ELECTRODYNAMICS OF A HOMOGENEOUS ANISOTROPIC AND DISPERSIVE MEDIUM

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A four-dimensional formalism is employed to determine the electromagnetic field strength and the induction produced by arbitrary sources. The field is found in a medium with given electric and magnetic permeability tensors. The formulas are valid also for a uniformly moving medium.

1. FOUR-DIMENSIONAL STATEMENT OF THE PROBLEM

We shall consider the electric current density $\mathbf{I}(\mathbf{r}, t)$ and the electric charge density $\rho(\mathbf{r}, t)$ to be given. It is required to find the electromagnetic field, that is, the field intensities $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ and the inductions $\mathbf{D}(\mathbf{r}, t)$ and $\mathbf{H}(\mathbf{r}, t)$. The three-dimensional statement of the problem leads to equations that are difficult to visualize; therefore, even if one seeks a solution in a medium at rest, it is more convenient to transform to the four-dimensional statement of the problem, inasmuch as the equations are then much clearer.

Since the connection between the inductions and the field strengths ceases to be an operator relation only when this connection exists between the Fourier amplitudes of these quantities, and since the imaginary character figures prominently in the Fourier expansion, it becomes inconvenient to use the four-dimensional Cartesian set of coordinates in which the time coordinate $x_i = ict$ also has an imaginary character. We shall make use below of the Galilean system of coordinates x^1, x^2, x^3, $x^0 = ct$, for which the metric tensor has the form

$$g_{ik} = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix}. \tag{1.1}$$

The electromagnetic field in a material medium is described by two antisymmetric 4-tensors of the second rank: the field strength 4-tensor F_{ik} and the induction 4-tensor Φ_{ik}. In addition to the field strength 4-tensor F_{ik}, we introduce the 4-tensor that is its dual:

$$\bar{F}^{ik} = {}^1/_2 \varepsilon^{iksr} F_{sr}, \tag{1.2}$$

where ϵ^{iksr} are the contravariant components of the unitary, completely antisymmetric pseudo 4-tensor (see, for example, [1], Par. 83; [2], Par. 22, and the end of Par. 37), the covariant components of which are such that

$$\varepsilon_{1230} = 1, \qquad \varepsilon^{iksr} = -\varepsilon_{iksr}.$$

Inasmuch as F_{sr} is a true 4-tensor then, in accord with (1.2), $\bar{F}^{ik}$ is a pseudo 4-tensor.

The equations of the electromagnetic field in material media have the following form in the four-dimensional representation:

$$\frac{\partial \Phi^{ik}}{\partial x^k} = \frac{4\pi}{c} I^i, \tag{1.3}$$

$$\frac{\partial \bar{F}^{ik}}{\partial x^k} = 0 \tag{1.4}$$

[see, for example, [3], Par. 33, Eqs. (270) and (271)]. Here I^i are the contravariant components of the 4-vector of the current density: I^1, I^2, I^3 form the 3-vector current density I, $I^0 = c\rho$. Equation (1.4) is satisfied if we set

$$\bar{F}^{ik} = \frac{\partial \mathfrak{N}^{iks}}{\partial x^s}, \tag{1.5}$$

where $\mathfrak{N}^{iks}$ is a 4-tensor that is antisymmetric in all its three indices. It is seen from (1.5) that it is a pseudotensor. Usually, in place of $\mathfrak{N}^{iks}$ there is introduced the 4-vector A_r that is dual to it, with the help of the relation

$$\mathfrak{N}^{iks} = \varepsilon^{iksr} A_r, \tag{1.6}$$

which is the 4-vector potential of the electromagnetic field. Inasmuch as $\mathfrak{N}^{iks}$ is a pseudo 4-tensor, then it is seen from (1.6) that A_r is a true 4-vector.

We note that the pseudo 4-tensor $\mathfrak{N}^{iks}$ is determined with an accuracy up to the component $\partial \Theta^{iksr}/\partial x^r$, where Θ^{iksr} is an arbitrary, completely antisymmetric pseudo 4-tensor. Indeed, by substituting the pseudo 4-tensor

$$\mathfrak{N}^{iks}+\frac{\partial}{\partial x^r}\Theta^{iksr},$$

on the right hand side of Eq. (1.5) in place of $\mathfrak{N}^{iks}$, we obtain the pseudo 4-tensor $\overline{F}^{ik}$ as before, inasmuch as

$$\frac{\partial^2}{\partial x^s\,\partial x^r}\Theta^{iksr}=0$$

as a consequence of the antisymmetry of Θ^{iksr} in the indices s and r.

Setting

$$\Theta^{iksr}=\Theta\varepsilon^{iksr}, \tag{1.7}$$

where Θ is a true 4-scalar, one can show that one can impose a single arbitrary scalar condition on the components of $\mathfrak{N}^{iks}$ (and consequently on the components of A_r also), which can only limit the arbitrariness in the selection of the 4-scalar Θ.

For the determination of the 4-vector A_r, one must use the equation (1.3), and also the connection of the 4-tensors Φ^{ik} and $\overline{F}^{ik}$. In a dispersive medium, this connection has the form

$$\Phi^{ik}=\hat{S}^{ik\cdot\cdot}_{\cdot\cdot sr}\overline{F}^{sr}, \tag{1.8}$$

where $\hat{S}^{ik\cdot\cdot}_{\cdot\cdot sr}$ is a pseudo 4-tensor operator.

The Fourier expansion yields

$$\Phi^{ik}(x^p)=\int\varphi^{ik}(k_s)\,e^{ik_qx^q}\,d^4k_m, \tag{1.9}$$

$$\overline{F}^{ik}(x^p)=\int\overline{f}^{ik}(k_s)\,e^{ik_qx^q}\,d^4k_m, \tag{1.10}$$

where k_1, k_2, k_3 are the components of the wave 3-vector $\mathbf{k}$, $k_0=-\omega/c$, ω is the frequency. Here $d^4k_m\equiv dk_1dk_2dk_3dk_0$ is the volume element in the space of the 4-vector k_i, $L(x^p)$ denotes a function of all four components of the 4-vector; for example,

$$\Phi^{ik}(x^p)\equiv\Phi^{ik}(x^1,x^2,x^3,x^0).$$

The connection between the Fourier amplitudes

$$\varphi^{ik}=S^{ik\cdot\cdot}_{\cdot\cdot sr}\overline{f}^{sr} \tag{1.8a}$$

exists not through the operator but through the c-number, which is a pseudo 4-tensor $S^{ik\cdot\cdot}_{\cdot\cdot sr}$.

The components of this pseudo 4-tensor in the fixed medium can be expressed in terms of the components of the 3-tensors of the dielectric constants $\epsilon^{\alpha\beta}$ and magnetic permeabilities $\mu^{\alpha\beta}$ if we compare the formulas

$$d^\alpha=\varepsilon^{\alpha\beta}e_\beta,\qquad h_\alpha=\mu_{\alpha\beta}^{-1}b^\beta$$

(which connect the Fourier amplitudes of the 3-vectors **D** and **E**, **H** and **B**) with the formula (1.8a), in which the components of the 4-tensors $\overline{f}^{sr}$ and φ^{ik} must be expressed by the 3-vectors **e**, **b** and **d**, **h** (see, for example, [1], the note on p. 283; [2], p. 95). As a result, we get

$$S^{\alpha\beta\cdot\cdot}_{\cdot\cdot\gamma\sigma}=0,\quad S^{\alpha\beta\cdot\cdot}_{\cdot\cdot\gamma0}={}^1\!/_2\,\varepsilon^{\alpha\beta\sigma}\mu^{-1}_{\sigma\gamma},$$
$$S^{\alpha0\cdot\cdot}_{\cdot\cdot\beta0}=0,\quad S^{\alpha0\cdot\cdot}_{\cdot\cdot\beta\gamma}={}^1\!/_2\varepsilon^{\alpha\sigma}\varepsilon_{\sigma\beta\gamma}. \tag{1.11}$$

Here and below, the Greek indices run through the values 1, 2, 3 and enumerate the spatial coordinates, while the Latin indices run through the values 1, 2, 3, 0 and enumerate all the coordinates of 4-space. The first 4-tensor permeability was introduced by Tamm. Expanding the 4-vector of the current density in a Fourier integral also,

$$I^i(x^p)=\int j^i(k_s)\,e^{ik_qx^q}\,d^4k_m, \tag{1.12}$$

we obtain the following complete set of equations [in place of (1.3), (1.4), (1.8)] for the Fourier amplitudes of the quantities characterizing the field:

$$\varphi^{ik}k_k=-i4\pi j^i/c, \tag{1.13}$$

$$\overline{f}^{ik}k_k=0, \tag{1.14}$$

$$\varphi^{ik}=S^{ik\cdot\cdot}_{\cdot\cdot sr}\overline{f}^{sr}. \tag{1.15}$$

2. SOLUTION OF THE SET OF EQUATIONS OF ELECTRODYNAMICS FOR THE FOURIER AMPLITUDES

Equation (1.14) can be satisfied by setting

$$\overline{f}^{ik}=iN^{iks}k_s, \tag{2.1}$$

where N^{iks} is a pseudo 4-tensor which is antisymmetric in all three indices and which appears in the Fourier amplitude of the pseudo 4-tensor $\mathfrak{N}^{iks}$ introduced in (1.5). As with $\mathfrak{N}^{iks}$ there is an ambiguity in the definition of N^{iks}. To be precise, one can add $T^{iksr}k_z$ to N^{iks} without changing $\overline{f}^{ik}$ in the process; here the pseudo 4-tensor T^{iksr} is arbitrary and is antisymmetric in all four indices. This is so because $T^{iksr}k_sk_r\equiv 0$ as the product of a symmetric 4-tensor k_sk_r and a 4-tensor that is antisymmetric in the indices s and r.

One can always set

$$T^{iksr}=T\varepsilon^{iksr}, \tag{2.2}$$

where T is a true 4-scalar, just as arbitrary as T^{iksr}. We introduce the 4-vector a_r, which is dual to the pseudo 4-tensor N^{iks}, by means of the relation

$$N^{iks}=\varepsilon^{iksr}a_r. \tag{2.3}$$

Inasmuch as one can add $T\epsilon^{iksr}k_r$ with the arbitrary 4-scalar T to N^{iks} without changing $\overline{f}^{ik}$ in the process, one can impose a completely arbitrary condition (which only limits our freedom in the selection of the 4-scalar T) on the component N^{iks} (and consequently on the component a_r).

The quantities T^{iksr}, T and a_r introduced here are the Fourier amplitudes corresponding to Θ^{iksr}, Θ, and the 4-vector potential A_r.

Substituting (2.3) in (2.1), forming φ^{ik} with the help of (1.15), and using (1.13), we get an equation for the 4-potential

$$T^{in}a_n = -4\pi j^i / c, \tag{2.4}$$

where

$$T^{in} = S^{ik}_{\cdot\cdot sr}\, \varepsilon^{srmn}k_k k_m = R^{ikmn}\, k_k k_m \tag{2.5}$$

is a true 4-tensor.

Here we have introduced the 4-tensor

$$R^{ikmn} = S^{ik}_{\cdot\cdot sr}\varepsilon^{srmn}. \tag{2.6}$$

Inasmuch as the following relation, similar to (1.2), holds between the mutually dual 4-tensors $\bar{f}^{ik}$ and f^{ik}:

$$\bar{f}^{ik} = {}^1/_2\varepsilon^{iksr}f_{sr}, \tag{2.7}$$

we see, substituting this in (1.15) that the 4-tensor introduced in (2.6) is the connection between φ^{ik} and f_{mn}:

$$\varphi^{ik} = {}^1/_2 R^{ikmn}f_{mn}. \tag{2.8}$$

The expressions for the components of the 4-tensor R^{ikmn} in terms of the components of the electric and magnetic 3-tensors have the form (in the medium at rest)

$$R^{\alpha\beta\gamma 0} = 0, \quad R^{\alpha\beta\gamma\sigma} = \varepsilon^{\alpha\beta\nu}\mu^{-1}_{\nu\lambda}\varepsilon^{\lambda\gamma\sigma},$$
$$R^{\alpha 0\beta\gamma} = 0, \quad R^{\alpha 0\beta 0} = -\varepsilon^{\alpha\beta}. \tag{2.9}$$

We note that, since the 4-tensor R^{ikmn} is antisymmetric in both the first and the second pair of its indices, then it follows from the definition of the 4-tensor T^{in} of (2.5) that

$$k_i T^{in} = T^{in}k_n = 0. \tag{2.10}$$

Hence (and also directly from (1.13) by virtue of the antisymmetric character of φ^{ik})

$$j^i k_i = 0, \tag{2.11}$$

which expresses the law of conservation of electric charge in terms of the Fourier amplitudes.

It must be expected from (2.11) that the four equations (2.4) are not independent, but this in turn must mean that the determinant $|T^{ik}|$, constructed from the components of the 4-tensor T^{ik}, has the value

$$|T^{ik}| = 0. \tag{2.12}$$

On the other hand, if we denote by the symbol $\Psi_{ik}^{(T^{rs})}$ the cofactor of the four-dimensional determinant $|T^{rs}|$, then the formal solution of the set (2.4) is

$$a_k = -\frac{4\pi}{c}\frac{j^i\Psi_{ik}^{(T^{rs})}}{|T^{rs}|}.$$

Inasmuch as the 4-vector a_k differs from zero, we must also have, if Eq. (2.12) is correct,

$$j^i\Psi_{ik}^{(T^{rs})} = 0. \tag{2.13}$$

Direct calculations confirm the correctness of Eqs. (2.12) and (2.13).

Since the expression mentioned above for a is seen to be indeterminate, of the type 0/0, it is then seen that it is impossible to solve the set of equations (2.4) without imposing an additional condition on the components of the 4-vector a_i. We spoke of this earlier. We now subject the 4-vector a_i to the condition

$$P^i a_i = 0, \tag{2.14}$$

where P^i is a 4-vector which can depend only on the permittivity and permeability and on the wave 4-vector k_i. By virtue of (2.14), regardless of the nature of 4-vector Q^i, we have the identity

$$Q^i P^k a_k \equiv 0.$$

(It is seen from what follows that the solution for a_k does not depend on Q^i.) Combining this identity with Eq. (2.4), we obtain

$$(T^{ik} + Q^i P^k)a_k = -4\pi j^i / c. \tag{2.15}$$

The solution of the set (2.15) is

$$a_k = -\frac{4\pi}{c}\frac{j^i\Psi_{ik}^{(T^{rs}+Q^rP^s)}}{|T^{rs}+Q^rP^s|}, \tag{2.16}$$

where the determinant in the denominator is composed of the four-dimensional matrix $T^{rs} + Q^rP^s$, and there is in the numerator the cofactor of this determinant, corresponding to the element with indices i and k.

Calculations lead to the following expression

$$a_k = -\frac{4\pi}{c}\frac{j^iG_{iq}}{V}\left(\delta_k{}^q - \frac{P^qk_k}{P^rk_r}\right), \tag{2.17}$$

where V is a 4-scalar having the form

$$V = \frac{2}{4!}k_aR^{abcd}\varepsilon_{cdnm}k_pR^{pqmg}k_g\varepsilon_{qbsr}k_lR^{lsnr}. \tag{2.18}$$

If we carry out the summation in (2.18) and use the values of the components from (2.9), then, after simple but tedious calculations (which are omitted here), an expression is obtained for the 4-scalar V (in the medium at rest):

$$V = k_0^4 |\varepsilon^{\alpha\beta}| + k_0^2 (g^\alpha \mu_{\alpha\beta}^{-1} p^\beta - T^{00} \varepsilon^{\alpha\beta} \mu_{\alpha\beta}^{-1}) + T^{00} \frac{k_\alpha \mu^{\alpha\beta} k_\beta}{|\mu^{\alpha\beta}|}. \quad (2.18a)$$

Here $|\epsilon^{\alpha\beta}|$ and $|\mu^{\alpha\beta}|$ are determinants composed respectively of the components of the 3-tensors $\epsilon^{\alpha\beta}$ and $\mu^{\alpha\beta}$,

$$p^\alpha = k_\beta \varepsilon^{\beta\alpha}, \quad g^\alpha = \varepsilon^{\alpha\beta} k_\beta, \quad T^{00} = k_\alpha \varepsilon^{\alpha\beta} k_\beta. \quad (2.18b)$$

For an isotropic medium at rest,

$$\varepsilon^{\alpha\beta} = \varepsilon g^{\alpha\beta}, \quad \mu^{\alpha\beta} = \mu g^{\alpha\beta},$$

therefore,

$$\mu_{\alpha\beta}^{-1} = \frac{1}{\mu} g_{\alpha\beta}, \quad p^\alpha = g^\alpha = \varepsilon k^\alpha, \quad T^{00} = \varepsilon k^2.$$

Substituting these expressions in (2.18a), we get for the isotropic, fixed medium,

$$V = \varepsilon (\varepsilon k_0^2 - k^2 / \mu)^2. \quad (2.18c)$$

The 4-tensor G_{ik} which enters into (2.17) is expressed in the following fashion:

$$G_{ik} = {}^1\!/_4 \varepsilon_{ibrs} k_l R^{lsnb} k_a R^{armt} \varepsilon_{mtnk}. \quad (2.19)$$

In the fixed medium, the components G_{ik} are expressed in terms of the electric and magnetic 3-tensors

$$G_{00} = k_\alpha \mu^{\alpha\beta} k_\beta / |\mu^{\alpha\beta}|,$$

$$G_{j\varkappa} = k_0^2 \psi_{j\varkappa}^{(\varepsilon)} + k_j p^\alpha \mu_{\alpha\varkappa}^{-1} - T^{00} \mu_{j\varkappa}^{-1} + {}^1\!/_2 (\mu_{j\alpha}^{-1} p^\alpha k_\varkappa - k_j \mu_{\varkappa\alpha}^{-1} p^\alpha),$$

$$G_{j0} = {}^1\!/_2 k_0 (k_j \varepsilon^{\alpha\beta} \mu_{\alpha\beta}^{-1} - \mu_{j\alpha}^{-1} p^\alpha),$$

$$G_{0\varkappa} = {}^1\!/_2 k_0 [k_\varkappa \varepsilon^{\alpha\beta} \mu_{\alpha\beta}^{-1} - \mu_{\varkappa\alpha}^{-1} p^\alpha + 2 (p^\alpha \mu_{\alpha\varkappa}^{-1} - g^\alpha \mu_{\alpha\varkappa}^{-1})]. \quad (2.19a)$$

In (2.19a), in addition to the notation which appears in (2.18b), we have

$$\psi_{j\varkappa}^{(\varepsilon)} = {}^1\!/_2 \varepsilon_{j\alpha\beta} \varepsilon_{\varkappa\gamma\sigma} \varepsilon^{\alpha\gamma} \varepsilon^{\beta\sigma},$$

which is the cofactor of the determinant $|\epsilon^{\alpha\beta}|$ corresponding to the element $\epsilon^{\alpha\beta}$.

As is seen from (2.19), and more explicitly from (2.19a), the 4-tensor G_{ik} is generally asymmetric. For an isotropic stationary medium, the components G_{ik} take the form

$$G_{00} = k^2/\mu^2,$$

$$G_{j\varkappa} = \varepsilon \left[\frac{k_j k_\varkappa}{\mu} + \left(k_0^2 \varepsilon - \frac{k^2}{\mu} \right) g_{j\varkappa} \right], \quad G_{0\varkappa} = G_{\varkappa 0} = \frac{\varepsilon}{\mu} k_0 k_\varkappa, \quad (2.19b)$$

from which it is seen that in the isotropic medium the 4-tensor G_{ik} is symmetric.

It follows from (2.17) that in the stationary field ($k_0 = 0$), the Fourier-amplitude of the scalar potential

$$a_0 = -\frac{4\pi}{c} \frac{j^i G_{i0}}{V}$$

does not depend on the additional condition, since the component containing the 4-vector P^i drops out. The same thing can be said about the scalar potential A_0.

It is not difficult to establish the fact that the solution of (2.17) satisfies the condition (2.14). The 4-vector P^i itself plays a purely auxiliary role: by its means we can avoid the difficulty presented by the fact that the solution of the set (2.4) leads to an indeterminacy. In the expression for the 4-tensors of the field strengths and inductions, as is pointed out below, P^i does not appear, but rather the expression for the 4-potential A_i found from $\overline{F}^{ik}$. We find the Fourier amplitude of the 4-tensor $\overline{f}^{ik}$ if we substitute the expression for a_i from (2.17) in the formula

$$\overline{f}^{ik} = i \varepsilon^{iksr} k_s a_r,$$

which is obtained upon substitution of (2.1) into (2.3). Here the component in a_r which is proportional to k_r drops out, since $\epsilon^{iksr} k_r k_s = 0$, and we get the formula

$$\overline{f}^{ik} = -i \frac{4\pi}{c} \frac{\varepsilon^{iksr} k_s j^n G_{nr}}{V}. \quad (2.20)$$

Forming φ^{ik} by means of (2.15) and using the definition (2.6), we find

$$\varphi^{ik} = -i \frac{4\pi}{c} \frac{R^{iksr} k_s j^n G_{nr}}{V}. \quad (2.21)$$

Equations (2.20) and (2.21) give a solution of the problem set up for the Fourier amplitudes of the 4-tensors of the field strengths and inductions.

3. FIELD STRENGTHS, INDUCTIONS AND POTENTIALS

To obtain the field strength 4-tensor $\overline{F}^{ik}$, it is necessary, in accord with (1.1), to multiply both sides of Eq. (2.20) by $e^{ik_q x^q}$ and integrate over the space of the 4-vector k_i. Moreover, if we reverse the expansion of (1.12)

$$j^n (k_i) = \frac{1}{(2\pi)^4} \int_{X^m} I^n (X^m) e^{-ik_q X^q} d^4 X^m, \quad (3.1)$$

we then obtain

$$\overline{F}^{ik} (x^p) = -i \frac{4\pi}{c (2\pi)^4} \int_{X^m} d^4 X^m I^n (X^m) \int_{k_f} \frac{d^4 k_f}{V} \varepsilon^{iksr} k_s G_{nr} e^{ik_q (x^q - X^q)}, \quad (3.2)$$

where x^p is the coordinate of the observation point and X^m is the source point.

In the same way, we get the following 4-tensor of the induction from (2.21):

$$\Phi^{ik}(x^p) = -\frac{4\pi}{c(2\pi)^4}\int_{X^m} d^4X^m I^n(X^m) \times \int_{k_f} \frac{d^4k_f}{V} R^{iksr} k_s G_{nr} e^{ik_q(x^q - X^q)}. \tag{3.3}$$

We note that

$$ik_s e^{ik_q(x^q - X^q)} = \frac{\partial}{\partial x^s} e^{ik_q(x^q - X^q)}.$$

Therefore, instead of (3.2), we can write

$$\bar{F}^{ik}(x^p) = \varepsilon^{iksr}\frac{\partial}{\partial x^s} \times \left[-\frac{4\pi}{c(2\pi)^4}\int_{X^m} d^4X^m I^n(X^m)\int_{k_f}\frac{G_{nr}}{V} e^{ik_q(x^q - X^q)} d^4k_f\right]. \tag{3.2a}$$

On the other hand, substituting (1.6) in (1.5), we find

$$\bar{F}^{ik} = \varepsilon^{iksr}\frac{\partial A_r}{\partial r^s}. \tag{3.4}$$

The components of the 4-tensor $\bar{F}^{ik}$ that differ from zero on the basis of (3.4) are expressed in the form

$$\bar{F}^{\alpha\beta} \equiv \varepsilon^{\alpha\beta\gamma}E_\gamma = \varepsilon^{\alpha\beta\gamma}\left(\frac{\partial A_0}{\partial x^\gamma} - \frac{\partial A_\gamma}{\partial x^0}\right)$$

$$\bar{F}^{\alpha 0} \equiv B^\alpha = \varepsilon^{\alpha\beta\gamma}\frac{\partial A_\gamma}{\partial x^\beta}. \tag{3.4a}$$

Comparing (3.4) with (3.2a), we find that the 4-vector potential at the point of observation can be computed by the formula

$$A_r(x^p) = -\frac{4\pi}{c(2\pi)^4}\int_{X^m} d^4X^m I^n(X^m)\int_{k_f}\frac{G_{nr}}{V} e^{ik_q(x^q - X^q)} d^4k_f. \tag{3.5}$$

In the same way, the 4-tensor of the inductions (3.3) can be written in the form

$$\Phi^{ik} = \frac{\partial L^{iks}}{\partial x^s}, \tag{3.6}$$

where the 4-tensor L^{iks} is determined by the formula

$$L^{iks}(x^p) = -\frac{4\pi}{c(2\pi)^4}\int_{X^m} d^4X^m I^n(X^m) \times \int_{k_f}\frac{R^{iksr}G_{nr}}{V} e^{ik_q(x^q - X^q)} d^4k_f. \tag{3.7}$$

Formulas (3.4)–(3.7) also give a solution for a uniformly moving medium.

The application of the general formula (3.5) to the stationary case of a fixed medium leads to the expression for the scalar potential:

$$\Phi(\mathbf{r}) = \frac{1}{|\varepsilon^{\alpha\beta}|^{1/2}}\int_{\mathbf{R}} [\varepsilon_{\alpha\beta}^{-1}(x^\alpha - X^\alpha)(x^\beta - X^\beta)]^{-1/2}\rho(\mathbf{R})\,d\mathbf{R} \tag{3.8}$$

and to the expression for the vector potential:

$$A_\gamma(\mathbf{r}) = \frac{1}{c}\mu_{\gamma\alpha}^{-1}|\mu^{\beta\sigma}|^{1/2} \times \int_{\mathbf{R}} [\mu_{\lambda\nu}^{-1}(x^\lambda - X^\lambda)(x^\nu - X^\nu)]^{-1/2} I^\alpha(\mathbf{R})\,d\mathbf{R}, \tag{3.9}$$

which were obtained previously[5, 6] by the direct solution of the stationary field equations.

Thus one can assume that the general formulas obtained above have withstood one of the possible tests.

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