2 > v_c$) and two values of f ($1 > f_1 > f_c$, $0 < f_2 < f_c$). However, thermodynamically more favored is a large value of f, and therefore values $f_2 < f_c$ and with them also values $v_2 > v_c$ cannot be realized at all in stationary states with specified values of the flux.

In the second case of a narrow slit, $\delta \gtrsim \pi$, formulas (2) and (3) yield

$$f^2 = \frac{4\delta^2}{3\pi^2}(1 - v^2)\left(1 - v^2 - \frac{\pi^2}{\delta^2}\right)\cos^2\frac{\pi y}{\delta} \tag{10}$$

from which, in analogy, with the derivation of (7)–(9), we obtain

$$v_c^2 = \frac{1}{3}\left(1 - \frac{\pi^2}{\delta^2}\right) = \frac{1}{3} v_m^2, \tag{11}$$

$$j_c = \frac{8}{9\sqrt{3}}\left(1 - \frac{\pi^2}{\delta^2}\right)^{3/2}\cos^2\frac{\pi y}{\delta}, \tag{12}$$

$$f_c^2 = \frac{8}{9}\left(1 - \frac{\pi^2}{\delta^2}\right)\cos^2\frac{\pi y}{\delta}. \tag{13}$$

Averaging of the quantities f_c^2 and j_c over the cross section yields

$$\overline{j_c^2} = \frac{16}{243}\left(1 - \frac{\pi^2}{\delta^2}\right)^3, \tag{12a}$$

$$\overline{f_c^2} = \frac{4}{9}\left(1 - \frac{\pi^2}{\delta^2}\right). \tag{13a}$$

4. We now proceed to derive an equilibrium equation for the function $fe^{i\varphi}$ averaged over a volume containing many pores. It must be borne in mind here that a narrow channel (pore) is characterized by the presence of a gradient of f transverse to the channel and by vanishing of f when $\delta = \delta_{0c}$, where δ is the width of the channel and δ_{c0}, its critical value for $\nabla\varphi = 0$, which has an order of magnitude of 3–5 times a_0 for different shapes of the channel cross section. We shall write the usual expression for the free energy per unit volume $F = E - 2j\nabla\varphi$ ($j = f^2\nabla\varphi$):

$$F = F_0 - f^2 + {}^1/_2 f^4 + (\nabla f)^2 - f^2(\nabla\varphi)^2, \tag{14}$$

where $F_0 \neq F_0(f, \nabla f)$ and it is implied that the pores are too narrow to allow motion of the normal component of the helium II.

The procedure for averaging (14), which we must follow, consists of two stages. We first average in a single channel, as a result of which the distribution of f transverse to the pore becomes smoothed out and only a relatively weak dependence of the average density of the superfluid component on the longitudinal coordinate remains. As to $\nabla\varphi$, it is in general constant across the channel, and is not subject to averaging during the first stage. More accurately speaking, it is averaged only in the sense that to each "elementary channel" there is assigned one and only one width δ. Then, after averaging over a volume containing many pores, we must "dilute" the superfluid component, spreading its density also over that part of the volume in which there is no liquid at all. This automatically solves the question of averaging the flux of the superfluid component, since the average flux is equal to the product of the average density by the velocity.

The first stage of the averaging leads to the expression

$$\overline{F} = \overline{F}_0 - a^2\overline{f^2} + \frac{c}{2}(\overline{f^2})^2 + (\nabla\sqrt{\overline{f^2}})^2 - \overline{f^2}(\nabla\varphi)^2,$$

$$a^2 = 1 - \delta_{c0}^2/\delta^2, \quad c = \overline{f^4}/(\overline{f^2})^2. \tag{15}$$

In writing out (15) we took account of the fact that the square of the gradient f transverse to the channel is proportional to the ratio $\overline{f^2}/\delta^2$. The choice of δ_{C0}^2 as the coefficient in this ratio is justified by the fact that it ensures the vanishing of $\overline{f^2}$ when $\delta = \delta_{C0}$. Indeed, according to (15), the equilibrium density of the superfluid component in a channel of width δ is determined for $\nabla \overline{f^2} = \nabla\varphi = 0$ by the equation

$$\overline{f^2} = a^2/c \tag{16}$$

and vanishes together with a^2 when $\delta = \delta_{C0}$ (the unit of measurement of $\overline{f^2}$ and the value of a^2 depend on the temperature). We note that for plane slits with $\delta \gtrsim \delta_{C0} = \pi$ the value of c is close to 1.5 and for cylindrical capillaries with $\delta \gtrsim \delta_C \approx 4.8$ it is of the order $c \approx 2.1$. In broader channels c becomes the closer to unity the stronger the inequality $\delta > \delta_{C0}$. When $\delta \to \infty$ we have $a^2 \to 1$, $c \to 1$, and at equilibrium $\overline{f^2} \to 1$.

During the second stage of the averaging it is necessary to introduce the quantity b—the volume of the pores per unit volume of the porous medium. It is easy to see that

$$\langle \overline{F} \rangle = b\overline{F}, \quad \langle \overline{f^2} \rangle = b\overline{f^2}.$$

The final expression for the mean free energy per unit volume of the porous medium takes the form

$$\langle \overline{F} \rangle = \langle \overline{F}_0 \rangle - a^2 \langle \overline{f^2} \rangle + \frac{c}{2b} \langle \overline{f^2} \rangle^2 + [\nabla (\langle \overline{f^2} \rangle)^{1/2}]^2 - \langle \overline{f^2} \rangle (\nabla\varphi)^2. \tag{17}$$

The equilibrium density of the superfluid component (per unit volume unit of the porous medium) is determined for $\nabla(\langle \overline{f^2} \rangle)^{1/2} = \nabla\varphi = 0$, in accordance with formula (17), by the expression

$$\langle \overline{f^2} \rangle = a^2 b / c, \tag{18}$$

which, as expected, vanishes not only when $a = 0$ ($\delta = \delta_{C0}$), but also when $b = 0$ (nonporous solid). It must be noted that we have considered earlier[1] the case $b = 1$, that is, it was implied (but not especially stipulated) that the pores occupy the overwhelming part of the volume of the porous matter, and the partitions between pores are quite thin.

To simplify the notation we shall henceforth leave out the averaging symbols and let f stand for the quantity $(\langle \overline{f^2} \rangle)^{1/2}$ and j^2 for the square of the average current, which we shall find convenient to separate in the formulas that follow, since it is a constant quantity. Then the expression for F takes the form

$$F = F_0 - a^2 f^2 + \frac{c}{2b} f^4 + (\nabla f)^2 - \frac{j^2}{f^2}, \tag{17a}$$

which coincides with formula (1) of [1] when $j = 0$ and $b = c = 1$.

5. Expression (17a) corresponds to the following equilibrium equation:

$$\Delta f + a^2 f - \frac{c}{b} f^3 - \frac{j^2}{f^3} = 0, \tag{19}$$

which defines the function f (for specified j) under different boundary conditions.

6. Let us consider the flow of a superfluid liquid in an unbounded porous medium. In this case $\Delta f = 0$, and the value $f = f(j^2)$ which is constant (all over space) is defined by the equation

$$f^6 - \frac{b}{c} a^2 f^4 + \frac{b}{c} j^2 = 0, \tag{20}$$

which has real positive solutions with respect to f^2 only when $j \le j_c$, where the current j_c is defined by

$$j_c^2 = \frac{4b^2}{27c^2} a^6. \tag{21}$$

For broad pores separated by thin partitions, $a = c = b = 1$ and formula (21) goes over into (8). For narrow plane channels, also separated by thin partitions,

$$b = 1, \quad c = 3/2, \quad a^2 = 1 - \pi^2/\delta^2,$$

and (21) goes over into (2a).

The dependence of f on j is described by the formula

$$f^2 = \begin{cases} \dfrac{b}{3c} a^2 + \dfrac{2b}{3c} a^2 \cos\dfrac{\varphi}{3} & \text{for} \quad j^2 < \dfrac{2b^2}{27c^2} a^6 \\ \dfrac{b}{3c} a^2 + \dfrac{2b}{3c} a^2 \cos\dfrac{\pi - \varphi}{3} & \text{for} \quad j^2 > \dfrac{2b^2}{27c^2} a^6 \end{cases}$$

$$\varphi = \arccos|1 - 27c^2 j^2 / 2b^2 a^6|, \quad 0 \leqslant \varphi \leqslant \pi/2. \tag{22}$$

When $j = 0$ we obtain (18). An increase in j decreases f. When j tends to j_c, the density of the superfluid component tends to the value

$$f_c^2 = \frac{2b}{3c} a^2. \tag{23}$$

Under suitable conditions, this formula yields expressions (9) or (13a).

Consequently, when the critical current is exceeded, a sudden vanishing of superfluidity takes place. At the same time, for sufficiently small a, the critical velocity

$$v_c = j_c / f_c^2 = a / \sqrt{3}$$

may turn out to be smaller than the critical velocity for vortex formation. Therefore experimental investigations of the flow of a superfluid liquid in a porous medium can afford an exceptional opportunity for observing the effect of destruction of superfluidity, not masked by vortex formation. To this end one can use a porous sample (for example, a tube filled with porous material) whose dimensions in terms of the characteristic length a_0 are much larger than a certain value which will be determined below.

7. If we do not use a closed porous system in the experiment just proposed, then we deal with the flow of a superfluid liquid from a free volume into a porous medium, and this may alter, generally speaking, the result of the preceding section. In this connection it is natural to consider the flow of a superfluid liquid in infinite space, half of which $(x < 0)$ is a free volume of helium II $(a = 1)$ and the second half $(x > 0)$ is a porous medium $(a < 1)$.

Inasmuch as $\Delta f = 0$ when $x = \pm\infty$, Eq. (20) determines the maximum and minimum values $f_{-\infty}$ and f_∞ of the quantity f, which depends on j^2 in accordance with (22), where we must put $a = b = c = 1$ in the case of $f_{-\infty}$.

To determine the dependence of f on x we use the first integral of (19):

$$\left(\frac{df}{dx}\right)^2 + a^2 f^2 - \frac{c}{2b} f^4 + \frac{j^2}{f^2} = \text{const}, \tag{24}$$

from which it follows that (inasmuch as $df/dx = 0$ for $x = \pm\infty$)

$$\left(\frac{2b}{c}\right)^{1/2} f \frac{df}{dx} = -\left\{(f^2 - f_{\pm\infty}{}^2)^2 \left[f^2 + 2\left(f_{\pm\infty}{}^2 - \frac{b}{c} a^2\right)\right]\right\}^{1/2} \tag{25}$$

and the integration leads to the formulas

$$\left[\frac{f^2 + 2(f_{-\infty}{}^2 - 1)}{3f_{-\infty}{}^2 - 2}\right]^{1/2} = -\operatorname{th}\left\{(3f_{-\infty}{}^2 - 2)^{1/2} \frac{x}{\sqrt{2}} - \operatorname{Arth}\left[\frac{f_0^2 + 2(f_{-\infty}{}^2 - 1)}{3f_{-\infty}{}^2 - 2}\right]^{1/2}\right\} \quad \text{for} \quad x < 0,$$

$$\left[\frac{f^2 + 2(f_\infty{}^2 - ba^2/c)}{3f_\infty{}^2 - 2ba^2/c}\right]^{1/2} = \operatorname{cth}\left\{\left(3f_\infty{}^2 - \frac{2b}{c} a^2\right)^{1/2} \left(\frac{c}{2b}\right)^{1/2} x + \operatorname{Arcth}\left[\frac{f_0^2 + 2(f_\infty{}^2 - ba^2/c)}{3f_\infty{}^2 - 2ba^2/c}\right]^{1/2}\right\} \quad \text{for} \quad x > 0. \tag{26}$$ *

Here f_0 is the value that f must have at $x = 0$ to ensure a smooth joining of the functions f defined in the left and right half-spaces.

It is easy to obtain with the aid of (25) the following equation for f_0:

$$(f_{-\infty}{}^2 - f_0^2)\,[f_0^2 + 2(f_{-\infty}{}^2 - 1)]^{1/2} = (c/b)^{1/2} (f_0^2 - f_\infty{}^2)\,[f_0^2 + 2(f_\infty{}^2 - ba^2/c)]^{1/2}. \tag{27}$$

When $j = 0$ we have $f_{-\infty} = 1$ and $f_\infty = a(b/c)^{1/2}$, and it follows from (26) and (27) that

$$f = \begin{cases} -\operatorname{th}\left[\dfrac{x}{\sqrt{2}} - \operatorname{Arth}\left(\dfrac{1 + a^2\sqrt{b/c}}{1 + \sqrt{c/b}}\right)^{1/2}\right], & x < 0 \\ \left(\dfrac{b}{c}\right)^{1/2} a \operatorname{cth}\left\{\dfrac{ax}{\sqrt{2}} + \operatorname{Arcth}\left[\dfrac{1 + a^2\sqrt{b/c}}{(1 + \sqrt{c/b})\, ba^2/c}\right]^{1/2}\right\}, & x > 0 \end{cases} \tag{26a}$$

These formulas show that proximity to the free volume of the liquid increases the density of the superfluid component in the porous medium. The "depth of penetration" of the influence of the free liquid on the liquid in the narrow channels is of the order of $1/a$, which is just the quantity which we promised to define at the end of Sec. 6. We emphasize that it does not depend on b or c and increases with decreasing δ.

It is interesting to note that when $a = 0$ ($\delta = \delta_{c0}$!) the influence of the free volume maintains the superfluidity in the porous medium, and the "depth of penetration" $1/a$ is infinite. Formulas (26a) yield in this case

$$f = \begin{cases} -\operatorname{th}\left(\dfrac{x}{\sqrt{2}} - \operatorname{Arth}\left(1 + \sqrt{\dfrac{c}{b}}\right)^{-1/2}\right), & x < 0 \\ \sqrt{\dfrac{2b}{c}} \dfrac{1}{x + [2\sqrt{b/c}(1 + \sqrt{b/c})]^{1/2}}, & x > 0 \end{cases} \tag{26b}$$

As already noted earlier,[1] these effects causing the increase of the density of the superfluid component bordering on a porous volume of helium II (to the extent that superfluidity occurs in pores having critical dimensions) are connected with the propagation of the wave field of the condensate in the neighboring regions. In other words, they have the same physical nature as the Josephson effect,[3,4] wherein current flows, without encountering resistance, through an insulator placed between two superconducting samples. This raises the question of the possibility of superfluid flow through a porous partition separating two volumes of helium II, and in particular also in the case when $a = 0$ ("insulator"). This question will be considered in the next section.

It must also be emphasized that the influence of the free volume on the density of the superfluid component in a porous substance (or in an unsaturated film) can be observed also directly, and not only by experimentally observing the analog of the Josephson effect. To this end it would be necessary to carry out measurements similar to the calorimetric experiments used to observe the shift of the λ-point and its vanishing upon reaching critical dimensions.[6] Under such conditions, contact with a large free volume of helium II should lead

*th ≡ tanh, cth ≡ coth, Arth ≡ $\tanh^{-1}$, Arcth ≡ $\coth^{-1}$.

to resumption of superfluidity (and appearance of a λ-transition).

Returning to the flow of liquid from a free volume of helium II into a porous substance, and in particular to formula (27), we note that it defines for any j in the interval $0 \le j \le j_c$ a value of f_0 which lies between $f_{-\infty}$ and f_∞. Therefore the condition for smooth continuity cannot give rise to a new critical situation, and the critical current is determined by the conditions in the porous medium ($x = \infty$), that is, by expression (21). The minimum density of the superfluid component in a porous substance is then not equal to zero, and $f_{\infty c}$ is determined by formula (23). Consequently when j_c is exceeded we again expect a change in the density of the superfluid component.

8. We now consider the flow of a superfluid liquid through a porous partition of thickness 2d in the x direction and extending to infinity in the two remaining directions. The free volume of the helium occupies the regions $x < -d$ and $x > d$. Since the problem is symmetrical, we can confine ourselves to a solution in the right half-space $x > 0$. The solution of (19) in the free volume $x > d$ is similar to the first formula in (26):

$$\left[\frac{f^2+2(f_\infty^2-1)}{3f_\infty^2-2}\right]^{1/2} = \operatorname{th}\left\{\left(\frac{3f_\infty^2-2}{2}\right)^{1/2}(x-d) + \operatorname{Arth}\left[\frac{f_d^2+2(f_\infty^2-1)}{3f_\infty^2-2}\right]^{1/2}\right\}, \tag{28}$$

where f_∞ is given by (22) with $a = b = c = 1$ and f_d denotes the value of f when $x = d$.

The solution of (19) in a porous partition is again determined with the aid of the first integral in (24), which under the conditions in question ($df/dx = 0$ when $x = 0$, but d^2f/dx^2 different from zero at this point) yields

$$\left(\frac{2b}{c}\right)^{1/2} f\frac{df}{dx} = \left\{(f^2-f_0^2)\left[f^4+\left(f_0^2-\frac{2b}{c}a^2\right)f^2+\frac{2b}{c}\frac{j^2}{f_0^2}\right]\right\}^{1/2}, \tag{29}$$

where f_0 is the value of f at the point $x = 0$ ($f_0 = f_{min}$).

Integrating (29), we obtain for the function f with $x < d$ implicit expressions containing elliptical integrals of the first kind:

$$\left(\frac{c}{2b}\right)^{1/2} x = \frac{1}{\sqrt{q_+}} F\left[\arcsin\left(\frac{f^2-f_0^2}{f^2-f_0^2+q_-}\right)^{1/2}, \left(\frac{q_+-q_-}{q_+}\right)^{1/2}\right] \tag{30a}$$

for

$$\left(f_0^2-\frac{2b}{c}a^2\right)^2-\frac{8b}{c}\frac{j^2}{f_0^2} = (q_+-q_-)^2 \geqslant 0, \qquad q_- \geqslant 0;$$

$$\left(\frac{2c}{b}\right)^{1/2} x = \frac{1}{\sqrt{p}} F\left[2\tan^{-1}\left(\frac{f^2-f_0^2}{p}\right)^{1/2}, \left(\frac{p-3f_0^2/2+ba^2/c}{2p}\right)^{1/2}\right] \tag{30b}$$

for

$$\left(f_0^2-\frac{2b}{c}a^2\right)^2-\frac{8b}{c}\frac{j^2}{f_0^2} < 0.$$

In (30a) and (30b) we introduce the notation:

$$p = \left[\left(\frac{3}{2}f_0^2-\frac{b}{c}a^2\right)^2+\frac{2b}{c}\frac{j^2}{f_0^2}-\frac{1}{4}\left(f_0^2-\frac{2b}{c}a^2\right)^2\right]^{1/2} = \sqrt{2\left(f_0^4-\frac{b}{c}a^2f_0^2+\frac{b}{c}\frac{j^2}{f_0^2}\right)},$$

$$q_\pm = \frac{3}{2}f_0^2-\frac{b}{c}a^2 \pm \frac{1}{2}\left[\left(f_0^2-\frac{2b}{c}a^2\right)^2-\frac{8b}{c}\frac{j^2}{f_0^2}\right]^{1/2}.$$

The condition for smooth joining (continuity of the derivative df/dx at the point $x = d$) is of the form

$$(f_\infty^2-f_d^2)[f_d^2+2(f_\infty^2-1)]^{1/2} = \left\{\frac{c}{b}(f_d^2-f_0^2)\left[f_d^4+\left(f_0^2-\frac{2b}{c}a^2\right)f_d^2+\frac{2b}{c}\frac{j^2}{f_0^2}\right]\right\}^{1/2}. \tag{31}$$

Formulas (29)–(31) enable us, in principle, to determine the values of f_0 and f_d for given values of d and j, and also to find for each value of d the critical value of the current j_c, above which formulas (30a) and (30b) no longer determine the sought implicit function f, that is, above which (19) has no solutions with $f^2 > 0$.

The latter problem can be solved not only by investigating the relations that violate the conditions of the theorem for the existence of the implicit function, but also by considering directly Eq. (19) at the point $x = 0$. Since f is minimal at this point, we have

$$f_0^6-\frac{b}{c}a^2f_0^4-\frac{b}{c}A^2f_0^2+\frac{b}{c}j^2 = 0, \tag{32}$$

where $A^2 = 0.5(d^2f^2/dx^2)_{x=0}$. Just as (20), Eq. (32) has positive solutions with respect to f_0^2 only when $j \le j_c$, where in this case

$$j_c^2 = \frac{2b^2}{27c^2}a^6+\frac{b}{3c}A^2a^2+\frac{2b^2}{27c^2}\left(a^4+\frac{3c}{b}A^2\right)^{3/2}. \tag{33}$$

Calculation of A from the obtained implicit solution is feasible but involves computational difficulties; these can be circumvented by recognizing that we are interested in the case of small a, and that under real conditions the quantity d (measured in units of a_0) is always very large. Then $f_0 \ll f_d$ and $A \ll 1$. But $p \equiv (2b/c)^{1/2}A$ and under the conditions in question the function F in formula (30b) is close to $F(\pi,0) = \pi$ when $x = d$.[1)]

[1)]When $j = 0$ the conditions for the applicability of formula (30a) are fulfilled, but an increase in j contributes to a change in the direction of the corresponding inequality. The critical situation sets in already after the transition from (30a) to (30b).

Thus, for sufficiently broad partitions

$$A^2 \approx \pi^4 b / 8cd^4, \tag{34}$$

whence

$$j_c^2 \approx \frac{2b^2}{27c^2} a^6 + \frac{\pi^4 b^2}{24c^2} \frac{a^2}{d^4} + \frac{2b^2}{27c^2}\left(a^4 + \frac{3\pi^4}{8d^4}\right)^{3/2}. \tag{35}$$

The corresponding value of $f_0(j_c^2) = f_{0c}$ is

$$f_{0c}^2 \approx \frac{b}{3c}\left[a^2 + \left(a^4 + \frac{3\pi^4}{8d^4}\right)^{1/2}\right]. \tag{36}$$

We emphasize once more that formulas (35) and (36) are valid for small a and large d.

In particular, when $d = \infty$ we have again formulas (21) and (23), and in the case of an "insulator," $a = 0$, we get

$$j_c^2 \approx \frac{\sqrt{6}\,\pi^6 b^2}{144c^2 d^6}. \tag{35a}$$

$$f_{0c}^2 \approx \frac{\sqrt{6}\,\pi^2 b}{12cd^2}. \tag{36a}$$

Formula (35a) ensures in principle the realizability of the direct analog of the Josephson dc effect in helium II. If, for example, we introduce a porous partition into an instrument similar to the paired vessels in which Daunt and Mendelssohn observed "inertial flow,"[7] then we can expect the flow effect to be conserved in the absence of a difference in levels.[2)] The liquid flux is determined in this case by (35) and will differ from zero even when $a = 0$.

It must be noted, however, that in the latter case ($a = 0$) the flux defined by formula (35a) is very small. If, reverting to dimensional notation, we denote the thickness of the partitions by $D = 2da_0$, then the flux J_c, measured in g-cm^{-2} sec, is given by the formula

$$J_c = \frac{6^{1/4}\pi^3\hbar^3 b}{3m\beta cD^3}. \tag{35b}$$

The lowering of the levels due to such a flux of the superfluid component is equal to

$$\Delta H = \frac{6^{1/4}\pi^2 b\hbar^3}{3m^2\alpha cD^3} B, \tag{37}$$

where $B = 2\pi Ht/R$, if one considers the outflow of liquid through the porous lateral surface of a cylinder with a radius R (H is the height of the liquid level, t is the time) and $B = St/r^2$ if the liquid flows through a partition with area S, and the level is registered in a capillary with radius r. The measurement of (37) is at the borderline of present experimental capabilities.

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